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STOCHASTIC SYSTEMS

On Continuous Random Processes with Fuzzy States

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Abstract—Continuous random processes with fuzzy states are studied. The properties of their numerical characteristics (expectations and correlation functions) corresponding to those of numerical random processes are established. The results obtained are based on the properties of fuzzy random variables. Applications to the problem of transforming a random signal with fuzzy states by a linear dynamic system are considered.

Keywords: continuous random processes, fuzzy states, fuzzy expectations, correlation functions

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1. INTRODUCTION

Fuzzy modeling is actively used in various applications with incomplete or weakly formalized initial data [1–3]. In particular, note the recent works on the fuzzy approach in control theory [4, 5].

On the other hand, when studying dynamic processes under limited initial information, a possible approach is to treat their parameters as realizations of some random processes [6].

This paper combines both approaches: continuous random processes with fuzzy states are investigated below. More precisely put, time and the set of possible fuzzy states are supposed to be continuous. (In other words, this set is uncountable.) The cross-section of a continuous fuzzy random process at any time instant is a fuzzy random variable. This study involves well-known results from the theory of fuzzy random variables [7–9] and the differentiability and integrability of fuzzy-valued functions ([10, 11] and [12, 13], respectively).

The properties established below are modifications of the well-known results [6, Chapter 1] on the expectations and correlation functions of standard continuous random processes. Seemingly, they have not been observed previously. In this case, an important role is played by fuzzy expectations (nonrandom fuzzy-valued functions of time) reflecting the trends of fuzzy random processes.

One application considered in this paper is the problem of transforming a fuzzy random signal by a linear dynamic system described by a linear differential equation with constant coefficients. Here, the characteristics of an input signal (fuzzy expectation and correlation function) are used to determine similar characteristics of an output signal. These results develop the well-known results for real stationary random processes; for example, see [6, Chapter 7].

Note the difference between the approach and results presented below and the works on random processes with discrete fuzzy states and continuous time (e.g., [14, 15]) that discussed the properties of the probabilities of discrete fuzzy states depending on time.

In what follows, a fuzzy number $\tilde{z}$ defined on the universal space R of real numbers will be understood as a set of ordered pairs $(x, \mu_{\tilde{z}}(x))$, where the membership function $\mu_{\tilde{z}} : R \to [0, 1]$ $\forall x \in R$ determines the grade of membership to the set $\tilde{z}$ [1, Chapter 5].

The interval representation of fuzzy numbers will be employed: to each fuzzy number we assign the set of its α-intervals.

As is known, the α-level set of a fuzzy number $\tilde{z}$ with a membership function $\mu_{\tilde{z}}(x)$ is defined as

$$Z_\alpha = \{x \,|\, \mu_{\tilde{z}}(x) \geqslant \alpha\} \quad (\alpha \in (0,1]), \quad Z_0 = cl\{x \,|\, \mu\tilde{z}(x) > 0\},$$

where cl indicates the closure of an appropriate set.

Assume that all α-levels of a fuzzy number are closed and bounded intervals on the entire real axis. Let $z^-(\alpha)$ and $z^+(\alpha)$ denote the left and right bounds of an interval $Z_\alpha : Z_\alpha = [z^-(\alpha), z^+(\alpha)]$. Sometimes, $z^-(\alpha)$ and $z^+(\alpha)$ are called the left and right α-indices of a fuzzy number, respectively. Suppose that they are measurable and bounded on $[0,1]$. The set of such fuzzy numbers will be denoted by J.

The sum of fuzzy numbers is understood as a fuzzy number whose indices are the sums of the corresponding indices of the terms. Multiplication of a fuzzy number by a positive number means multiplying its indices by this number. Multiplication by a negative real number means multiplying the indices by this number and reversing them. Equality for fuzzy numbers is understood as equality for all the corresponding α-indices $\forall\, \alpha \in [0,1]$.

The set J can be metrized in various ways. In particular, for fuzzy numbers $\tilde{z}_1, \tilde{z}_2 \in J$, one metric is given by

$$d(\tilde{z}_1, \tilde{z}_2) = \left(\int_0^1 \left((z_1^-(\alpha) - z_2^-(\alpha))^2 + (z_1^+(\alpha) - z_2^+(\alpha))^2 \right) d\alpha \right)^{1/2}, \tag{1}$$

where $z_i^-(\alpha)$ and $z_i^+(\alpha)$ are the left and right indices of $\tilde{z}_i$ $(i = 1, 2)$, respectively; for example, see [16].

2. FUZZY EXPECTATION, THE EXPECTATION AND COVARIANCE OF FUZZY RANDOM VARIABLES

Let (Ω, Σ, P) be a probability space with Ω as the set of elementary events, Σ as a σ-algebra consisting of the subsets of the set Ω, and P as a probability measure.

Consider a mapping $\widetilde{X} : \Omega \to J$. For a fixed event $\omega \in \Omega$, its α-level intervals $X_\alpha(\omega)$ are given by

$$X_\alpha(\omega) = \left\{ r \in R : \mu_{\widetilde{X}(\omega)}(r) \geqslant \alpha \right\} \quad \alpha \in (0,1], \quad X_0(\omega) = cl \left\{ r \in R : \mu_{\widetilde{X}(\omega)}(r) > 0 \right\},$$

where $\mu_{\widetilde{X}(\omega)}(r)$ is the membership function of a fuzzy number $\widetilde{X}(\omega)$. An interval $X_\alpha(\omega)$ can be written as $X_\alpha(\omega) = [X^-(\omega,\alpha), X^+(\omega,\alpha)]$, where the bounds $X^-(\omega,\alpha)$ and $X^+(\omega,\alpha)$ are called the left and right indices of the mapping $\widetilde{X}$, respectively.

A mapping $\widetilde{X} : \Omega \to J$ is called a fuzzy random variable (FRV) if the real-valued functions $X^\pm(\omega,\alpha)$ are measurable for all $\alpha \in [0,1]$; for example, see [7, 8]. In this case, the indices are real random variables for any $\alpha \in [0,1]$.

