CONTROL IN TECHNICAL SYSTEMS

The Guaranteeing Estimation Method to Calibrate a Gyro Unit

P. A. Akimov[*,a] and A. I. Matasov[*,b]

[*]*Moscow State University, Faculty of Mechanics and Mathematics,*
Laboratory of Control and Navigation, Moscow, Russia
e-mail: [a]akmpavel@rambler.ru, [b]alexander.matasov@gmail.com

Abstract—This paper is devoted to the guaranteeing estimation method with application to the calibration problem of a gyro unit. Mathematical models are constructed to describe the kinematics of the gyro unit on a test bench. The applicability limits and errors of the models are investigated. A numerical solution procedure is developed for guaranteeing estimation problems based on their reduction to l_1-approximation problems.

Keywords: the guaranteeing approach to estimation, convex optimization, gyro unit, calibration, Python

1. INTRODUCTION

This paper proposes a calibration procedure for a unit of angular rate sensors (ARSs), i.e., gyros, based on the guaranteeing estimation method [1–5]. The purpose of calibration is to estimate parametric errors (biases, scale factors, and misalignment angles) in a unit consisting of three ARSs. These parameters are determined through a series of measurements on a bench with high-accuracy control of the angular rate and orientation of the unit. During calibration, rotation modes are selected and the resulting signals are processed.

Many traditional calibration methods involve dynamic models for estimating unknown parameters based on measurements of inertial navigation system (INS) sensors (accelerometers and gyros); for details, see [6–8]. Nevertheless [9], in some cases, it is preferable to determine ARS errors without involving information from accelerometers. Such a situation arises, first, when the accuracy of accelerometers is too low to use their signals in bench tests and, second, when calibrating laser gyros on vibration suspensions. The Kalman filter and the least squares method [10] are the main tool for estimation. The calibration problem includes a large number of unknown parameters (sensor errors, bench errors) that nonlinearly affect the measurement results. Therefore, two questions are essential here as follows. How can one construct a mathematical model of bench tests to consider all these factors? How can one minimize the impact of errors on the estimation result? A possible answer to these questions is presented in this paper, which continues the earlier research on the application of the guaranteeing approach in inertial navigation; for example, see [11], where the calibration of an accelerometer unit was considered. However, in contrast to the cited paper, this method is applied below to a different class of systems with a large number of unknown parameters and nonlinear effects. This class requires constructing other models and leads to other, structurally more complex estimation problems.

The guaranteeing estimation method allows estimating unknown parameters under the "worst-case" realizations of measurement errors with minimal estimation accuracy. In this case, typical

maximum values of the bench errors are used instead of probabilistic hypotheses. The resulting solution sets the directions of rotation of the gyro unit on the bench, i.e., explicitly describes the optimal calibration modes.

This paper is organized as follows. In Section 1, the mathematical model of bench tests of the gyro unit is constructed; the transition from the basic kinematic equations to several linear models with signal averaging is performed; the applicability limits of these models are investigated. Sections 2 and 3 formulate the guaranteeing estimation problems and the corresponding auxiliary variational l_1-approximation problems that can be solved numerically. In Section 4, we determine optimal experiment plans, apply the estimation algorithms to model problems, and analyzing the accuracy of the resulting solutions.

2. MATHEMATICAL MODEL

2.1. Basic Assumptions and Kinematic Relations for a Gyro Unit

Consider a mathematical model describing the basic kinematic relations of a gyro unit on a test bench during calibration [12]. Let an experiment be conducted at the point M rigidly coupled to the Earth on its surface. We introduce the following notations:

$Mz = Mz_1z_2z_3$ is the instrumental frame rigidly coupled to the gyro unit;

$Mx = Mx_1x_2x_3$ is the frame rigidly coupled to the bench base fixed relative to the Earth;

$D(t)$ is the orthogonal orientation matrix of Mz relative to Mx. By definition of an orientation matrix, for any vector l, its coordinates in the reference frames Mz, Mx have the relation $l_z = D(t)\,l_x$ and the rows of the orientation matrix consist of the coordinates of the basis vectors of Mz in the frame Mx;

$\Omega(t)$ is the angular rate vector of the gyro unit relative to the bench;

$\omega(t)$ is the absolute angular rate vector of the gyro unit;

u_x are the coordinates of the angular rate of the Earth in the frame Mx and u is the absolute angular rate of the Earth.

The absolute angular rate of the gyro unit is described in projections onto Mx by the relation

$$\omega_x(t) = \Omega_x(t) + u_x.$$

In projections onto the axis of the frame Mz, this equality takes the form

$$\omega_z(t) = D(t)\,(\Omega_x(t) + u_x). \tag{1}$$

At the initial time instant, the orientation matrix D is known with some accuracy. We denote this estimate $D(0)$ by D_{init}:

$$D(0) = D_{\text{init}}(I_3 + \hat{\beta}), \quad \beta = (\beta_1, \beta_2, \beta_3)^{\text{T}}, \quad \hat{\beta} = \begin{pmatrix} 0 & \beta_3 & -\beta_2 \\ -\beta_3 & 0 & \beta_1 \\ \beta_2 & -\beta_1 & 0 \end{pmatrix}, \tag{2}$$

$$|\beta_i| \leqslant \beta_{\max}, \quad i = 1, 2, 3.$$

The initial alignment errors of the gyro unit, i.e., the small rotation angles β_i, are unknown but their absolute values are bounded by $\beta_{\max}$. Throughout this paper, I_n stands for an identity matrix of dimensions $n \times n$ and $\hat{\beta}$ denotes the skew-symmetric matrix constructed from the vector β according to the above rule.

For the gyro unit, the output signals (measurements) are the readings of each ARS, i.e., the components of the vector $\omega_z(t)$. Let us introduce the measurement model

$$\zeta(t) = \omega_z(t) + \Gamma\omega_z(t) + \nu_0 + \delta\nu(t). \tag{3}$$

Here: $\zeta(t) \in \mathbf{R}^3$ are measurement values; $\Gamma \in \mathbf{R}^{3 \times 3}$ is an unknown matrix describing the scale factor errors and the orientation errors of the sensitivity axes; $\nu_0 \in \mathbf{R}^3$ are unknown zero biases in the sensor readings; finally, $\delta\nu(t) \in \mathbf{R}^3$ are nonparametric measurement errors (fluctuations). Without loss of generality, the matrix Γ is supposed symmetric [5].

The calibration problem consists in determining the values of Γ and ν_0 from the set of available measurements $\zeta(t)$. Note that the angular rate $\Omega_x(t)$ is controlled on the bench, and the optimal strategy of the unit's motion on the bench is one purpose of the mathematical calibration problem. Let the angular rate $\Omega_x(t)$ be written as

$$\Omega_x(t) = s'(t)w,$$

where $s'(t) \in \mathbf{R}$ denotes the angular rate and $w \in \mathbf{R}^3$ is the unit direction vector of the angular rate of Mz in projections onto Mx. The considerations below concern a special case of motion of the gyro unit on the bench that consists of several same-type experiments. Within each of them, the unit is rotated about a fixed axis with a given angular rate; then the unit is placed in a new position, a new direction of its rotation is set, and the experiment is repeated. The direction of the rotation axis and the angular rate are known with some errors. In other words, the function $s(t) \geqslant 0$ and vector y ($\|y\|_2 = 1$) are given in the expressions

$$s'(t) = s(t) + \varepsilon(t), \quad w = (I_3 + \hat{\alpha})y, \quad \Omega_x(t) = (s(t) + \varepsilon(t))(I_3 + \hat{\alpha})y, \tag{4}$$

which relate them to their true counterparts. The small rotation angles $\alpha \in \mathbf{R}^3$ and the corresponding skew-symmetric matrix $\hat{\alpha}$ determine the unknown errors in the rotation vector of the gyro unit whereas the scalar function $\varepsilon(t)$ determines the error in the angular rate value. As in the case of the angles β, the maximum possible values for α are known: $|\alpha_i| \leqslant \alpha_{\max}$, $i = 1, 2, 3$.

At each test stage, the errors α and β are constant but nonidentical in different experiments: by assumption, the programmed (target) angular rate and the unit orientation are set independently in each experiment. Thus, several series of measurements $\zeta(t)$ are formed that correspond to different rotation modes and different error realizations.

2.2. Linearization of the Equations and Signal Averaging

Substituting (1) and (4) into (3) yields the measurements

$$\zeta(t) = (I_3 + \Gamma)\, D(t)\, (s'(t)w + u_x) + \nu_0 + \delta\nu(t). \tag{5}$$

In addition to the signal $\zeta(t)$, two angular rate components, $D(t)u_x$ and $s'(t)D(t)w$, as well as the uncertain errors $\delta\nu(t)$ and $\varepsilon(t)$ depend on time. This section considers a mathematical model corresponding to the rotation mode of the gyro unit with a directionally constant angular rate on a time interval T. The calibration procedure will consist of a sequence of such rotation modes with different directions.

Next, we construct a "time-averaged" analog of the measurement equation by considering the averaging effect when Mz rotates relative to Mx. Averaging means calculating arithmetic means from a series of measurements on a long time interval T (several tens of minutes), during which the system makes multiple complete revolutions.

Recall that $D(t)$ is the transition matrix from Mx to Mz, and Mz is rigidly coupled to the gyro unit rotating relative to Mx with the angular rate Ω_x. The transition from Mx to Mz consists of three stages as follows.

