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CONTROL IN TECHNICAL SYSTEMS

Calibration of a 3D Sensor under Its Orientation Constraint

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Abstract—Three-dimensional (3D) sensors usually require a calibration procedure. In some cases, scale factor errors depend on the signs of the projections of the vector input signal onto the sensitivity axes of the sensor. To eliminate the ambiguity of scale factor errors, the angular positions of the sensor can be restricted so that the corresponding projections have a definite sign. This paper presents an analytical solution of the optimal calibration problem for a 3D sensor under a constraint on its angular positions.

Keywords: 3D sensor, calibration, the guaranteeing approach to estimation

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1. INTRODUCTION

Consider a three-dimensional (3D) sensor designed to measure some vector physical quantity. Examples of such quantities are a specific force acting on the sensitive mass of an accelerometer unit or the electric (magnetic) field strength. Let the body of the 3D sensor be associated with the so-called instrumental (right orthogonal) frame with the origin in the conditional center of its sensitive element. Assume that the 3D sensor consists of three one-dimensional (1D) sensors whose sensitivity axes in ideal are perpendicular to each other and are directed along the axes of the instrumental frame.

The model of 3D sensor readings has the form

$$f' = (I_3 + \Gamma) f_z + \Delta f^0 + \varrho'' \tag{1}$$

with the following notations: $f' \in \mathbb{R}^3$ are the sensor readings; $I_3 \in \mathbb{R}^{3\times 3}$ is an identity matrix; $\Gamma \in \mathbb{R}^{3\times 3}$ is the error matrix of the sensor (its diagonal elements characterize scale factor errors, whereas the off-diagonal ones characterize the small angular deviations of the sensitivity axes of the 1D sensors from the axes of the instrumental frame); $f_z \in \mathbb{R}^3$ is the physical quantity vector in the form of projections onto the instrumental frame; $\Delta f^0 \in \mathbb{R}^3$ is the systematic biases of the sensor readings; finally, $\varrho'' \in \mathbb{R}^3$ is the fluctuation component of the measurement errors.

The vector f_z is a desired signal to be extracted from the triple f' of its measured components. The parametric errors Γ and Δf^0 are obstacles on this way. The goal of calibration is to determine Γ and Δf^0. After calibration (i.e., after solving the corresponding parameter estimation problem), the impact of these quantities can be compensated in an obvious way.

During the calibration procedure, the 3D sensor is placed on a test bench in different angular positions relative to a fixed vector input signal; a system of equations is compiled for finding Γ and Δf^0. One of the main difficulties is to design a set of such angular positions. Since calibration experiments are rather difficult technically, the number of angular positions should be reduced.

The initial sensor readings contain a significant high-frequency error. Therefore, several series of sensor readings are considered: in each series, the angular position of the sensor does not change, and all readings are then averaged. This considerably reduces the fluctuation component of the measurement errors. By assumption, the relation (1) describes the averaged measurements.

For example, when calibrating an accelerometer unit [1], the physical quantity under consideration is the difference between the specific force acting on the sensitive mass of the unit from the suspension and the specific gravitational force; if the unit is stationary relative to the Earth (static tests), then the quantity is equal to the acceleration of gravity at the point of experiments with opposite sign. In this case, the traditional model of accelerometer unit readings [2] is defined by (1). The calibration problem is particularly topical for sensors in inertial navigation systems; for example, see [3–8].

In some cases, the models of sensor readings are heterogeneous: the scale factor errors of such a sensor depend on the signs of the projections of the vector input signal onto the sensitivity axes of the sensor [7]. To eliminate the ambiguity of scale factor errors, the angular positions of the sensor can be restricted so that the corresponding projections of a test input signal have a definite sign. This circumstance considerably complicates the estimation problem arising in the mathematical formalization of the calibration problem.

The nature of the fluctuation component of measurement errors (after natural averaging) in electromechanical instruments is quite diverse and difficult to formalize. In a series of calibration experiments, the traditional white-noise model usually has no serious justification. Moreover, even the statistical stability assumption for error components (and thus the presence of statistical characteristics) is often questionable. With much greater certainty the fluctuation component vector of measurement errors can be considered bounded by known value. The spectrum of these components after averaging usually has no intelligible model. Therefore, these components will be supposed to be bounded in absolute value by a known constant (taking any values within these bounds). Such an assumption leads to the guaranteeing approach to the corresponding estimation problem.

The guaranteeing approach to estimation in the so-called a priori formulation was pioneered in the classical works [9–11]; also, see [12]. It was further developed in [13, 14] for solving space ballistics problems; in this context, also see [15, 16]. In a slight modification, the guaranteeing approach is a convenient tool to formalize the calibration problem of 3D sensors. In the case of an accelerometer unit without the asymmetry of scale factor errors, the guaranteeing approach was applied in [17–19].

The scalarization method will be used below to calibrate an accelerometer unit in the case of inaccurate information about the angular positions of a test bench. This method radically reduces the undesirable influence of errors in the knowledge of the angular orientation of the bench. For the first time in the available literature, it was presented in [20]; also, see [18, 21–23]).