We will study the class $\mathcal{X}$ of all fuzzy random variables for which the indices $X^-(\omega,\alpha)$ and $X^+(\omega,\alpha)$ are square summable functions on $\Omega \times [0,1]$.

Note that an FRV can be interpreted as a random element in the metric space J with the metric (1).

For an FRV $\widetilde{X}(\omega)$, let

$$x^-(\alpha) = \int_\Omega X^-(\omega,\alpha)dP, \quad x^+(\alpha) = \int_\Omega X^+(\omega,\alpha)dP. \tag{2}$$

A fuzzy number with α-indices (2) is called the fuzzy expectation of an FRV $\widetilde{X}$; for example, see [9]. In what follows, it will be denoted by $M(\widetilde{X})$ and its indices by $[M(\widetilde{X})]_\alpha^\pm$.

For FRVs $\widetilde{X}_1$ and $\widetilde{X}_2$ on the set $\mathcal{X}$, we consider the metric

$$\rho(\widetilde{X}_1, \widetilde{X}_2) = \left(\int_0^1 \int_\Omega \left((X_1^-(\omega,\alpha) - X_2^-(\omega,\alpha))^2 + (X_1^+(\omega,\alpha) - X_2^+(\omega,\alpha))^2 \right) dP d\alpha \right)^{1/2}. \quad (3)$$

The fuzzy expectation defined by (2) has similar properties as the expectations of real random variables.

Proposition 1 (e.g., see [17, 18]). *The fuzzy expectation defined by* (2) *possesses the following properties:*

1. *If* $\widetilde{X}(\omega) = \widetilde{X}$ *for almost all* $\omega \in \Omega$, *then* $M(\widetilde{X}) = \widetilde{X}$ (*idempotency*).
2. *The fuzzy expectation* $M : \mathcal{X} \to J$ *is additive, i.e.,* $M(\widetilde{X}_1 + \widetilde{X}_2) = M(\widetilde{X}_1) + M(\widetilde{X}_2)\ \forall \widetilde{X}_1, \widetilde{X}_2 \in \mathcal{X}$.
3. *The fuzzy expectation* $M : \mathcal{X} \to J$ *is homogeneous, i.e.,* $M(C\widetilde{X}) = CM(\widetilde{X})\ \forall \widetilde{X} \in \mathcal{X}$, $\forall C \in R$.
4. *The fuzzy expectation* $M : \mathcal{X} \to J$ *is continuous as a mapping from* $\mathcal{X}$ *with the metric* (3) *into* J *with the metric* (1).
5. *The fuzzy expectation* $M(\widetilde{X})$ *of a given FRV* $\widetilde{X} \in \mathcal{X}$ *satisfies the inequality*

$$\rho^2(\widetilde{X}, M(\widetilde{X})) = \int_0^1\int_\Omega \left(\left(X^-(\omega,\alpha) - [M(\widetilde{X})]_\alpha^- \right)^2 + \left(X^+(\omega,\alpha) - [M(\widetilde{X})]_\alpha^+ \right)^2 \right) dP d\alpha$$
$$\leqslant \int_0^1\int_\Omega \left((X^-(\omega,\alpha) - V_\alpha^-)^2 + (X^+(\omega,\alpha) - V_\alpha^+)^2 \right) dP d\alpha = \rho^2(\widetilde{X}, \widetilde{V}) \quad \forall \widetilde{V} \in J,$$

where $X^\pm(\omega,\alpha)$ *and* $V_\alpha^\pm$ *are the indices of the FRV* $\widetilde{X}(\omega)$ *and the fuzzy number* $\widetilde{V}$, *respectively* (*the extremal property*).

Consider now the defuzzification of fuzzy expectations.

According to [19], within the interval approach, the mean value of a fuzzy number $\tilde{z}$ is given by

$$z_{\text{mean}} = \frac{1}{2} \int_0^1 (z^-(\alpha) + z^+(\alpha)) d\alpha, \quad (4)$$

where $z^\pm(\alpha)$ are the indices of $\tilde{z}$.

In view of (4), the expectation $m(\widetilde{X})$ of an FRV $\widetilde{X} \in \mathcal{X}$ is defined as an averaging functional by

$$m(\widetilde{X}) = \frac{1}{2} \int_0^1 \left([M(\widetilde{X})]^-(\alpha) + [M(\widetilde{X})]^+(\alpha) \right) d\alpha, \quad (5)$$

where $M^\pm(\alpha)$ are the indices of the fuzzy expectation $M(\widetilde{X})$ given by (2). As a matter of fact, this is a defuzzification method for fuzzy expectations.

The next result is immediate from the definition (5) and Proposition 1.

Proposition 2 (e.g., see [17, 18]). *The expectation* (5) *of an FRV possesses the following properties:*

1. *If* $\widetilde{X}(\omega) = \widetilde{X}$ *for almost all* $\omega \in \Omega$, *then* $m(\widetilde{X}) = X_{\text{mean}}$ (*idempotency*).
2. *The expectation* $m : \mathcal{X} \to R$ *is additive, i.e.,* $m(\widetilde{X}_1 + \widetilde{X}_2) = m(\widetilde{X}_1) + m(\widetilde{X}_2)\ \forall \widetilde{X}_1, \widetilde{X}_2 \in \mathcal{X}$.

3. *The expectation* $m : \mathcal{X} \to R$ *is homogeneous, i.e.,* $m(C\widetilde{X}) = Cm(\widetilde{X})\ \forall\, \widetilde{X} \in \mathcal{X},\ \forall\, C \in R$.