1. Transition from Mx to Mx_{fix}, the stationary frame relative to the bench base, whose unit basis vector $e_{\text{fix}3}$ coincides with w in direction. We denote by $D_{\text{fix}} = (d_{\text{fix}1}; d_{\text{fix}2}; d_{\text{fix}3})$ the corresponding

transition matrix; its third row is $d_{\text{fix3}} = w^{\text{T}}$, and the first and second rows are orthogonal to it and to each other and can be chosen in any suitable way.

2. Rotation about the axis $Mx_{\text{fix3}} = Mz_{\text{cir3}}$ with the angular rate $s'(t)$, translating Mx_{fix} into the frame Mz_{cir} rigidly coupled to the gyro unit. We denote by $\psi(t)$ the time-dependent rotation angle in the plane $Mx_{\text{fix1}}x_{\text{fix2}}$. Therefore, $\frac{d\psi(t)}{dt} = s'(t)$, and the transition matrix from Mx_{fix} to Mz takes the form

$$D_{\text{cir}}(t) = \begin{pmatrix} \cos\psi(t) & -\sin\psi(t) & 0 \\ \sin\psi(t) & \cos\psi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3. Transition from Mz_{cir} to Mz through an inexactly known orthogonal matrix. For convenience of further calculations, this matrix is represented as $D' = (d_1', d_2', d_3')$.

Thus, the matrix $D(t)$ can be written as a product of the fixed and time-dependent transition matrices:

$$D(t) = D'D_{\text{cir}}(t)D_{\text{fix}}.$$

By definition, the first two rows of the matrix D_{fix} are orthogonal to w. Using this fact, we obtain the following expression for $s'(t)D(t)w$:

$$s'(t)D(t)w = s'(t)D'D_{\text{cir}}(t)D_{\text{fix}}w = s'(t)D' \begin{pmatrix} \cos\psi(t) & -\sin\psi(t) & 0 \\ \sin\psi(t) & \cos\psi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = s'(t)d_3'. \quad (6)$$

By the definition of the transition matrix D', the column d_3' consists of the projections of the unit basis vectors of the instrumental frame onto the axis Mz_{cir3}. Since the direction of Mz_{cir3} does not change in time and the rotation is about this axis, the projections of the unit basis vectors of the instrumental frame onto this direction will also remain constant. Therefore, d_3' can be determined from the a priori information (2):

$$d_3' = D(t)w = D(0)w = D_{\text{init}}(I_3 + \hat{\beta})w. \quad (7)$$

In other words, when rotating about a fixed axis, the direction of the vector $D(t)\Omega_x(t)$ remains constant; its averaging yields the vector $s'D_{\text{init}}(I_3 + \hat{\beta})w$, where s' is the mean value of $s'(t)$.

Due to the motion mode under consideration, the averaging result for the vector $D(t)u_x$ will have a special structure as follows.

Lemma 1. *Let the angular rate of the rotating frame be described by the function $s'(t) = s + \varepsilon(t)$, the "programmed" angular rate s be constant and $s > \varepsilon_{\max}$, and the rotation occur about a fixed direction w. Then, under the time averaging of the signal, the projections of the Earth's rotation rate onto the axes of Mz are described by*

$$\bar{u}_z = D_{\text{init}}(I_3 + \hat{\beta})ww^{\text{T}}u_x + u^{\perp}.$$

In addition, the unknown vector $u^{\perp}$ is orthogonal to the vector $D_{\text{init}}(I_3 + \hat{\beta})w$ and its components can be estimated as

$$|u_i^{\perp}| \leqslant u \left(\frac{4}{T(s - \varepsilon_{\max})} + C\frac{\varepsilon_{\max}}{s} \right) \stackrel{\text{def}}{=} u_{\max},$$

where C is a bounded value depending on the initial and final values of the rotation angle on the bench.

Table 1. The orders of values for model parameters

Parameter	The order of value
$\alpha, \alpha_{\max}$	1' $\approx 2.9 \times 10^{-4}$
$\beta, \beta_{\max}$	5' $\approx 1.5 \times 10^{-3}$
$\varepsilon(t)$	5×10^{-6} 1/s
$\varepsilon, \varepsilon_{\max}$	1×10^{-8} 1/s
s	17.5 1/s
T	600–1200 s
Γ_{ii}	5×10^{-3} (5×10^{-5})
$\Gamma_{ij}, \ i \neq j$	5×10^{-3} (5×10^{-5})
ν_0	2.4×10^{-7} 1/s (5×10^{-8} 1/s)
$\nu_{\max}$	1.2×10^{-8} 1/s
u	7.292115×10^{-5} 1/s

We obtain an explicit form of the measurement equation by passing to the mean values in (5) and using the expressions (4) and (7) with Lemma 1:

$$\tilde{\zeta} = (I_3 + \Gamma) \left(s' D_{\text{init}}(I_3 + \hat{\beta})w + D_{\text{init}}(I_3 + \hat{\beta}) \, ww^{\text{T}}u_x + u^{\perp} \right) + \nu_0 + \delta\tilde{\nu}$$

$$= (I_3 + \Gamma) D_{\text{init}}(I_3 + \hat{\beta}) \left((s + \varepsilon)(I_3 + \hat{\alpha})y + (I_3 + \hat{\alpha})yy^{\text{T}}(I_3 + \hat{\alpha})^{\text{T}}u_x \right) \qquad (8)$$

$$+ (I_3 + \Gamma) \, u^{\perp} + \nu_0 + \delta\tilde{\nu}.$$

Here, ε (the mean value of the noise $\varepsilon(t)$) and the error $\delta\tilde{\nu}$ (the mean value of the noise $\delta\nu(t)$) are supposed to be bounded:

$$|\varepsilon| \leqslant \varepsilon_{\max}, \quad |\delta\tilde{\nu}_j| \leqslant \nu_{\max}, \quad j = 1, 2, 3,$$

where a known constant $\nu_{\max}$ characterizes the a priori knowledge of the gyro error.

Depending on the scales of the variables α, β, Γ, ε, and ν, the measurement model can be simplified in different ways by neglecting one or another group of variables. In Table 1 below, we fix the characteristic scales of the model parameters corresponding to the typical accuracies of benches and gyros as well as the accuracy requirements for estimating the parameters Γ and ν_0 (indicated in parentheses).

Note that in the case under consideration, the values $\nu_{\max}$ and $\varepsilon_{\max}$ are much smaller than the characteristic amplitudes of $\delta\nu(t)$ and $\varepsilon(t)$. This corresponds to the averaging of the original signal. The approach proposed in this paper is applicable to other scales of variables as well; the original measurement model can be simplified in other ways depending on the real problem.

With the selected values, the terms in the measurements expression are divided into several groups: non-small terms, such as sy and $\tilde{\zeta}$; terms with a linear dependence on the small parameters α, β, ε, and u; negligibly small second- and third-order infinitesimals not exceeding $\nu_{\max}$; nonlinear terms that cannot be neglected due to their dependence on $s\Gamma D_{\text{init}}\hat{\alpha}$, $s\Gamma D_{\text{init}}\hat{\beta}$, $sD_{\text{init}}\hat{\alpha}\hat{\beta}$, $\Gamma u^{\perp}$, $u^{\perp}$ and (possibly) higher values than $\nu_{\max}$.

After eliminating the small terms, Eq. (8) takes the form

$$\tilde{\zeta} = D_{\text{init}} \left(s(I_3 + \hat{\alpha} + \hat{\beta})y + s\hat{\beta}\hat{\alpha}y + \varepsilon y + (I_3 + \hat{\alpha} + \hat{\beta})yy^{\text{T}}u_x - yy^{\text{T}}\hat{\alpha}u_x \right)$$

$$+ u^{\perp} + \Gamma u^{\perp} + \Gamma D_{\text{init}}(sy + yy^{\text{T}}u_x) + \Gamma D_{\text{init}}s(\hat{\alpha} + \hat{\beta})y + \nu_0 + \delta\tilde{\nu}.$$

The terms $sy^{\mathrm{T}}\hat{\beta}\hat{\alpha}y$ and $\Gamma D_{\mathrm{init}}s(\hat{\alpha}+\hat{\beta})y$ can be neglected if their value is comparable with the unremovable noise $\delta\tilde{\nu}$, i.e., $\|sy^{\mathrm{T}}\hat{\beta}\hat{\alpha}y\|_{\infty} \leqslant \nu_{\max}$ and $\|\Gamma D_{\mathrm{init}}s(\hat{\alpha}+\hat{\beta})y\|_{\infty} \leqslant \nu_{\max}$. The components of the matrix Γ have a known scale: $|\Gamma_{ij}| \leqslant \Gamma_{\max}$; see Table 1. Therefore, we introduce an additional constraint on the angular rate s:

$$s \leqslant \frac{\nu_{\max}}{2\max\{\alpha_{\max}\beta_{\max}, \Gamma_{\max}(\alpha_{\max}+\beta_{\max})\}} \overset{\mathrm{def}}{=} s_{\max}. \tag{9}$$

In other words, the errors α and β will have a smaller effect on the estimation result when rotating the gyro unit on the bench with a lower angular rate.