In this study, we analytically solve the optimal calibration problem of a 3D sensor within the guaranteeing approach under a significant constraint on the admissible angular positions of the sensor.

2. REDUCING CALIBRATION TO THE MOMENT PROBLEM

2.1. Applying the Scalarization Method

We normalize the relation (1) by dividing it by the modulus of the test signal g, which is assumed to be precisely known for simplicity. (Of course, the inaccuracy of this parameter can be considered, but it will not in principle affect the resulting conclusions.) Then the original sensor readings model (1) can be written as

$$\frac{f'}{g} = (I_3 + \Gamma)\, n_z + \varepsilon + \varrho', \qquad \|n_z\| = 1, \tag{2}$$
$$n_z = \frac{f_z}{g}, \qquad \varepsilon = \frac{\Delta f^0}{g}, \qquad \varrho' = \frac{\varrho''}{g}.$$

The approximate value of the unit vector n_z, fixed relative to the bench base, is determined by measuring the rotation angles of the bench. Let $n \in \mathbb{R}^3$ denote the approximate value of n_z; then n is precisely known and belongs to the unit sphere. The error in the bench orientation (up to second-order infinitesimals) is described by a skew-symmetric matrix $\hat\alpha$ with unknown small elements:

$$n_z = (I_3 + \hat\alpha)\, n, \qquad \hat\alpha = \begin{pmatrix} 0 & \alpha_3 & -\alpha_2 \\ -\alpha_3 & 0 & \alpha_1 \\ \alpha_2 & -\alpha_1 & 0 \end{pmatrix}, \qquad |\alpha_i| \ll 1, \quad i = 1, 2, 3.$$

Assume that each component of the measurement errors in the accelerometer readings $\varrho' = \mathrm{col}(\varrho'_1, \varrho'_2, \varrho'_3)$ is bounded by a given value σ. After introducing the new precisely known value $z'(n) = g^{-1} f'(n) - n \in \mathbb{R}^3$, the relation (2) can be written as

$$z'(n) = (\Gamma + \hat\alpha(n))n + \varepsilon + \varrho'(n), \qquad |\varrho'_i(n)| \leqslant \sigma, \quad i = 1, 2, 3. \tag{3}$$

(Again, the accuracy is within second-order infinitesimals.) The essential difference between the formulation of the calibration problem in this paper and those considered earlier is an additional important constraint on the admissible orientations of the 3D sensor. To eliminate the ambiguity of the parameters describing scale factor errors, the components of the orientation vector $n = \mathrm{col}(n_1, n_2, n_3) \in \mathbb{R}^3$ will be supposed nonnegative. Note that other distributions of their signs can be treated by analogy. In other words, let

$$n \in S^+ = S \cap \mathbb{R}^3_+,$$

where S is the unit sphere and $\mathbb{R}^3_+$ is the nonnegative octant.

When calibrating on relatively coarse benches, the unknown term $\hat\alpha(n)n$, different for each successive angular position, significantly affects the estimation accuracy: it cannot be neglected. Within the scalarization method, the 3D measurements (3) are replaced by the 1D scalar measurement

$$z(n) = n^\mathsf{T} z'(n) = n^\mathsf{T}(\Gamma + \hat\alpha(n))n + n^\mathsf{T}\varepsilon + n^\mathsf{T}\varrho(n) = n^\mathsf{T}\Gamma\, n + n^\mathsf{T}\varepsilon + n^\mathsf{T}\varrho'(n), \qquad n \in S^+. \tag{4}$$

This voluntarily eliminates the effect of the unknown noise $\alpha(n)$ since it is obvious that $n^\mathsf{T}\hat\alpha(n)n = 0$ [17, 18]. Evidently, the new measurement noise $\varrho(n) = n^\mathsf{T}\varrho'(n) \in \mathbb{R}^1$ can be estimated from above as

$$|\,\varrho(n)| \leqslant \sqrt{3}\,\sigma, \qquad n \in S^+. \tag{5}$$

A refined estimate for the new measurement noise $\varrho(n)$ will be presented in Section 5.

According to formula (4), the off-diagonal elements of the matrix Γ enter it not separately but in the form of the corresponding sums that characterize the mutual skewness of the sensitivity axes. Thus, the calibration problem turns into the following estimation problem: on the continuum of all measurements

$$z(n) = H^\mathsf{T}(n)q + \varrho(n), \qquad n \in S^+, \tag{6}$$

where

$$H(n) = \mathrm{col}\big(n_1^2, n_2^2, n_3^2, n_1 n_2, n_1 n_3, n_2 n_3, n_1, n_2, n_3\big) \in \mathbb{R}^9,$$

it is required to estimate the components of the unknown parameter vector

$$q = \mathrm{col}(\Gamma_{11}, \Gamma_{22}, \Gamma_{33}, \Gamma_{12} + \Gamma_{21}, \Gamma_{13} + \Gamma_{31}, \Gamma_{23}+\Gamma_{32}, \varepsilon) \in \mathbb{R}^9 \tag{7}$$

against the background of the bounded noise (5). Sometimes the matrix Γ is initially specified as lower-triangular or symmetric. Then the ambiguity in finding Γ is automatically eliminated.