4. *The expectation* $m : \mathcal{X} \to R$ *is continuous.*

5. *For* $\widetilde{X} \in \mathcal{X}$, *the expectation* $m(\widetilde{X})$ *satisfies the inequality*

$$\rho\left(\widetilde{X}, m(\widetilde{X})\right) \leqslant \rho(\widetilde{X}, a) \quad \forall\, a \in R,$$

where $a_\alpha^- = a_\alpha^+ = a$ *for a real number* a *(the extremal property).*

The covariance of FRVs $\widetilde{X}$ and $\widetilde{Y}$ is defined by [20]

$$cov(\widetilde{X}, \widetilde{Y}) = \frac{1}{2}\int_0^1 \left(cov(X_\alpha^-, Y_\alpha^-) + cov(X_\alpha^+, Y_\alpha^+)\right)\, d\alpha; \qquad (6)$$

the variance of an FRV $\widetilde{X}$, by the formula $D(\widetilde{X}) = cov(\widetilde{X}, \widetilde{X})$. In (6), the covariances of real random variables $X_\alpha^\pm$ and $Y_\alpha^\pm$ are defined in the standard way [21, Chapter I]:

$$cov(X_\alpha^\pm, Y_\alpha^\pm) = E(X_\alpha^\pm - E(X_\alpha^\pm))(Y_\alpha^\pm - E(Y_\alpha^\pm)).$$

Hereinafter, the symbol E stands for the expectation of a scalar random variable: $E\xi = \int_\Omega \xi(\omega)\, dP$ for a random variable $\xi(\omega)$.

Proposition 3 [20]. *The covariance* (6) *of an FRV possesses the following properties:*

1. $cov(\widetilde{X}, \widetilde{Y}) = cov(\widetilde{Y}, \widetilde{X})$, $\forall\, \widetilde{X}, \widetilde{Y} \in \mathcal{X}$ *(symmetry).*
2. $cov(\widetilde{X} + \widetilde{Y}, \widetilde{Z}) = cov(\widetilde{X}, \widetilde{Y}) + cov(\widetilde{Y}, \widetilde{Z})$, $\forall\, \widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathcal{X}$ *(additivity).*
3. $cov(C_1\widetilde{X}, C_2\widetilde{Y}) = C_1C_2cov(\widetilde{X}, \widetilde{Y})$, $\forall\, \widetilde{X}, \widetilde{Y} \in \mathcal{X}$, $\forall\, C_1, C_2 \in R : C_1C_2 > 0$ *(positive homogeneity).*
4. $D(C\widetilde{X}) = C^2D(\widetilde{X})$, $\forall\, \widetilde{X} \in \mathcal{X}$, $\forall\, C \in R$.
5. $D(\widetilde{X} + \widetilde{Y}) = D(\widetilde{X}) + D(\widetilde{Y}) + 2cov(\widetilde{X}, \widetilde{Y})$, $\forall\, \widetilde{X}, \widetilde{Y} \in \mathcal{X}$.

3. CONTINUOUS RANDOM PROCESSES WITH FUZZY STATES

Let $[t_0, T]$ be an extended segment of the real axis. As in Section 2, (Ω, Σ, P) is a probability space and $\mathcal{X}$ is the set of FRVs with square summable indices on $\Omega \times [0, 1]$.

A continuous random process with fuzzy states or a fuzzy random process (FRP) is a mapping $\widetilde{X} : [t_0, T] \to \mathcal{X}$, i.e., a function $\widetilde{X}(\omega, t)$ with FRVs as its values $\forall\, t \in [t_0, T]$. The α-indices of an FRP $\widetilde{X}(\omega, t)$ will be denoted by $X_\alpha^\pm(\omega, t)$.

For the FRPs considered below, the functions $X_\alpha^\pm(\omega, t)$ are jointly square summable on $\Omega \times [0, 1] \times [t_0, T]$.

For each $t \in [t_0, T]$, the fuzzy expectation $M(\widetilde{X}(\omega, t))$ of an FRP $\widetilde{X}(\omega, t)$ is defined as the fuzzy expectation of the corresponding FRV, i.e., a fuzzy-valued function with the indices $\left[M(\widetilde{X}(\omega, t))\right]_\alpha^\pm = \int_\Omega X_\alpha^\pm(\omega, t)\, dP$.

The definition of the fuzzy expectation of an FRP and the properties of the fuzzy expectations of FRVs imply the following result.

Proposition 4. 1. *The fuzzy expectation of a nonrandom fuzzy-valued function* $\tilde{z} : [t_0, T] \to J$ *coincides with this function:* $M(\tilde{z}(t)) = \tilde{z}(t)$.

2. *A nonrandom scalar function* $\varphi : [t_0, T] \to R$ *can be factored outside the fuzzy expectation sign:*

$$M(\varphi(t)\widetilde{X}(t)) = \varphi(t)M(\widetilde{X}(t)),$$

where $\widetilde{X}(t)$ *is a fuzzy random process.*

3. *The fuzzy expectation of the sum of two FRPs equals the sum of their fuzzy expectations:*

$$M(\widetilde{X}(t) + \widetilde{Y}(t)) = M(\widetilde{X}(t)) + M(\widetilde{Y}(t)).$$

4. *For an FRP $\widetilde{X}(t)$, the fuzzy expectation $M(\widetilde{X}(t))$ satisfies the inequality*

$$\int_{t_0}^{T} \rho^2 \left(\widetilde{X}(t), M(\widetilde{X}(t))\right) dt \leqslant \int_{t_0}^{T} \rho^2 \left(\widetilde{X}(t), \widetilde{V}(t)\right) dt.$$

(*the extremal property*). *Here $\widetilde{V} : [t_0, T] \to J$ is an arbitrary nonrandom fuzzy-valued function with square summable indices on $[t_0, T]$ and the metric ρ is given by* (3).

In accordance with (5), the expectation of an FRP $\widetilde{X}(t)$ is defined as

$$m(\widetilde{X}(t)) = \frac{1}{2} \int_0^1 \left([M(\widetilde{X}(t))]_\alpha^- + [M(\widetilde{X}(t))]_\alpha^+\right) d\alpha.$$

Note that Proposition 2 holds for $m(\widetilde{X}(t))$ $\forall\, t \in [t_0, T]$.

This paper involves the mean-square differentiability of a scalar random process [21, Chapter II]. A scalar random process $\xi(t)$ is said to be mean-square differentiable at a point $t \in R$ if there exists a random variable $\xi'(t)$ such that the expectation

$$E \left| \frac{\xi(t+h) - \xi(t)}{h} - \xi'(t) \right|^2 \to 0 \quad \text{as} \quad h \to 0.$$

Also, the derivative of a fuzzy-valued function will be used; for example, see [10, 11].