Under too slow rotation, it may turn out that the averaging error (the term $\Gamma u^{\perp}$) exceeds the required estimation accuracy. Hence, we obtain the second constraint on the parameter s: $\|\Gamma u^{\perp}\|_{\infty} \leqslant \nu_{\max}$. Due to the a priori known scale of the components of the matrix $\Gamma_{\max}$ and Lemma 1,

$$\|\Gamma u^{\perp}\|_{\infty} \leqslant 3\Gamma_{\max}\, u\, \left(\frac{4}{(s-\varepsilon_{\max})T} + C\frac{\varepsilon_{\max}}{s}\right) \leqslant \nu_{\max}.$$

The term $\Gamma_{\max}\, C\, u\, \varepsilon_{\max}/s$ is small compared to $\nu_{\max}$. Hence, the constraint on s takes the form

$$3\Gamma_{\max}\, u\, \frac{4}{(s-\varepsilon_{\max})T} \leqslant \nu_{\max}, \quad \text{or} \quad s \geqslant \frac{12\Gamma_{\max}u}{\nu_{\max}T} + \varepsilon_{\max} \overset{\mathrm{def}}{=} s_{\min}. \tag{10}$$

The effect of $u^{\perp}$ can be compensated by radically increasing T. However, see below, this is achieved in a different way (through scalarization).

In addition to rotation with the angular rate sy, another mode of bench tests is possible: the ARS unit is stationary relative to the bench base and the gyros measure the angular rate of the Earth's rotation. In this case, the expression for measurements can be obtained from (5) by substituting $s'=0$, $s=0$, $\varepsilon(t)=0$, and $D(t)=D_{\mathrm{init}}(I_3+\hat{\beta})$ and passing to the averaged signals $\tilde{\zeta}$:

$$\tilde{\zeta} = (I_3+\Gamma)\, D_{\mathrm{init}}(I_3+\hat{\beta})\, u_x + \nu_0 + \delta\tilde{\nu} = D_{\mathrm{init}}\, u_x + \Gamma\, D_{\mathrm{init}}u_x + D_{\mathrm{init}}\hat{\beta}\, u_x + \nu_0 + \delta\tilde{\nu}$$

within second-order infinitesimals.

Summarizing the results of this section, we formulate the averaged gyro signal model with the constraints (9) and (10):

$$\begin{aligned}
\tilde{\zeta} = {}& D_{\mathrm{init}}\left(s(I_3+\hat{\alpha}+\hat{\beta})y + \varepsilon y + (I_3+\hat{\alpha}+\hat{\beta})yy^{\mathrm{T}}u_x - yy^{\mathrm{T}}\hat{\alpha}u_x\right) \\
& + \Gamma D_{\mathrm{init}}(sy + yy^{\mathrm{T}}u_x) + u^{\perp} + \nu_0 + \delta\tilde{\nu}, \quad s \in \{0\}\cup[s_{\min}, s_{\max}].
\end{aligned} \tag{11}$$

2.3. Measurement Models and Scalarization

In Eq. (11), the input information is the terms $\tilde{\zeta}$ and $D_{\mathrm{init}}(sy + yy^{\mathrm{T}}u_x)$, whereas the "useful signal" is the terms $\Gamma D_{\mathrm{init}}(sy + yy^{\mathrm{T}}u_x) + \nu_0$. The measurement errors consist of the vector $\delta\tilde{\nu}$ arising when averaging the fluctuation noise $\delta\tilde{\nu}(t)$, and unknown systematic errors due to bench inaccuracy (varying with each new bench test). After rearranging the known terms to the left-hand side of Eq. (11), we obtain the linear measurement model

$$z(s,y) = \Gamma D_{\mathrm{init}}(sy + yy^{\mathrm{T}}u_x) + \nu_0 + r + \delta\nu', \tag{12}$$

with the "measurements" $z=z(s,y)$ and their errors $r=r(s,y,\alpha,\beta,\varepsilon)$ and $\delta\nu'$ given by

$$z = \tilde{\zeta} - D_{\mathrm{init}}(sy + yy^{\mathrm{T}}u_x), \quad r = D_{\mathrm{init}}\left(s(\hat{\alpha}+\hat{\beta})y + \varepsilon y + (\hat{\alpha}+\hat{\beta})yy^{\mathrm{T}}u_x - yy^{\mathrm{T}}\hat{\alpha}u_x\right), \tag{13}$$

$$\delta\nu' = \delta\tilde{\nu} + u^{\perp}, \quad |\delta\nu'_j| \leqslant \nu_{\max} + u_{\max} \overset{\mathrm{def}}{=} \nu'_{\max}, \quad j=1,2,3. \tag{14}$$

For definiteness, this model will be called three-dimensional (3D) (since $z(s, y) \in \mathbf{R}^3$) or Model-1.

In model (12)–(14), the component $\delta\tilde{\nu}$ has no intelligible spectrum due to averaging and is therefore exhaustively described by the inequality $|\delta\tilde{\nu}_j| \leqslant \nu_{\max}$. The component $u^\perp$ is not equally arbitrary; see Lemma 1. Therefore, the constraint $|\delta\nu'_j| \leqslant \nu'_{\max}$ in (14) is coarse.

We use the scalarization method [11]: the original 3D measurement equations are multiplied by a known vector (in this case, $\tilde{y} = D_{\text{init}}y$). After this step, all terms representing the product of a skew-symmetric matrix by the vector y in r are reduced. The scalar product $y^{\text{T}}D_{\text{init}}^{\text{T}}u^\perp$ can also be considered approximately equal to 0 by Lemma 1:

$$0 = w^{\text{T}}(I_3 - \hat{\beta})D_{\text{init}}^{\text{T}}u^\perp = y^{\text{T}}(I_3 - \hat{\alpha})(I_3 - \hat{\beta})D_{\text{init}}^{\text{T}}u^\perp \approx y^{\text{T}}D_{\text{init}}^{\text{T}}u^\perp \qquad (15)$$

(within the infinitesimals of order $(\alpha_{\max} + \beta_{\max})u(\frac{4}{T(s-\varepsilon_{\max})} + C\frac{\varepsilon_{\max}}{s})$). This approximation accuracy is sufficient because

$$(\alpha_{\max} + \beta_{\max})u\left(\frac{4}{T(s - \varepsilon_{\max})} + C\frac{\varepsilon_{\max}}{s}\right) \leqslant \nu_{\max}$$

for the scales of the parameters $\alpha_{\max}$, $\beta_{\max}$, s, T, and $\varepsilon_{\max}$.

Consequently, it becomes possible to pass to a one-dimensional (scalar) measurement model with smaller scale errors:

$$z_{\text{scal}} = z_{\text{scal}}(s, y) = \tilde{y}^{\text{T}}\Gamma D_{\text{init}}(sy + yy^{\text{T}}u_x) + \tilde{y}^{\text{T}}\nu_0 + r_{\text{scal}} + \tilde{y}^{\text{T}}\delta\tilde{\nu}, \qquad (16)$$

where

$$z_{\text{scal}} = \tilde{y}^{\text{T}}\tilde{\zeta} - s - y^{\text{T}}u_x, \quad r_{\text{scal}} = \varepsilon - y^{\text{T}}\hat{\alpha}u_x. \qquad (17)$$

This measurement model will be called scalar or Model-2. It covers the specifics of the term containing the value $u^\perp$, which almost vanishes during scalarization (see (15)). Therefore, the model for the measurement noise z_{scal} is more adequate in this case than for Model-1 (the 3D model), which explains its better accuracy.

3. GUARANTEED ESTIMATION: PROBLEM STATEMENTS

Following the ideas presented in [5, 11], we obtain the unknown matrix Γ and vector ν_0 through guaranteeing estimation. The vector of unknown parameters in this estimation problem consists of the components of the errors Γ and ν_0:

$$\gamma = (\Gamma_{11}, \Gamma_{21}, \Gamma_{31}, \Gamma_{12}, \Gamma_{22}, \Gamma_{32}, \Gamma_{13}, \Gamma_{23}, \Gamma_{33})^{\text{T}}, \quad q = (\gamma^{\text{T}}, \nu_{01}, \nu_{02}, \nu_{03})^{\text{T}} \in \mathbf{R}^{12}.$$

It is required to estimate the scalar value $a^{\text{T}}q$ with a given vector $a \in \mathbf{R}^{12}$. For example, $a = (1, 0, \dots, 0)^{\text{T}}$ when estimating the component Γ_{11}.

The desired estimate is a linear functional of the measurements:

$$l(\Phi) = \int\limits_{(y,s)\in \text{S}} \Phi_0^{\text{T}}(y, s)z(y, s)dyds + \sum_{k=1}^{K} \Phi^T(k)z(y(k), s(k)),$$

where the integral is taken over the set

$$\text{S} = \{y \in \mathbf{R}^3 : \|y\|_2 = 1\} \times \{\{0\} \cup [s_{\min}, s_{\max}]\},$$

$\Phi_0(\cdot)\colon \mathrm{S} \to \mathbf{R}^3$, $\Phi^{(k)} \in \mathbf{R}^3$, and the aggregate $\{y(k), s(k)\}_{k=1}^K$ specifies the set of isolated positions and angular rates of rotation within the experiment. (For brevity, the set over which the integral is taken will be omitted below.) To each element (y, s) we assign the measurements $z(y, s)$ obtained by processing the gyro signals during rotation with the corresponding angular rate. The structure of measurements can be defined by Model-1 or Model-2 (see the previous section). Let us first formulate the guaranteeing estimation problem for the 3D model and then, by analogy, for the scalar model.

With a formal notation using the Dirac delta function, $l(\Phi)$ can be compactly written as

$$l(\Phi) = \int\limits_{y,s} \Phi^{\mathrm{T}}(y, s)z(y, s)dyds, \quad \Phi(y, s) = \Phi_0(y, s) + \sum_{k=1}^K \Phi^T(k)\,\delta(y - y(k), s - s(k)).$$

We denote by $\mathcal{F}$ the set of all such functions $\Phi(\cdot)$.