An FRP $\widetilde{X}(\omega, t)$ with α-intervals $[X_\alpha^-(\omega, t), X_\alpha^+(\omega, t)]$ is said to be differentiable at a point t (in the Seikkala sense) if its α-indices are mean-square differentiable with respect to t as scalar random processes and $\frac{\partial}{\partial t} X_\alpha^-(\omega, t)$ and $\frac{\partial}{\partial t} X_\alpha^+(\omega, t)$ are the lower and upper α-indices of some FRV called the derivative (cf. with [11] for a fuzzy-valued function). In this case, the time derivative of an FRP $\widetilde{X}(t)$ will be denoted by $\widetilde{X}'(t) = \frac{\partial}{\partial t} \widetilde{X}(\omega, t)$.

Theorem 1. *Let an FRP $\widetilde{X}(t)$ be differentiable for $t \in (t_0, T)$. Assume that $\forall\, \alpha \in [0, 1]$ there exist summable functions $\varphi_\alpha^\pm(\omega)$ on Ω such that $\left|\frac{\partial}{\partial t} X_\alpha^\pm(\omega, t)\right| \leqslant \varphi_\alpha^\pm(\omega)$ $\forall\, t \in (t_0, T)$, $\omega \in \Omega$. Then the fuzzy expectation of the derivative of this FRP equals the derivative of its fuzzy expectation:*

$$M\widetilde{X}'(t) = \left(M\widetilde{X}(t)\right)'. \tag{7}$$

Indeed, by the definition of derivative and fuzzy expectation, $\left[M(\widetilde{X}'(t))\right]_\alpha^\pm = \int_\Omega \frac{\partial}{\partial t} X_\alpha^\pm(\omega, t)\, dP$ $\forall\, \alpha \in [0, 1]$.

Under the conditions of Theorem 1, the derivative on the right-hand side of this equality can be taken outside the sign of the integral; see the theorem on differentiation by a parameter under the sign of a Lebesgue integral. Then, using the interval test of equality for fuzzy numbers, we obtain (7).

The considerations below will employ the integral of a fuzzy-valued function; for example, see [12, 13].

The integral of an FRP $\widetilde{X}(\omega, t)$ between the limits of a segment $[t_0, T]$ is an FRV $\widetilde{Y}(\omega) = \int_{t_0}^T \widetilde{X}(\omega, t)\, dt$ with the α-indices

$$Y_\alpha^\pm(\omega) = \int_{t_0}^{T} X_\alpha^\pm(\omega, t)\, dt,$$

where $X_\alpha^{\pm}(t)$ denote the corresponding α-indices of the FRP $\widetilde{X}(t)$ and the integrals of the random processes $X_\alpha^{\pm}(t)$ are understood in the mean-square sense [21, Chapter II]. If such an integral exists, then the FRP is said to be integrable on $[t_0, T]$.

Theorem 2. *Let $\widetilde{X}(\omega, t)$ be an integrable FRP on $[t_0, T]$. Then*

$$M\left(\int_{t_0}^{T} \widetilde{X}(\omega, t)\, dt\right) = \int_{t_0}^{T} M\left(\widetilde{X}(\omega, t)\right)\, dt. \tag{8}$$

Indeed, by the definition of fuzzy expectation, the left-hand side of (8) has the α-indices $\left[M\left(\int_{t_0}^{T} \widetilde{X}(\omega, t)\, dt\right)\right]_\alpha^{\pm} = \int\limits_\Omega \left[\int_{t_0}^{T} \widetilde{X}(\omega, t)\, dt\right]_\alpha^{\pm} dP = \int\limits_\Omega \left(\int_{t_0}^{T} \widetilde{X}_\alpha^{\pm}(\omega, t) dt\right) dP$. Since the random variables $X_\alpha^{\pm}(\omega, t)$ are square summable on $\Omega \times [t_0, T]$, the order of integration in the last expression can be interchanged by Fubini's theorem. As a result, based on the definition of fuzzy expectation,

$$\left[M\left(\int_{t_0}^{T} \widetilde{X}(\omega, t)\, dt\right)\right]_\alpha^{\pm} = \int_{t_0}^{T}\left(\int_\Omega X_\alpha^{\pm}(\omega, t)\, dP\right) dt = \int_{t_0}^{T} \left[M(\widetilde{X}(\omega, t))\right]_\alpha^{\pm} dt = \left[\int_{t_0}^{T} M\left(\widetilde{X}(\omega, t)\right) dt\right]_\alpha^{\pm}.$$

Then formula (8) follows from the equality of the corresponding indices $\forall\, [0, 1]$.

Example 1. Let numerical random processes $\xi_i(\omega, t)$ $(i = 1, 2, 3;\ \omega \in \Omega,\ t \in [t_0, T])$ be square summable on $\Omega \times [t_0, T]$ and $\xi_1(\omega, t) < \xi_2(\omega, t) < \xi_3(\omega, t)$ for all $\omega \in \Omega$, $t \in [t_0, T]$.

Consider an FRP $\widetilde{X}(t)$ in which the fuzzy number $\widetilde{X}(\omega, t)$ for each $\omega \in \Omega$, $t \in [t_0, T]$, has the triangular form $(\xi_1(\omega, t), \xi_2(\omega, t), \xi_3(\omega, t))$. In other words, for any $\omega \in \Omega$, $t \in [t_0, T]$, the membership function $\widetilde{X}(\omega, t)$ is described by

$$\mu_{\omega,t}(x) = \begin{cases} \dfrac{x - \xi_1(\omega, t)}{\xi_2(\omega, t) - \xi_1(\omega, t)} & \text{if } x \in [\xi_1(\omega, t), \xi_2(\omega, t)]; \\ \dfrac{x - \xi_3(\omega, t)}{\xi_2(\omega, t) - \xi_3(\omega, t)} & \text{if } x \in [\xi_2(\omega, t), \xi_3(\omega, t)]; \\ 0 & \text{otherwise.} \end{cases}$$

In this case,

$$X_\alpha^{-}(\omega, t) = (1 - \alpha)\xi_1(\omega, t) + \alpha\xi_2(\omega, t), \quad X_\alpha^{+}(\omega, t) = (1 - \alpha)\xi_3(\omega, t) + \alpha\xi_2(\omega, t).$$

Then the fuzzy expectation $M(\widetilde{X}(t))$ is given by the formulas for the α-indices

$$[M(\widetilde{X})]_\alpha^{-}(t) = (1 - \alpha)\int_\Omega \xi_1(\omega, t)\, dP + \alpha\int_\Omega \xi_2(\omega, t)\, dP = (1 - \alpha)E\xi_1(t) + \alpha E\xi_2(t) \quad \forall\, \alpha \in [0, 1]$$

and

$$[M(\widetilde{X})]_\alpha^{+}(t) = (1 - \alpha)\int_\Omega \xi_3(\omega, t)\, dP + \alpha\int_\Omega \xi_2(\omega, t)\, dP = (1 - \alpha)E\xi_3(t) + \alpha E\xi_2(t) \quad \forall\, \alpha \in [0, 1],$$

where E denotes the expectation of a real random variable.

Example 2. Within Example 1, let the random processes $\xi_i(\omega, t)$, $i = 1, 2, 3$, be mean-square differentiable with respect to t. In addition, assume that the derivatives satisfy the relation $\frac{\partial}{\partial t}\xi_1(\omega, t) \leqslant \frac{\partial}{\partial t}\xi_2(\omega, t) \leqslant \frac{\partial}{\partial t}\xi_3(\omega, t)$ for all $\omega \in \Omega$, $t \in [t_0, T]$. Then the derivative $\widetilde{X}'(t)$ of the FRP $\widetilde{X}(t)$ has the triangular form $\left(\frac{\partial}{\partial t}\xi_1(\omega, t), \frac{\partial}{\partial t}\xi_2(\omega, t), \frac{\partial}{\partial t}\xi_3(\omega, t)\right)$.

In particular, let random variables $\xi_1(\omega) < \xi_2(\omega) < \xi_3(\omega)$ be given. Consider the triangular fuzzy process $(e^t\xi_1(\omega), e^t\xi_2(\omega), e^t\xi_3(\omega))$. Its derivative exists and coincides with the initial FRP.

Example 3. Within Example 1, let the random processes $\xi_i(\omega, t)$ be integrable in t on a segment $[t_0, T]$. Then the FRP $\int_{t_0}^{T} \widetilde{X}(t)dt$ has the triangular form $\left(\frac{\partial}{\partial t}\xi_1(\omega,t)dt, \frac{\partial}{\partial t}\xi_2(\omega,t)dt, \frac{\partial}{\partial t}\xi_3(\omega,t)dt\right)$.

In what follows, we present the covariance function of an FRP and its properties. The covariance function of an FRP $\widetilde{X}(t)$ is the value

$$K_{\widetilde{X}}(t,s) = cov(\widetilde{X}(t), \widetilde{X}(s)) = \frac{1}{2}\int_0^1 \left(K_{X_\alpha^-}(t,s) + K_{X_\alpha^+}(t,s)\right) d\alpha. \tag{9}$$

Here, $K_{X_\alpha^-}(t,s)$ and $K_{X_\alpha^+}(t,s)$ are the covariance functions of random processes $X_\alpha^-(\omega,t)$ and $X_\alpha^+(\omega,t)$, respectively:

$$K_{X_\alpha^\pm}(t,s) = E\left[(X_\alpha^\pm(\omega,t) - EX_\alpha^\pm(\omega,t))(X_\alpha^\pm(\omega,s) - EX_\alpha^\pm(\omega,s))\right]. \tag{10}$$

The variance of an FRP $\widetilde{X}(t)$ is given by $D_{\widetilde{X}}(t) = K_{\widetilde{X}}(t,t)$.

Note that (9) is a fuzzy modification of the conventional covariance function of scalar random processes; for example, see [21, Chapter II].

The expressions (9) and (10) and the properties of the covariance of an FRV (Proposition 3) lead to the following result.

Proposition 5. *The covariance function of an FRP possesses the following properties:*

1. *For a continuous FRP* $\widetilde{X}(t)$, $K_{\widetilde{X}}(t_1,t_2) = K_{\widetilde{X}}(t_2,t_1)\ \forall\, t_1,t_2 \in [t_0,T]$ *(symmetry).*

2. *If* $\widetilde{X}(t)$ *is a continuous FRP and* $\varphi(t)$ *is a nonrandom numerical function, then the covariance function* $K_{\widetilde{Y}}(t_1,t_2)$ *of the FRP* $\widetilde{Y}(t) = \varphi(t)\widetilde{X}(t)$ *has the form* $K_{\widetilde{Y}}(t_1,t_2) = \varphi(t_1)\varphi(t_2)K_{\widetilde{X}}(t_1,t_2)$.

3. *If* $\widetilde{Y}(t) = \widetilde{X}(t) + \varphi(t)$, *then* $K_{\widetilde{Y}}(t_1,t_2) = K_{\widetilde{X}}(t_1,t_2)$.

4. $|K_{\widetilde{X}}(t_1,t_2)| \leqslant \sqrt{D_{\widetilde{X}}(t_1)D_{\widetilde{X}}(t_2)}$.

The next result characterizes the connection between the correlation functions of a differentiable FRP and its derivative.