Consider the guaranteeing estimation problem for the scalar parameter $a^{\mathrm{T}}q$: find an estimator Φ minimizing the (guaranteeing estimation) error [11]

$$I(\Phi) \to \inf_{\Phi \in \mathcal{F}}, \tag{18}$$

where the objective functional $I(\Phi)$ is the supremum of the error $|l(\Phi) - a^{\mathrm{T}}q|$, i.e.,

$$I(\Phi) = \sup_{(q, \alpha, \beta, \varepsilon, \delta\nu') \in \mathcal{B}'} |l(\Phi) - a^{\mathrm{T}}q|. \tag{19}$$

This supremum is calculated over $\mathcal{B}'$, the set of all admissible values of the unknown parameters $(q, \alpha, \beta, \varepsilon, \delta\tilde{\nu})$:

$$q \in \mathbf{R}^{12}, \quad |\alpha_j| \leqslant \alpha_{\max}, \quad |\beta_j| \leqslant \beta_{\max}, \quad |\delta\nu_j'| \leqslant \nu_{\max}', \quad j = 1, 2, 3, \quad |\varepsilon| \leqslant \varepsilon_{\max}. \tag{20}$$

The solution of problem (18)–(20) determines the optimal plan of the experiment. In practice, it is often necessary to estimate each component of the vector q. For these purposes, 12 separate problems of the form (20) are solved; in each of them, only one component a is nonzero.

For the guaranteeing estimation problem, an equivalent l_1-approximation problem can be formulated and numerically solved.

Proposition 1. *A function $\Phi(y, s) \in \mathcal{F}$ is the solution of the guaranteeing estimation problem* (18)–(20) *for Model-1* (12)–(13) *if and only if it is the solution of the optimization problem*

$$\int \left(\nu_{\max}'\|\Phi\|_1 + \alpha_{\max}\|C_\alpha\Phi\|_1 + \beta_{\max}\|C_\beta\Phi\|_1 + \varepsilon_{\max}\left|y^{\mathrm{T}}D_{\mathrm{init}}^{\mathrm{T}}\Phi\right|\right)dyds \to \inf_{\Phi \in \mathcal{F}} \tag{21}$$

subject to the constraints

$$\begin{pmatrix} \int v(y, s) \otimes \Phi\, dyds \\ \int \Phi\, dyds \end{pmatrix} = a \tag{22}$$

with the following notations:

$$\begin{aligned} v &= D_{\mathrm{init}}(sy + yy^{\mathrm{T}}u_x), \\ C_\alpha &= \left(s\hat{y} + y^{\mathrm{T}}u_x\hat{y} - \hat{u}_xyy^{\mathrm{T}}\right)D_{\mathrm{init}}^{\mathrm{T}}, \\ C_\beta &= \left(s\hat{y} + y^{\mathrm{T}}u_x\hat{y}\right)D_{\mathrm{init}}^{\mathrm{T}}. \end{aligned} \tag{23}$$

Here, the symbol $\otimes$ stands for the Kronecker product; the vector v and the matrices C_α and C_β are functions of s, y. The proof of Proposition 1 is given in the Appendix.

Similarly, we formulate the guaranteeing estimation problems for Model-2 by defining the one-dimensional estimator

$$\chi(y, s) = \chi_0(y, s) + \sum_{k=1}^{K} \chi(k) \, \delta(y - y(k), s - s(k)).$$

The resulting estimate of the unknown scalar parameter $a^{\mathrm{T}} q$ has the form

$$l(\chi) = \int_{y, s} \chi(y, s) z_{\mathrm{scal}}(y, s) dy ds.$$

We denote by $\mathcal{X}$ the set of all such functions $\chi(y, s)$ with the described structure.

Proposition 2. *A function $\chi(y, s) \in \mathcal{X}$ is the solution of the guaranteeing estimation problem (18)–(20) for Model-2 (16)–(17) if and only if it is the solution of the optimization problem*

$$\int \left(\nu_{\max} \| D_{\mathrm{init}} y \|_1 + \alpha_{\max} \| \hat{u}_x y \|_1 + \varepsilon_{\max} \right) |\chi(y, s)| \, dy ds \to \inf_{\chi \in \mathcal{X}} \tag{24}$$

subject to the constraints

$$\int \chi(y, s) \begin{pmatrix} v(y, s) \otimes D_{\mathrm{init}} y \\ D_{\mathrm{init}} y \end{pmatrix} dy ds = a. \tag{25}$$

Proposition 2 is established by analogy to Proposition 1.

4. DISCRETE OPTIMIZATION PROBLEMS

In the variational l_1-approximation problem (21)–(22), the desired variable is the vector function $\Phi(y, s)$ whose argument takes a continuum set of values. This fact complicates numerical solution. For similar guaranteeing estimation problems, it was proved [5, 11] that the optimal estimator Φ differs from zero on a finite set of points. In this paper, we will not obtain an analytical solution: consider a discrete analog of the above problems and solve them numerically instead. Let a function $\Phi(y, s)$ be nonzero on a finite set of points $\{y(k), s(k)\}_{k=1}^{K}$ corresponding to different values of the angular rate vector of the unit on the bench and take values $\Phi(k)$ at them. This discrete set of positions can be specified by introducing spherical coordinates to define the vector y and choose a "value grid" for latitude and longitude with a given step.

Transition from integrals to finite sums in (21)–(22) yields the optimization problem

$$\sum_{k=1}^{K} \Big(\nu'_{\max} \| \Phi(k) \|_1 + \alpha_{\max} \| C_\alpha(k) \Phi(k) \|_1 + \beta_{\max} \| C_\beta(k) \Phi(k) \|_1$$

$$+ \varepsilon_{\max} | y^{\mathrm{T}}(k) D_{\mathrm{init}}^{\mathrm{T}} \Phi(k) | \Big) \to \inf_{\Phi(1), \dots, \Phi(K)} \tag{26}$$

subject to the constraints

$$\begin{pmatrix} \sum_{k=1}^{K} v(k) \otimes \Phi(k) \\ \sum_{k=1}^{K} \Phi(k) \end{pmatrix} = a, \tag{27}$$

where the vectors $v(k)$ and the matrices $C_\alpha(k)$ and $C_\beta(k)$ depend on the known arguments $y(k)$ and $s(k)$:

$$v(k) = D_{\mathrm{init}}(s(k)y(k) + y(k)y^{\mathrm{T}}(k)u_x),$$

$$C_\alpha(k) = \left(s(k)\hat{y}(k) + y^{\mathrm{T}}(k)u_x\hat{y}(k) - \hat{u}_x y(k)y^{\mathrm{T}}(k) \right) D_{\mathrm{init}}^{\mathrm{T}},$$

$$C_\beta(k) = \left(s(k)\hat{y}(k) + y^{\mathrm{T}}(k)u_x\hat{y}(k) \right) D_{\mathrm{init}}^{\mathrm{T}}.$$

Note that in this problem, the discrete value grid $\{y(k), s(k)\}_{k=1}^{K}$ is considered a priori given and only the values of $\Phi(k)$ have to be found. Multiplying the objective function (26) by the constant $\nu_{\max}'^{-1}$ and introducing the notations

$$\frac{\alpha_{\max}}{\nu_{\max}'} C_\alpha(k)\Phi(k) = x_\alpha(k), \quad \frac{\beta_{\max}}{\nu_{\max}'} C_\beta(k)\Phi(k) = x_\beta(k), \quad \frac{\varepsilon_{\max}}{\nu_{\max}'} y^{\mathrm{T}}(k)D_{\mathrm{init}}^{\mathrm{T}}\Phi(k) = x_\varepsilon(k),$$

$$x = \left(\Phi^{\mathrm{T}}(1), \dots, \Phi^{\mathrm{T}}(K), x_\alpha^{\mathrm{T}}(1), \dots, x_\alpha^{\mathrm{T}}(K), x_\beta^{\mathrm{T}}(1), \dots, x_\beta^{\mathrm{T}}(K), x_\varepsilon(1), \dots, x_\varepsilon(K) \right)^{\mathrm{T}} \in \mathbf{R}^{10K}, \tag{28}$$

we write problem (26)–(27) in a compact form corresponding to the classical l_1-approximation problem

$$\|x\|_1 \to \inf_{x \in \mathbf{R}^{10K}} \tag{29}$$

subject to the linear constraints $A_{eq}x = a_{eq}$.

The matrix and vector from the constraint equation can be represented in the block form:

$$A_{eq} = \begin{pmatrix} A_\alpha & I_{3K} & 0_{3K \times 3K} & 0_{K \times K} \\ A_\beta & 0_{3K \times 3K} & I_{3K} & 0_{K \times K} \\ A_\varepsilon & 0_{K \times 3K} & 0_{K \times 3K} & I_K \\ A_\Phi & 0_{12 \times 3K} & 0_{12 \times 3K} & 0_{12 \times K} \end{pmatrix} \in \mathbf{R}^{(7K+12) \times 10K}, \quad a_{eq} = \begin{pmatrix} 0_{7K \times 1} \\ a \end{pmatrix} \in \mathbf{R}^{7K+12},$$

where, due to (27) and (28),

$$A_\alpha = \frac{\alpha_{\max}}{\nu_{\max}} \begin{pmatrix} C_\alpha(1) & 0_{3 \times 3} & \dots & 0_{3 \times 3} \\ 0_{3 \times 3} & C_\alpha(2) & \dots & 0_{3 \times 3} \\ \dots & \dots & \ddots & \dots \\ 0_{3 \times 3} & \dots & 0_{3 \times 3} & C_\alpha(K) \end{pmatrix},$$

$$A_\beta = \frac{\beta_{\max}}{\nu_{\max}} \begin{pmatrix} C_\beta(1) & 0_{3 \times 3} & \dots & 0_{3 \times 3} \\ 0_{3 \times 3} & C_\beta(2) & \dots & 0_{3 \times 3} \\ \dots & \dots & \ddots & \dots \\ 0_{3 \times 3} & \dots & 0_{3 \times 3} & C_\beta(K) \end{pmatrix},$$

$$A_\varepsilon = \frac{\varepsilon_{\max}}{\nu_{\max}} \begin{pmatrix} y^{\mathrm{T}}(1)D_{\mathrm{init}}^{\mathrm{T}} & 0_{1 \times 3} & \dots & 0_{1 \times 3} \\ 0_{1 \times 3} & y^{\mathrm{T}}(2)D_{\mathrm{init}}^{\mathrm{T}} & \dots & 0_{1 \times 3} \\ \dots & \dots & \ddots & \dots \\ 0_{1 \times 3} & \dots & 0_{1 \times 3} & y^{\mathrm{T}}(K)D_{\mathrm{init}}^{\mathrm{T}} \end{pmatrix},$$

$$A_\Phi = \begin{pmatrix} v_1(1) & v_1(2)I_3 & \dots & v_1(K)I_3 \\ v_2(1)I_3 & v_2(2)I_3 & \dots & v_2(K)I_3 \\ v_3(1)I_3 & v_3(2)I_3 & \dots & v_3(K)I_3 \\ I_3 & I_3 & \dots & I_3 \end{pmatrix}.$$

A discrete analog of the scalarized model (24)–(25) can be formulated by analogy. In this problem, it is required to minimize the sum of the moduli of the components of the unknown vector under linear equality constraints:

$$\sum_{k=1}^{K} \left(\nu_{\max}\|D_{\mathrm{init}}y(k)\|_1 + \alpha_{\max}\|\hat{u}_x y(k)\|_1 + \varepsilon_{\max}\right)|\chi(k)| \to \inf_{\chi(1),\dots,\chi(K)} \tag{30}$$

subject to

$$\sum_{k=1}^{K} \chi(k) \begin{pmatrix} v(k) \otimes D_{\mathrm{init}}y(k) \\ D_{\mathrm{init}}y(k) \end{pmatrix} = a. \tag{31}$$

In matrix form, the problem is written as

$$\|x_\chi\|_1 \to \inf_{x_\chi \in \mathbf{R}^K} \tag{32}$$

subject to the constraints $A_\chi x_\chi = a$.

Here, we adopt the notations

$$x_{\chi k} = \rho_k \chi(k), \quad \rho_k = \nu_{\max}\|D_{\mathrm{init}}y(k)\|_1 + \alpha_{\max}\|\hat{u}_x y(k)\|_1 + \varepsilon_{\max}, \quad k = 1,\dots,K;$$

$$A_\chi = \begin{pmatrix} \rho_1^{-1}v_1(1)D_{\mathrm{init}}y(1) & \rho_2^{-1}v_1(2)D_{\mathrm{init}}y(2) & \dots & \rho_K^{-1}v_1(K)D_{\mathrm{init}}y(K) \\ \rho_1^{-1}v_2(1)D_{\mathrm{init}}y(1) & \rho_2^{-1}v_2(2)D_{\mathrm{init}}y(2) & \dots & \rho_K^{-1}v_2(K)D_{\mathrm{init}}y(K) \\ \rho_1^{-1}v_3(1)D_{\mathrm{init}}y(1) & \rho_2^{-1}v_3(2)D_{\mathrm{init}}y(2) & \dots & \rho_K^{-1},v_3(K)D_{\mathrm{init}}y(K) \\ \rho_1^{-1}D_{\mathrm{init}}y(1) & \rho_2^{-1}D_{\mathrm{init}}y(2) & \dots & \rho_K^{-1}D_{\mathrm{init}}y(K) \end{pmatrix} \in \mathbf{R}^{12 \times K}.$$

Such convex optimization problems can be solved by various numerical methods, e.g., the interior point method [13], ADMM [14], and the method of variationally weighted quadratic approximations [5]. Unlike the problem for the 3D model (29), problems (32) have a smaller dimension of the unknown vector and constraint matrices (7–10 times less variables and constraints). Therefore, they better suit numerical solution in the case of large values K.

Thus, the optimal estimators $\Phi(k)$ and $\chi(k)$ obtained by solving the l_1-approximation problems (29) or (32) yield the target values of the angular rates $s(k)y(k)$ of the gyro unit on the bench. As a rule, a small number of angular rate positions correspond to non-zero values of $\Phi(k)$ or $\chi(k)$. (This is a common property of guaranteeing estimation solutions; for example, see justification in the book [5].) We denote this subset by $\mathcal{K}$, $\mathcal{K} \subset \{1,\dots,K\}$.

The guaranteeing estimation algorithm for the ARS unit errors is a series of steps. At each step, the following operations are carried out for each $k \in \mathcal{K}$:

(1) The gyro unit is rotated with the angular rate $s(k)y(k)$, and the set of gyro readings $\zeta(t) \in \mathbf{R}^3$ is formed.

(2) The signal $\zeta(t)$ is averaged on the time interval T of the fixed-rate rotation:

$$\tilde{\zeta} = \left(\sum_{t=0}^{\mathrm{T}} \zeta(t)\right)/(T+1).$$

(3) According to (13)–(17), the measurements $z(y(k), s(k))$ and $z_{\mathrm{scal}}(y(k), s(k))$ are formed for the linear estimation models.

Then the unknown parameter $a^{\mathrm{T}}q$ is estimated as

$$\sum_{k\in\mathcal{K}} \Phi^{\mathrm{T}}(k)z(y(k), s(k)) \quad \text{or} \quad \sum_{k\in\mathcal{K}} \chi(k)z_{\mathrm{scal}}(y(k), s(k)).$$

5. NUMERICAL EXAMPLES

Consider several numerical examples illustrating the application of the guaranteeing estimation methods proposed in this paper. The practical implementation of the algorithm includes several stages as follows: solving the guaranteeing estimation problems; modeling the signal, i.e., the measurements $\zeta(t)$ for given "true" values of the errors and the unknown parameters Γ and ν_0; building the estimates Γ and ν_0 and comparing them with the "true" values. The corresponding code was implemented in Python and standard procedures from CVXPY[1] were used to solve the l_1-approximation problems (29) and (32).

The typical values of the errors and model parameters were selected according to Table 1. The admissible limits for the angular rate were defined by formulas (9) and (10): $s_{\min} = 1.25\,°/\text{s}$ and $s_{\max} = 3.28\,°/\text{s}$. In the model example, we supposed the following: the absolute angular rate takes the value $s = 0$ (no rotation) and two values from the segment $[s_{\min}, s_{\max}]$, i.e., $s_1 = 1.5\,°/\text{s}$ and $s_2 = 2\,°/\text{s}$; the vectors $y(k)$ are uniformly located on the unit sphere; $D_{\text{init}} = I_3$. Let us describe the resulting solutions for each group of the unknown parameters Γ and ν_0.

For the full and scalarized models, the optimal estimators for the diagonal components Γ_{ii} have the form

$$\Phi_{ii}(y,s) = \chi_0 e_i \delta(y - e_i, s - s_2) - \chi_0 e_i \delta(y + e_i, s - s_2),$$
$$\chi_{ii}(y,s) = \chi_0 \delta(y - e_i, s - s_2) + \chi_0 \delta(y + e_i, s - s_2),$$

where e_i is a unit vector with ith component equal to 1 and χ_0 is some value numerically determined in the solution of the optimization problem.

In other words, to estimate, e.g., the component Γ_{11} (the scaling factor of the first gyro's error), it is necessary to carry out two series of measurements, rotating the unit along the sensitivity axis of this gyro with the maximum angular rate $s = s_2$ first in one direction ($y(1) = (1,0,0)^{\text{T}}$) and then in the other ($y(2) = (-1,0,0)^{\text{T}}$).

For the off-diagonal elements $\Gamma_{12} = \Gamma_{21}$, the optimal estimators are given by

$$\Phi_{12}(y,s) = \begin{pmatrix} \Phi_1 \\ \Phi_1 \\ 0 \end{pmatrix} \delta(y - e(\pi/4), s - s_2) + \begin{pmatrix} \Phi_2 \\ -\Phi_1 \\ 0 \end{pmatrix} \delta(y - e(3\pi/4), s - s_2)$$
$$+ \begin{pmatrix} -\Phi_2 \\ -\Phi_2 \\ 0 \end{pmatrix} \delta(y - e(5\pi/4), s - s_2) + \begin{pmatrix} -\Phi_1 \\ \Phi_2 \\ 0 \end{pmatrix} \delta(y - e(7\pi/4), s - s_2),$$
$$\chi_{12}(y,s) = \chi_0 \delta(y - e(\pi/4), s - s_2) + \chi_0 \delta(y - e(3\pi/4), s - s_2)$$
$$+ \chi_0 \delta(y - e(5\pi/4), s - s_2) + \chi_0 \delta(y - e(7\pi/4), s - s_2)$$

with the following notations: $e(\theta) = (\cos\theta, \sin\theta, 0)^{\text{T}}$ are the unit vectors corresponding to the rotation by the angle θ in the plane $e_1 e_2$; Φ_1 and Φ_2 are the values numerically determined in the solution of the optimization problem; χ_0 is the same value as for the diagonal elements. The estimators for the components $\Gamma_{13} = \Gamma_{31}$ and $\Gamma_{23} = \Gamma_{32}$ are determined by analogy within the rearrangements of the indices in the corresponding vectors. Thus, the optimal experiment for estimating the misalignment angles between the gyro sensitivity axes consists of four series of measurements; in each of them, rotation is performed along the bisector of the angle between the coordinate axes with the maximum admissible angular rate (by absolute value).