Theorem 3. *Let the second derivatives* $\frac{\partial^2 K_{X_\alpha^-}(t,s)}{\partial t\partial s}$ *and* $\frac{\partial^2 K_{X_\alpha^+}(t,s)}{\partial t\partial s}$ *of the covariance functions* (10) *of an FRP* $\widetilde{X}(t)$ *be defined and jointly continuous in the variables* t, s, α. *Then the covariance function* $K'_{\widetilde{X}}(t,s)$ *of the derivative* $\widetilde{X}'(t)$ *of the FRP* $\widetilde{X}(t)$ *is given by*

$$K'_{\widetilde{X}}(t,s) = \frac{\partial^2 K_{\widetilde{X}}(t,s)}{\partial t\partial s}. \tag{11}$$

Proof. By definition (9),

$$K_{\widetilde{X}'}(t,s) = \frac{1}{2}\int_0^1 \left(K_{(\widetilde{X}_\alpha^-)'}(t,s) + K_{(\widetilde{X}_\alpha^+)'}(t,s)\right) d\alpha.$$

According to the well-known property of (scalar) random processes,

$$K_{(\widetilde{X}_\alpha^-)'}(t,s) = \frac{\partial^2 K_{X_\alpha^-}(t,s)}{\partial t\partial s}, \quad K_{(\widetilde{X}_\alpha^+)'}(t,s) = \frac{\partial^2 K_{X_\alpha^+}(t,s)}{\partial t\partial s}.$$

Then $K_{\widetilde{X}'}(t,s) = \frac{1}{2}\int_0^1 \frac{\partial^2}{\partial t\partial s}\left(K_{X_\alpha^-}(t,s) + K_{X_\alpha^+}(t,s)\right) d\alpha$. Taking the second mixed variable outside the integral sign yields formula (11).

Note that the last operation is valid due to the joint continuity of $\frac{\partial^2 K_{X_\alpha^-}(t,s)}{\partial t \partial s}$ and $\frac{\partial^2 K_{X_\alpha^+}(t,s)}{\partial t \partial s}$ in the variables t, s, α.

Consider the integral $\widetilde{Y}(t) = \int_{t_0}^{t} \widetilde{X}(\omega, s)\, ds$ of an FRP $\widetilde{X}(t)$ with a variable upper limit.

Theorem 4. *Let the covariance functions $K_{X_\alpha^\pm}(t,s)$ of the α-indices $X_\alpha^\pm(t)$ of an FRP $\widetilde{X}(t)$ be jointly summable in the variables t, s, α. Then the covariance function of the integral $\widetilde{Y}(t)$ is*

$$K_{\widetilde{Y}}(t,s) = \int_{t_0}^{t} \int_{t_0}^{s} K_{\widetilde{X}}(\tau_1, \tau_2)\, d\tau_1 d\tau_2. \tag{12}$$

Proof. By definition (9), $K_{\widetilde{Y}}(t,s) = \frac{1}{2} \int_0^1 \left(K_{Y_\alpha^-}(t,s) + K_{Y_\alpha^+}(t,s) \right) d\alpha$. Due to the definition of a fuzzy integral, $Y_\alpha^\pm(t) = \int_{t_0}^{t} X_\alpha^\pm(\tau)\, d\tau$. Using the well-known property of the integral of a scalar random process, we obtain

$$K_{Y_\alpha^-}(t,s) = \int_{t_0}^{t} \int_{t_0}^{s} K_{X_\alpha^-}(\tau_1, \tau_2)\, d\tau_1 d\tau_2, \quad K_{Y_\alpha^+}(t,s) = \int_{t_0}^{t} \int_{t_0}^{s} K_{X_\alpha^+}(\tau_1, \tau_2)\, d\tau_1 d\tau_2.$$

Hence, $K_{\widetilde{Y}}(t,s) = \frac{1}{2} \int_0^1 \left(\int_{t_0}^{t} \int_{t_0}^{s} (K_{X_\alpha^-}(\tau_1,\tau_2) + K_{X_\alpha^+}(\tau_1,\tau_2)\right) d\tau_1 d\tau_2)\, d\alpha$. The desired result (12) is established by interchanging the order of integration on the right-hand side (based on Fubini's theorem) and employing (9).

Note that the last operation is valid due to the joint summability of $K_{X_\alpha^-}(\tau_1, \tau_2)$ and $K_{X_\alpha^+}(\tau_1, \tau_2)$ in the variables τ_1, τ_2, α.

4. TRANSFORMATION OF A FUZZY RANDOM SIGNAL BY A LINEAR DYNAMIC SYSTEM

Consider some device A (Fig. 1) with continuous random signals $y(t)$ and $z(t)$ at its input and output, respectively.

Device A is called a linear dynamic system if the relationship between the input and output signals is described by an nth order differential equation with constant coefficients.

The literature (e.g., see [6, Chapter 7]) considers the problem of establishing connections between the numerical characteristics (expectations and covariance functions) of the input and output random signals. Assuming the stationarity of the random signals, this problem is solved using the frequency response of the system, the direct and inverse Fourier transforms, and the Wiener–Khinchin theorem. A random process is called stationary (in the broad sense) if its expectation does not depend on time and the covariance function depends only on the difference of the arguments.

Consider a similar problem when the input and output signals are continuous random signals with fuzzy states (fuzzy random signals, FRSs). In this case, stationarity in any sense is not supposed. In contrast to well-known techniques, the Green function method is used below. The proposed approach will be illustrated on examples.

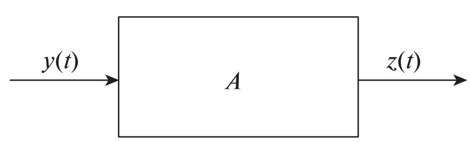

Fig. 1.

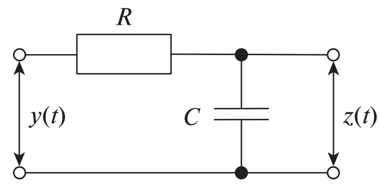

Fig. 2.

Example 4. Consider an RC circuit (Fig. 2) described by the differential equation

$$z'(t) + \beta z(t) = \beta y(t), \quad \beta = \frac{1}{RC} > 0,$$

where R and C denote the resistance and capacity, respectively.

Let a continuous FRS $\widetilde{Y}(t)$ be supplied to the input. We will determine the connection between the fuzzy expectations (as well as expectations) of the output and input signals of this system. By convention, the output FRS $\widetilde{X}(t)$ satisfies the fuzzy random differential equation

$$\widetilde{X}'(t) + \beta\widetilde{X}(t) = \beta\widetilde{Y}(t). \tag{13}$$

Note that fuzzy differential equations were studied, e.g., in [22–24].