---

[1] An open source Python-embedded modeling language for convex optimization problems;
https : //web.stanford.edu/ boyd/papers/pdf/cvxpy_rewriting.pdf.

Table 2. Guaranteed estimation errors

Model	Variable		
	Γ_{ii}	$\Gamma_{ij},\ i \neq j$	ν_{0i}
The order of value	$[0.7;\ 1.3] \times 10^{-3}$	$[4;\ 6] \times 10^{-3}$	$[2;3] \times 10^{-7}$
Required accuracy	5×10^{-5}	5×10^{-5}	5×10^{-8}
Model-1	2.01×10^{-4}	5.67×10^{-4}	7.01×10^{-6}
Model-2	2.23×10^{-6}	3.45×10^{-6}	5.16×10^{-8}

Table 3. Average estimation errors

Model	Variable		
	Γ_{ii}	$\Gamma_{ij},\ i \neq j$	ν_{0i}
The order of value	$[0.7;\ 1.3] \times 10^{-3}$	$[4;\ 6] \times 10^{-3}$	$[2;3] \times 10^{-7}$
Model-1	$1.25 \times 10^{-6}\ \ (0.13\%)$	$1.33 \times 10^{-6}\ \ (0.02\%)$	$4.88 \times 10^{-8}\ \ (22\%)$
Model-2	$1.20 \times 10^{-6}\ \ (0.13\%)$	$1.24 \times 10^{-6}\ \ (0.03\%)$	$2.89 \times 10^{-8}\ \ (14\%)$

The optimal estimators for the zero biases ν_{0i} have the following structure:

$$\Phi_{\nu i}(y,s) = \phi_1 e_i \delta(y - e_i, s - s_2) + \phi_2 e_i \delta(y + e_i, s - s_2) + \phi_3 e_i \delta(y - e_i, s - s_1) + \phi_4 e_i \delta(y + e_i, s - s_1),$$

$$\chi_{\nu i}(y,s) = \chi_1 \delta(y - e_i, s - s_2) - \chi_2 \delta(y + e_i, s - s_2) + \chi_3 \delta(y - e_i, s - s_1) - \chi_4 \delta(y + e_i, s - s_1),$$

where the values $\phi_i > 0$ and $\chi_i > 0$ are numerically determined in the solution of the optimization problem. They are close to $1/4$.

Clearly, the rotation directions and weight coefficients obtained by guaranteeing estimation have a simple geometric structure: the optimal motion modes imply rotation with the maximum admissible angular rate, in the direction coinciding (within the initial alignment error) either with the gyro sensitivity axes or with the bisectors of the angles between these axes. A similar result was established using guaranteeing estimation in the calibration problem of an accelerometer unit [11]. However, unlike the latter problem, the mathematical model of measurements in this paper is significantly more complex and depends on a larger number of parameters. Therefore, it is difficult to justify the optimal structure of the estimator analytically, and numerical methods are employed to find solutions.

The main quality indicator of the solutions is the guaranteeing estimation error, which does not depend on particular realizations of errors and measurements. Table 2 shows the guaranteeing estimation errors corresponding to the optimal estimators for the components Γ and ν_0.

For the model parameters given in Table 1, the required guaranteeing estimation accuracy is achieved for all components of the matrix Γ within Model-2. Model-1 can lead to errors of about 10% of the parameter estimated; for the components ν_{0i}, the required accuracy is achieved within Model-2.

We present the estimation results for the components Γ, ν_0 from a series of numerical experiments with this procedure under particular realizations of the systematic errors and noises in the measurements. The "true" values of these parameters and errors α, β, $\varepsilon(t)$, and $\delta\nu(t)$ were outputted using a random number generator. For each parameter Γ_{ij}, ν_{0i}, the modeling and estimation procedure was repeated 20 times to evaluate typical deviations of the estimate from the true value. Table 3 shows the average absolute ($|\Gamma_{ij}^0 - \Gamma_{ij}|$) and relative ($|\Gamma_{ij}^0 - \Gamma_{ij}|/|\Gamma_{ij}|$, in parentheses) deviations of the estimates obtained with Models-1 and -2 from the true values. Unlike guaranteeing errors, these deviations do not determine an upper bound on the error but characterize both the accuracy of the models and solutions of the estimation problems. This is important because the input information $\zeta(t)$ is constructed using a nonlinear model of the original signal (3), and the
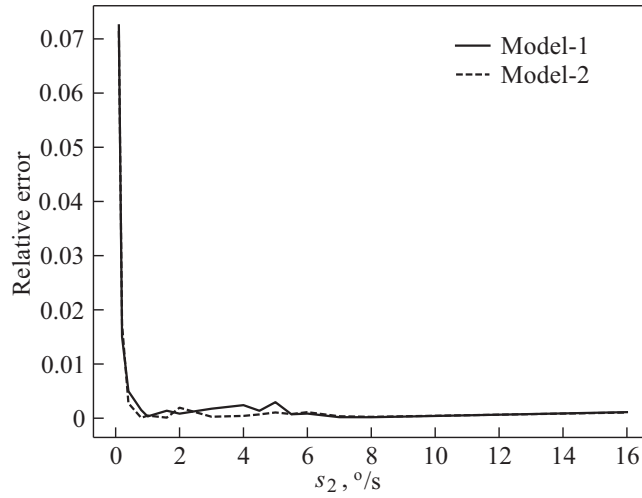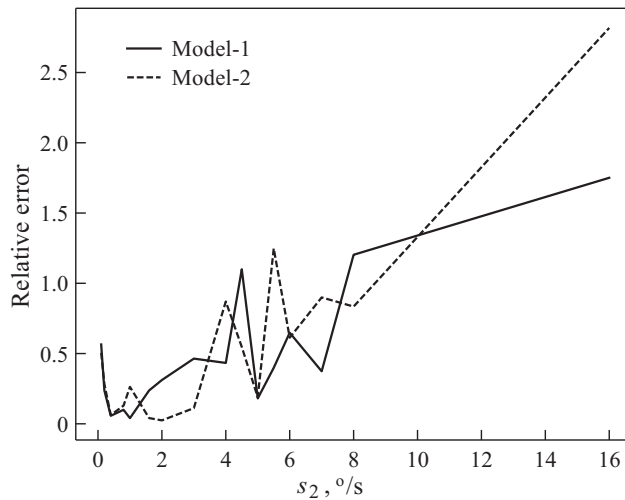
Fig. 1.

Fig. 2.

optimal solution is the result of applying guaranteeing estimation to linearized Models-1 and -2. In other words, this experiment reflects the effect of different factors on the estimation result: the errors and noises in the measurements and the errors due to the transition to simpler linear models.

According to the series of numerical examples, the estimates of the unknown parameters based on the proposed algorithm with the measurement information are close to their "true" values and the deviations lie within the guaranteeing estimation errors.

Let us illustrate how the solution accuracy depends on the choice of the angular rate s_2. Consider an experiment in which, for the same pre-selected values Γ and ν_o, the unknown parameters are estimated using the two models described above for 16 different values of s_2. The relative accuracy of the resulting estimates is presented in the graphs below: the estimation errors for Γ_{11} (Fig. 1; for Γ_{12} the results are similar) and the estimation errors for ν_{01} (Fig. 2).

According to the graph, the critical drop in accuracy is observed under the minimum values of the angular rate (less than the threshold $s_{\min}$ obtained in Section 2.2). When estimating the matrix Γ, the approaches appear to be insensitive to the increase in the angular rate s_2. However, when estimating ν_0, the accuracy deteriorates as s_2 increases, especially in the case of exceeding the threshold $s_2 > s_{\max} = 3.28$ °/s $= 0.057$ 1/s.

6. CONCLUSIONS

This paper has developed a calibration procedure for a gyro unit with three main ideas as follows: derivation of linear measurement models for averaged signals, scalarization, and guaranteeing estimation. The guaranteeing estimation problems have been reduced to discrete l_1-approximation problems, which are solved using numerical algorithms. An important advantage is that this procedure yields an optimal experiment plan as a result of solving the estimation problem. Guaranteed estimation leads to simple-structure solutions: from a large set of admissible directions and angular rates, an optimal combination contains 2–4 rotation modes. This calibration procedure and the corresponding software implementations can be extended with minimum changes to more complex systems, e.g., the ones with a limited number of rotation directions and with temperature-dependent sensor errors.

Besides inertial navigation, the approach proposed above or its modifications can be used in other applications requiring an optimal set of measurements to estimate unknown parameters or an experiment plan from a set of admissible scenarios.

APPENDIX

Proof of Lemma 1. On the time interval T the rotation occurs about a fixed direction. Therefore, the rotation matrix D_{cir} is decomposed by averaging as follows:

$$\bar{D}_{\mathrm{cir}} = \bar{D}_{\mathrm{cir1}} + \bar{D}_{\mathrm{cir2}}, \quad \bar{D}_{\mathrm{cir1}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \bar{D}_{\mathrm{cir2}} = \begin{pmatrix} c_1 & -c_2 & 0 \\ c_2 & c_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A.1}$$

where c_i is the result of the time averaging of the functions $\sin\psi(t)$ and $\cos\psi(t)$.

Due to formulas (6), (7), and (A.1), in the course of averaging, the vector u_z is represented as the sum of two terms, one proportional to w and the other orthogonal to $D(0)w$:

$$\bar{u}_z = \bar{D}u_x = D'\bar{D}_{\mathrm{cir1}}D_{\mathrm{fix}}u_x + D'\bar{D}_{\mathrm{cir2}}D_{\mathrm{fix}}u_x$$

$$= D'\begin{pmatrix} 0_{1\times3} \\ 0_{1\times3} \\ w^{\mathrm{T}} \end{pmatrix}u_x + u^\perp = d_3'w^{\mathrm{T}}u_x + u^\perp = D_{\mathrm{init}}(I_3 + \hat{\beta})ww^{\mathrm{T}}u_x + u^\perp, \tag{A.2}$$

where $u^\perp = D'\bar{D}_{\mathrm{cir2}}D_{\mathrm{fix}}u_x$.

The orthogonality of $u^\perp$ to the direction $D(0)w = D_{\mathrm{init}}(I_3 + \hat{\beta})w$ can be established using formulas (2) and (6): $D(0)w = D'D_{\mathrm{cir}}(0)D_{\mathrm{fix}}w = D'(0,0,1)^{\mathrm{T}}$; the corresponding scalar product is explicitly calculated as

$$w^{\mathrm{T}}D(0)^{\mathrm{T}}u^\perp = (0,0,1)D'^{\mathrm{T}}D'\begin{pmatrix} c_1 & -c_2 & 0 \\ c_2 & c_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}D_{\mathrm{fix}}u_x = (0,0,1)^{\mathrm{T}}\begin{pmatrix} c_1 & -c_2 & 0 \\ c_2 & c_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}D_{\mathrm{fix}}u_x = 0.$$

Using the component c_1 as an example, we explain the idea of estimating from above the result of the time averaging of the function $\cos\psi(t)$. Consider the continuous case of averaging:

$$c_1 = \frac{1}{T}\int_0^{\mathrm{T}}\cos\psi(t)dt.$$

The dynamics of the angle ψ are described by a differential equation and constraints on the functions on its right-hand side:

$$\frac{d\psi(t)}{dt} = s + \varepsilon(t), \quad \psi(0) = \psi_0, \quad |\varepsilon(t)| \leqslant \varepsilon_{\max}, \quad s + \varepsilon(t) > 0.$$

The change of variables $t = t(\psi)$, $\epsilon(\psi) = \varepsilon(t(\psi))$, $|\epsilon(\psi)| \leqslant \varepsilon_{\max}$, in the integral yields

$$\int\limits_0^T \cos\psi(t)dt = \int\limits_{\psi_0}^{\psi(T)} \frac{\cos\psi}{s + \epsilon(\psi)} d\psi.$$

This integral can be written as the sum of integrals on the half-periods of the function $\cos\psi$ (intervals where the function has a fixed sign) and two integrals corresponding to the time intervals at the beginning and end of the interval $[\psi_0, \psi(T)]$. For example, if $\psi_0 < \pi/2$, this interval is represented as follows:

$$[\psi_0, \psi(T)] = [\psi_0, \pi/2] \cup [\pi/2, 3\pi/2] \cup [3\pi/2, 5\pi/2] \cup \ldots \cup [\pi/2 + 2\pi n_{\mathrm{cir}}, \psi(T)],$$

where n_{cir} is the number of complete revolutions of the system about the axis of rotation and the length of the last interval does not exceed π, i.e., $\pi/2 + 2\pi n_{\mathrm{cir}} \leqslant \psi(T) \leqslant 3\pi/2 + 2\pi n_{\mathrm{cir}}$.

The integrand on each such interval has a fixed sign, and the maximum value of the integrand (hence, that of the integral) is achieved at $\epsilon(\psi) = -\mathrm{sgn}(\cos\psi)\,\varepsilon_{\max}$:

$$\int \frac{\cos\psi}{s + \epsilon(\psi)} d\psi \leqslant \int \frac{\cos\psi}{\min_{|\epsilon| \leqslant \varepsilon_{\max}}(s + \epsilon)} d\psi = \int \frac{\cos\psi}{s - \mathrm{sgn}(\cos\psi)\,\epsilon_{\max}} d\psi.$$

Therefore, each integral can be estimated bilaterally (from below and above):

$$\left| \int\limits_{\psi_0}^{\pi/2} \frac{\cos\psi}{s + \epsilon(\psi)} d\psi \right| \leqslant \frac{2}{s - \varepsilon_{\max}}, \quad \left| \int\limits_{\pi/2 + 2\pi n_{\mathrm{cir}}}^{\psi(T)} \frac{\cos\psi}{s + \epsilon(\psi)} d\psi \right| \leqslant \frac{2}{s - \varepsilon_{\max}},$$

$$\frac{-2}{s - \varepsilon_{\max}} \leqslant \int\limits_{\pi/2}^{3\pi/2} \frac{\cos\psi}{s + \epsilon(\psi)} d\psi \leqslant \frac{-2}{s + \varepsilon_{\max}}, \quad \frac{2}{s + \varepsilon_{\max}} \leqslant \int\limits_{3\pi/2}^{5\pi/2} \frac{\cos\psi}{s + \epsilon(\psi)} d\psi \leqslant \frac{2}{s - \varepsilon_{\max}}.$$

As a result, the absolute value of the integral on the averaging interval admits the following upper bound:

$$\left| \int\limits_{\psi_0}^{\psi(T)} \frac{\cos\psi}{s + \varepsilon(\psi)} d\psi \right| \leqslant \left| \int\limits_{\psi_0}^{\pi/2} \frac{\cos\psi}{s + \varepsilon(\psi)} d\psi \right| + \left| \int\limits_{\pi/2 + 2\pi n_{\mathrm{cir}}}^{\psi(T)} \frac{\cos\psi}{s + \varepsilon(\psi)} d\psi \right|$$

$$+ \left| \sum_{j=1}^{n_{\mathrm{cir}}} \left(\int\limits_{\pi/2}^{3\pi/2} \frac{\cos\psi}{s + \varepsilon(\psi)} d\psi + \int\limits_{3\pi/2}^{5\pi/2} \frac{\cos\psi}{s + \varepsilon(\psi)} d\psi \right) \right|$$

$$\leqslant \frac{4}{s - \varepsilon_{\max}} + \left| \sum_{j=1}^{n_{\mathrm{cir}}} \frac{-2}{s + \varepsilon_{\max}} + \frac{2}{s - \varepsilon_{\max}} \right| \leqslant \frac{4}{s - \varepsilon_{\max}} + \frac{n_{\mathrm{cir}}\,4\varepsilon_{\max}}{(s + \varepsilon_{\max})(s - \varepsilon_{\max})}.$$

The angular rate and the number of complete revolutions of the system are related by

$$sT = 2\pi\,n_{\mathrm{cir}} + \Delta\psi$$

for some $\Delta\psi \leqslant 2\pi$. Consequently,

$$|c_1| = \left| \frac{1}{T} \int_0^{\mathrm{T}} \cos\psi(t)dt \right| \leqslant \frac{4}{T\,(s-\varepsilon_{\max})} + \frac{n_{\mathrm{cir}}\,4\varepsilon_{\max}}{T\,(s^2-\varepsilon_{\max}^2)}$$

$$= \frac{4}{T\,(s-\varepsilon_{\max})} + \frac{(sT-\Delta\psi)\,4\varepsilon_{\max}}{2\pi\,sT\,s(1-\varepsilon_{\max}^2/s^2)} = \frac{4}{T\,(s-\varepsilon_{\max})} + \frac{2\,(1-\Delta\psi/(sT))}{\pi\,(1-\varepsilon_{\max}^2/s^2)}\,\frac{\varepsilon_{\max}}{s}.$$

Thus, we obtain

$$|c_1| \leqslant \frac{4}{T\,(s-\varepsilon_{\max})} + C\frac{\varepsilon_{\max}}{s},$$

where the parameter $C = \frac{2}{\pi\,(1-\varepsilon_{\max}^2/s^2)}$ is an upper bound for the fraction $\frac{2(1-\Delta\psi/(sT))}{\pi\,(1-\varepsilon_{\max}^2/s^2)}$.