In view of the additivity and homogeneity of the expectation of an FRP (Proposition 4) and Theorem 1, taking the fuzzy expectation of both sides of (13) gives

$$(M\widetilde{X})'(t) + \beta M\widetilde{X}(t) = \beta M\widetilde{Y}(t).$$

By the definition of equality for fuzzy numbers and operations between them, the definition of derivative for α-indices, and $\beta > 0$, this equation is equivalent to the following set of relations for the indices:

$$\frac{\partial}{\partial t}(M\widetilde{X})^{\pm}_{\alpha}(t) + \beta(M\widetilde{X})^{\pm}_{\alpha}(t) = \beta(M\widetilde{Y})^{\pm}_{\alpha}(t) \quad \forall\,\alpha \in [0,1]. \tag{14}$$

In addition, assume that the functions $(M\widetilde{Y})^{\pm}_{\alpha}(t)$ are bounded in t on the entire real axis. According to [25, Chapter II], there exists a unique and asymptotically Lyapunov stable solution of Eq. (14) that is bounded on the entire real axis. This solution has the form

$$(M\widetilde{X})^{\pm}_{\alpha}(t) = \beta\int\limits_{-\infty}^{\infty} G_1(t-s)(M\widetilde{Y})^{\pm}_{\alpha}(s)\,ds = \beta\int\limits_{-\infty}^{t} e^{-\beta(t-s)}(M\widetilde{Y})^{\pm}_{\alpha}(s)\,ds \quad \forall\,\alpha \in [0,1], \tag{15}$$

where $G_1 = \begin{cases} e^{-\beta t} & \text{for} \;\; t > 0; \\ 0 & \text{for} \;\; t < 0 \end{cases}$ is the Green function of the problem on bounded solutions of the (scalar) Eq. (14).

Formula (15) characterizes the connection between the fuzzy expectations of the input and output FRSs of the system described by Eq. (13).

Due to (15) and (5), the expectations of the input and output FRSs of system (13) have the connection

$$m\left(\widetilde{X}(t)\right) = \beta\int\limits_{-\infty}^{t} e^{-\beta(t-s)} m\left(\widetilde{Y}(s)\right)\,ds.$$

By the definition of equality for fuzzy numbers and the definition of derivative for FRPs, from (13) it follows that the α-indices of $\widetilde{X}(t)$ satisfy

$$(X_\alpha^{\pm})'(t) + \beta X_\alpha^{\pm}(t) = \beta Y_\alpha^{\pm}(t) \quad \forall\, \alpha \in [0,1]. \tag{16}$$

The relation (16) can be treated as an equation in the Hilbert space $\mathcal{H}$ of all random variables with a finite second moment. In addition, assume that the functions $Y_\alpha^{\pm}(t)$ are bounded in t in the space $\mathcal{H}$ on the entire real axis. According to [25, Chapter II], there exists a unique and asymptotically Lyapunov stable solution of Eq. (16) that is bounded on the entire real axis. This solution has the form

$$X_\alpha^{\pm}(t) = \beta \int\limits_{-\infty}^{t} e^{-\beta(t-s)} Y_\alpha^{\pm}(s) ds \quad \forall\, \alpha \in [0,1]. \tag{17}$$

Note that $X_\alpha^{\pm}(t)$ in (17) determine the α-indices of the fuzzy number $\widetilde{X}(t)$. In particular, $X_\alpha^{+}(t)$ and $X_\alpha^{-}(t)$ are monotonically nonincreasing and nondecreasing in α, respectively, due to the monotonicity of the integral and the corresponding properties of the α-indices $Y_\alpha^{+}(s)$ and $Y_\alpha^{-}(s)$, respectively; for details, see [1, Chapter 5].

Now, we calculate the covariance function of the output FRP $\widetilde{X}$ of the system described by Eq. (13). (Recall that this process is bounded on the entire real axis.) By (10) and (17), the covariance function $K_{X_\alpha^{\pm}}(t,s)$ has the form

$$K_{X_\alpha^{\pm}}(t,s) = \beta^2 E\left[\left(\int\limits_{-\infty}^{t} e^{-\beta(t-\tau_1)} Y_\alpha^{\pm}(\omega,\tau_1)\, d\tau_1 - E\int\limits_{-\infty}^{t} e^{-\beta(t-\tau_1)} Y_\alpha^{\pm}(\omega,\tau_1)\, d\tau_1\right)\right.$$
$$\left.\times\left(\int\limits_{-\infty}^{s} e^{-\beta(s-\tau_2)} Y_\alpha^{\pm}(\omega,\tau_2)\, d\tau_2 - E\int\limits_{-\infty}^{s} e^{-\beta(s-\tau_2)} Y_\alpha^{\pm}(\omega,\tau_2)\, d\tau_2\right)\right].$$

Interchanging the expectation (E) and integration operations in the inner parentheses yields

$$\begin{aligned} K_{X_\alpha^{\pm}}(t,s) &= \beta^2 E\left[\left(\int\limits_{-\infty}^{t} e^{-\beta(t-\tau_1)} \left(Y_\alpha^{\pm}(\omega,\tau_1) - E(Y_\alpha^{\pm}(\omega,\tau_1))\right) d\tau_1\right)\right. \\ &\quad \left.\times\left(\int\limits_{-\infty}^{s} e^{-\beta(s-\tau_2)} \left(Y_\alpha^{\pm}(\omega,\tau_2) - E(Y_\alpha^{\pm}(\omega,\tau_2))\right) d\tau_2\right)\right] \\ &= \beta^2 \int\limits_{-\infty}^{t}\int\limits_{-\infty}^{s} e^{-\beta(t-\tau_1)} e^{-\beta(s-\tau_2)} K_{Y_\alpha^{\pm}}(\tau_1,\tau_2)\, d\tau_1 d\tau_2. \end{aligned} \tag{18}$$