Proof of Proposition 1. We transform the integrand of the objective function into problem (18) by substituting formulas (12) and (13) with the additional notation $v = v(s,y) = D_{\mathrm{init}}(sy+yy^{\mathrm{T}}u_x)$:

$$\Phi^{\mathrm{T}}z = \Phi^{\mathrm{T}}\left(\Gamma D_{\mathrm{init}}(sy+yy^{\mathrm{T}}u_x) + \nu_0 + r + \delta\nu'\right)$$

$$= \Phi^{\mathrm{T}}\Gamma v + \Phi^{\mathrm{T}}\nu_0 + \Phi^{\mathrm{T}}\delta\nu' + \Phi^{\mathrm{T}}D_{\mathrm{init}}\left(-s(\hat{y}\alpha+\hat{y}\beta) + \varepsilon y + y^{\mathrm{T}}u_x(\hat{\alpha}+\hat{\beta})y + yy^{\mathrm{T}}\hat{u}_x\alpha\right)$$

$$= (v\otimes\Phi)^{\mathrm{T}}\gamma + \Phi^{\mathrm{T}}\nu_0 + \Phi^{\mathrm{T}}\delta\nu' + \Phi^{\mathrm{T}}D_{\mathrm{init}}\left(-s(\hat{y}\alpha+\hat{y}\beta) + \varepsilon y - y^{\mathrm{T}}u_x(\hat{y}\alpha+\hat{y}\beta) + yy^{\mathrm{T}}\hat{u}_x\alpha\right).$$

Hence,

$$\Phi^{\mathrm{T}}z = (v\otimes\Phi)^{\mathrm{T}}\gamma + \Phi^{\mathrm{T}}\nu_0 + \Phi^{\mathrm{T}}\delta\nu' + \varepsilon\Phi^{\mathrm{T}}D_{\mathrm{init}}y$$
$$+ \Phi^{\mathrm{T}}D_{\mathrm{init}}\left(-s\hat{y} - y^{\mathrm{T}}u_x\hat{y} + yy^{\mathrm{T}}\hat{u}_x\right)\alpha + \Phi^{\mathrm{T}}D_{\mathrm{init}}\left(-s\hat{y} - y^{\mathrm{T}}u_x\hat{y}\right)\beta. \tag{A.3}$$

These formulas involve, first, the properties of matrix operations

$$\Phi^{\mathrm{T}}\Gamma v = (\Phi^{\mathrm{T}}\otimes v^{\mathrm{T}})\gamma = (v\otimes\Phi)^{\mathrm{T}}\gamma, \quad \hat{\alpha}y = -\hat{y}\alpha$$

and, second, the possibility of transferring the scalar product $y^{\mathrm{T}}u_x$ to the other part of the corresponding multiplier group: $\hat{\alpha}yy^{\mathrm{T}}u_x = -y^{\mathrm{T}}u_x\,\hat{y}\alpha$.

Let us define the matrices C'_α and C'_β :

$$C'_\alpha = D_{\mathrm{init}}\left(-s\hat{y} - y^{\mathrm{T}}u_x\hat{y} - y^{\mathrm{T}}u_x\hat{y}\right), \quad C'_\beta = D_{\mathrm{init}}\left(-s\hat{y} - y^{\mathrm{T}}u_x\hat{y}\right).$$

Then the right-hand side of (A.3) is represented as a function that linearly depends on the variables $q, \alpha, \beta, \varepsilon$, and $\delta\tilde{\nu}$:

$$\Phi^{\mathrm{T}}(y,s)z(y,s) = (v\otimes\Phi)^{\mathrm{T}}\gamma + \Phi^{\mathrm{T}}\nu_0 + \Phi^{\mathrm{T}}\delta\nu' + \Phi^{\mathrm{T}}C'_\alpha\alpha + \Phi^{\mathrm{T}}C'_\beta\beta + \varepsilon\Phi^{\mathrm{T}}D_{\mathrm{init}}y. \tag{A.4}$$

Substituting formula (A.4) into the original objective functional (19) yields

$$I(\Phi) = \sup_{(q,\alpha,\beta,\varepsilon,\delta\nu')\in\mathcal{B}'} |l(\Phi) - a^{\mathrm{T}}q|$$

$$= \sup_{(q,\alpha,\beta,\varepsilon,\delta\nu')\in\mathcal{B}'} \left| \int \left((v\otimes\Phi)^{\mathrm{T}}\gamma + \Phi^{\mathrm{T}}\nu_0 + \Phi^{\mathrm{T}}\delta\nu' + \Phi^{\mathrm{T}}C'_\alpha\alpha + \Phi^{\mathrm{T}}C'_\beta\beta + \varepsilon\Phi^{\mathrm{T}}D_{\mathrm{init}}y\right)dyds - a^{\mathrm{T}}q \right|.$$

Since $q = \mathrm{col}(\gamma, \nu_0)$, the function $l(\Phi) - a^{\mathrm{T}}q$ linear depends on q, and the multiplier at q is

$$\left(\begin{array}{c} \int v \otimes \Phi \, dyds \\ \int \Phi \, dyds \end{array} \right) - a.$$

Therefore, if condition (22) is violated, we have $\sup_{q \in \mathbf{R}^{12}} |l(\Phi) - a^{\mathrm{T}}q| = +\infty$ for a fixed Φ and arbitrary admissible $\alpha, \beta, \varepsilon$, and $\delta\nu'$. Consequently,

$$\sup_{(q,\alpha,\beta,\varepsilon,\delta\nu') \in \mathcal{B}'} |l(\Phi) - a^{\mathrm{T}}q| = \sup_{(q,\alpha,\beta,\varepsilon,\delta\nu') \in \mathcal{B}'} \left| \int \left(\Phi^{\mathrm{T}}\delta\tilde{\nu} + \Phi^{\mathrm{T}}C'_\alpha\alpha + \Phi^{\mathrm{T}}C'_\beta\beta + \varepsilon\Phi^{\mathrm{T}}D_{\mathrm{init}}y \right) dyds \right|.$$

In other words, it is necessary to maximize the absolute value of a linear function where each term depends on only one variable not figuring in the other terms. This means that the maximum can be found independently in each of the variables. For a fixed Φ, the maximum is determined in an explicit form:

$$\sup_{\alpha: |\alpha_i| \leqslant \alpha_{\max}} \int \Phi^{\mathrm{T}}C'_\alpha \alpha \, dyds = \sup_{\alpha: |\alpha_i| \leqslant \alpha_{\max}} \int \left(\sum_{i=1}^{3} (C'^{\mathrm{T}}_\alpha \Phi)_i \alpha_i \right) dyds$$

$$= \sum_{i=1}^{3} \sup_{\alpha_i: |\alpha_i| \leqslant \alpha_{\max}} \int (C'^{\mathrm{T}}_\alpha \Phi)_i \alpha_i \, dyds$$

$$= \int \left(\sum_{i=1}^{3} \alpha_{\max} \, \mathrm{sgn}((C'^{\mathrm{T}}_\alpha \Phi)_i) \, (C'^{\mathrm{T}}_\alpha \Phi)_i \right) dyds = \int \|C_\alpha \Phi\|_1 \, dyds.$$

A similar chain of considerations applies to the other terms in the objective functional (18). Thus, the explicit calculation of the supremum of the original objective functional finally leads to the optimization problem (21)–(22).

REFERENCES

1. Lidov, M.L., On the A Priori Accuracy of Parameter Estimation by Means of the Least Squares Method, *Kosm. Issled.*, 1964, vol. 2, no. 5, pp. 713–715.

2. Krasovskii, N.N., On the Theory of Controllability and Observability of Linear Dynamic Systems, *J. Appl. Math. Mech.*, 1964, vol. 28, no. 1, pp. 1–14.

3. Lidov, M.L., Minimax Estimation Methods, *Preprint No. 71 of Keldysh Inst. of Applied Mathematics, Russ. Acad. Sci.*, Moscow, 2010.

4. Matasov, A.I., *Estimators for Uncertain Dynamic Systems*, Dordrecht–Boston–London: Springer Science+Business Media, 2013.

5. Akimov, P.A., Derevyankin, A.V., and Matasov, A.I., *Garantiruyushchii podkhod i l_1-approksimatsiya v zadachakh otsenivaniya parametrov BINS pri stendovykh ispytaniyakh* (Guaranteeing Approach and l_1-norm Approximation in the Problems of SDINS Parameter Estimation under Bench Testing), Moscow: Mosk. Gos. Univ., 2012.

6. Kozlov, A.V., Parusnikov, N.A., Vavilova, N.B., Tarygin, I.E., and Golovan, A.A., Dynamic Calibration of Assembled Strapdown Inertial Navigation Systems on a Rate Table, *Izv. SFedU. Engin. Sci.*, 2018, no. 1, pp. 241–257.

7. Vavilova, N.B., Vasineva, I.A., Golovan, A.A., Kozlov, A.V., Papusha, I.A., and Parusnikov, N.A., The Calibration Problem in Inertial Navigation, *J. Math. Sci.*, 2021, vol. 253, no. 6, pp. 818–836.

8. Veremeenko, K.K. and Galay, I.A., Calibration Algorithm Development for Inertial Navigation System with Use of Two-Axis Motion Simulator, *Trudy MAI*, 2013, no. 63, pp. 1–14.

9. Tarygin, I.E., Calibration of the Thermal Model of an Inertial Measurement Unit with Three Angular Rate Sensors, *Gyroscopy Navig.*, 2020, vol. 11, pp. 25–33.

10. Kailath, T., Sayed, A.H., and Hassibi, B., *Linear Estimation*, New Jersey: Prentice Hall, 2000.

11. Matasov, A.I., Variational Problems for Calibrating an Accelerometer Unit, *Autom. Remote Control*, 2019, vol. 80, no. 12, pp. 2135–2151.

12. Golovan, A.A. and Parusnikov, N.A., *Matematicheskie osnovy navigatsionnykh sistem, Ch. I: Matematicheskie modeli inertsial'noi navigatsii* (Mathematical Foundations of Navigation Systems, Part I: Mathematical Models of Inertial Navigation), Moscow: MAKS Press, 2011.

13. Boyd, S. and Vandenberghe, L., *Convex Optimization*, Cambridge: Cambridge University Press, 2004.

14. Boyd, S., Parikh, N., Chu, E., Peleato, B., and Eckstein, J., Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers, *Foundations and Trends in Machine Learning*, 2010, vol. 3, no. 1, pp. 1–122.

This paper was recommended for publication by O.A. Stepanov, a member of the Editorial Board