Due to (18) and (9), the covariance functions of the output and input FRSs of system (13) are related by

$$\begin{aligned} K_{\widetilde{X}}(t,s) &= \frac{\beta^2}{2}\int\limits_{-\infty}^{t}\int\limits_{-\infty}^{s} e^{-\beta(t-\tau_1)} e^{-\beta(s-\tau_2)} \int\limits_{0}^{1}\Big(K_{Y_\alpha^{-}}(\tau_1,\tau_2) + K_{Y_\alpha^{+}}(\tau_1,\tau_2)\Big)\, d\alpha d\tau_1 d\tau_2 \\ &= \beta^2 \int\limits_{-\infty}^{t}\int\limits_{-\infty}^{s} e^{-\beta(t-\tau_1)} e^{-\beta(s-\tau_2)} K_{\widetilde{Y}}(\tau_1,\tau_2) d\tau_1 d\tau_2. \end{aligned}$$

The convergence of these improper integrals and the validity of the considerations above are ensured by the exponential estimates of the corresponding integrands.

Example 5. An FRS $\widetilde{Y}(t)$ is supplied to the input of a linear dynamic system described by the differential equation

$$z''(t) + a_1 z'(t) + a_2 z(t) = y(t). \tag{19}$$

It is required to characterize its output FRS $\widetilde{X}(t)$.

From (19) we have the fuzzy differential equation

$$\widetilde{X}''(t) + a_1 \widetilde{X}'(t) + a_2 \widetilde{X}(t) = \widetilde{Y}(t). \tag{20}$$

By analogy with Example 4, the expectation $M(\widetilde{X})(t)$ of the output signal satisfies the differential equation

$$(M\widetilde{X})''(t) + a_1 (M\widetilde{X})'(t) + a_2 M(\widetilde{X})(t) = M(\widetilde{Y})(t).$$

Let the coefficients be $a_1, a_2 > 0$. Then, for the α-indices of the expectation,

$$\left[(M\widetilde{X})^{\pm}_{\alpha}\right]''(t) + a_1 \left[(M\widetilde{X})^{\pm}_{\alpha}\right]'(t) + a_2 (M\widetilde{X})^{\pm}_{\alpha}(t) = M(\widetilde{Y})^{\pm}_{\alpha}(t) \quad \forall\, \alpha \in [0,1]. \tag{21}$$

Let the functions $(M\tilde{Y})^{\pm}_{\alpha}(t)$ be bounded in t on the entire real axis. In addition, suppose that the roots of the characteristic equation $\lambda^2 + a_1\lambda + a_2 = 0$ corresponding to (19) are real, negative, and $\lambda_1 < \lambda_2 < 0$. Then there exists a unique and asymptotically Lyapunov stable solution of Eq. (21) that is bounded on the entire real axis. This solution has the form

$$(M\widetilde{X})^{\pm}_{\alpha}(t) = \int\limits_{-\infty}^{t} G_2(t-s)(M\widetilde{Y})^{\pm}_{\alpha}(s)\, ds \quad \forall\, \alpha \in [0,1], \tag{22}$$

where G_2 is the Green function of the problem on bounded solutions of the scalar Eq. (19); for example, see [26, Chapter 2, § 8]. Under the assumptions accepted above, it has the form

$$G_2(t) = \begin{cases} (e^{\lambda_2 t} - e^{\lambda_1 t})(\lambda_2 - \lambda_1)^{-1} & \text{for} \quad t > 0; \\ 0 & \text{for} \quad t < 0. \end{cases}$$

Formula (22) characterizes the connection between the fuzzy expectations of the input and output FRSs of the system described by Eq. (20).

Assume that the α-indices $Y^{\pm}_{\alpha}(t)$ are bounded in the space $\mathcal{H}$ on the entire real axis. Following the considerations of Example 4, we easily arrive at

$$\begin{aligned} K_{\widetilde{X}}(t,s) &= \frac{1}{2} \int\limits_{-\infty}^{t} \int\limits_{-\infty}^{s} G_2(t-\tau_1) G_2(s-\tau_2) \times \int\limits_{0}^{1} \left(K_{\widetilde{Y}^{-}_{\alpha}}(\tau_1,\tau_2) + K_{\widetilde{Y}^{+}_{\alpha}}(\tau_1,\tau_2) \right) d\alpha d\tau_1 d\tau_2 \\ &= \int\limits_{-\infty}^{t} \int\limits_{-\infty}^{s} G_2(t-\tau_1) G_2(s-\tau_2) K_{\widetilde{Y}}(\tau_1,\tau_2)\, d\tau_1 d\tau_2. \end{aligned} \tag{23}$$

Formula (23) characterizes the connection between the correlation functions of the input and output FRSs of the system described by Eq. (20).

Note that within Examples 4 and 5, the output processes are asymptotically Lyapunov stable in the sense that their α-indices are asymptotically Lyapunov stable; for example, see [25, Chapter II]. Only such processes are physically realizable. Other approaches to the stability of solutions of fuzzy differential equations are also considered in the literature; for example, see [3, Chapter 8].

5. CONCLUSIONS

The starting point for this study is the paper [20], where the covariances of FRVs were discussed and the covariance function of an FRP was introduced. Such functions have been investigated above.

The properties of the fuzzy expectations of FRPs (Proposition 4) and covariance functions (Proposition 5) naturally follow from the corresponding properties of the fuzzy expectations and covariances of fuzzy random variables (Propositions 1–3).

The essential content and scientific novelty of this paper are Theorems 1–4, which present the characteristics of differentiable and integrable FRPs. Their proof involves the definitions and properties of differentiability and integrability of fuzzy-valued functions. Theorems 1–4 generalize the well-known results for standard continuous random processes; for example, see [6]. Also, note the extremal property of the fuzzy expectations of FRPs (Proposition 4), which seems new to the author.

Examples 1–3 are illustrative. Examples 4–5 show the possible use of this theory in applications, particularly the problem of transforming an FRS by a linear dynamic system. The results of Section 4 can be extended to the case of periodic and almost periodic FRSs, including the problem of spectral decompositions of FRSs.

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