Remark 1. Strictly speaking, to ensure the nonnegative projections of the input signal onto the axes of the instrumental frame, it is necessary to require that the components of the vector n_z, not n, be nonnegative. For simplicity, however, this detail will be ignored when solving the optimal estimation problem below. Since $n_z \approx n$, appropriate minor corrections to the optimal plan can be made after solving the estimation problem.

2.2. Method of Guaranteeing Estimation

Consider the scalar measurements (6), (5), where the noise $\varrho(n)$ is a Lebesgue integrable function defined everywhere on S^+. The problem is to estimate the scalar value $l = a^\mathsf{T} q \in \mathbb{R}^1$, where q is determined by (7), for various given vectors $a \in \mathbb{R}^9$. In this problem, $a = e^{(\nu)}$, where $e^{(\nu)}$ is a unit basis vector from $\mathbb{R}^9$ with 1 on the νth place. (This corresponds to estimating each component of q [18].) For $l = a^\mathsf{T} q$, linear estimators have the form

$$\tilde{l} = \int\limits_{S^+} \Phi_0(n)\, z(n)\, dS + \sum_{k=1}^{M} \Phi^{[k]} z(n^{[k]}), \tag{8}$$

where integration is performed over the surface S^+, $\Phi_0(n)\colon S^+ \to \mathbb{R}^1$ is some Lebesgue integrable weight function, $\Phi^{[k]} \in \mathbb{R}^1$, $n^{[k]} \in S^+$, $k = 1, \ldots, M$, are some values and orientation vectors, respectively, and M is an arbitrary natural number. In contrast to the conventional one, this estimator contains not only an integral term but also terms depending on measurements at certain orientations.

For simplicity, let

$$\tilde{l} = \int\limits_{S^+} \Phi(n)\, z(n)\, dS, \qquad \Phi(n) = \Phi_0(n) + \sum_{k=1}^{M} \Phi^{[k]} \delta(n - n^{[k]}),$$

where, formally, $\int_{S^+} f(n)\delta(n - n^{[k]})dS = f(n^{[k]})$, i.e., $\delta(n - n^{[k]})$ is the Dirac delta function. The set of all such functions $\Phi(n)$ is denoted by $\mathcal{F}$.

The value

$$I(\Phi(n)) = \sup_{q \in \mathbb{R}^9,\, |\varrho(n)| \leqslant \sqrt{3}\,\sigma} |\tilde{l} - l| \tag{9}$$

is called the guaranteed estimation error.

With a chosen estimator, this is the maximum value of the estimation error for all possible values of the uncertain factors. It is necessary to find the weight function $\Phi(n)$ minimizing the guaranteed estimation error, i.e., to solve the minimax problem

$$\inf_{\Phi(n) \in \mathcal{F}} \; \sup_{q \in \mathbb{R}^9,\, |\varrho| \leqslant \sqrt{3}\,\sigma} |\tilde{l} - l|. \tag{10}$$

This problem is called the optimal guaranteeing estimation problem [9–11]. Thus, for solving the calibration problem, we have to solve 9 separate problems for all a indicated above. Interestingly, the use of nonlinear estimators in addition to the linear ones (8) does not reduce the guaranteed estimation error; for details, see [16, 24–27]. In other words, only the linear estimators can be considered.

Straightforward calculations yield

$$I(\Phi(n)) = \begin{cases} \sqrt{3}\,\sigma \int\limits_{S^+} |\Phi(n)|\, dS & \text{if } \int\limits_{S^+} H(n)\Phi(n)\, dS = a \\ \infty & \text{otherwise.} \end{cases}$$

Therefore, the optimal calibration problem reduces to the moment problem

$$I_0 = \inf_{\Phi \in \mathcal{F}} \int_{S^+} |\Phi(n)|\, dS \tag{11}$$

subject to the unbiasedness condition

$$\int\limits_{S^+} H(n)\Phi(n)\, dS = a, \qquad H(n) \in \mathbb{R}^m \qquad (m = 9). \tag{12}$$

(Hereinafter, the constant factor $\sqrt{3}\,\sigma$ is omitted.) As the main content, this paper obtains an analytical (explicit-form) solution of the moment problem (11), (12) for $m = 9$ unit vectors of the form $a = a_{(\nu)} = \mathrm{col}(0, \ldots, 0, 1, 0 \ldots, 0)$, where 1 stands on the νth place, $\nu = 1, \ldots, 9$.

3. SOLVING THE MOMENT PROBLEM

3.1. Preliminaries

To solve the moment problem (11), (12), it is convenient to use its dual counterpart [28–30] in the form

$$I^0 = \sup_{\lambda \in \mathbb{R}^m} a^{\mathsf{T}}\lambda \qquad (m = 9) \tag{13}$$

subject to the condition

$$|H^{\mathsf{T}}(n)\lambda| \leqslant 1, \qquad n \in S^+. \tag{14}$$

When considering the dual problem, the original one is often called primal.

Theorem 1. *Let the function $H(n) \in \mathbb{R}^m$ be continuous on S^+. Then the following assertions are true.*

1. *For the primal problem* (11), (12), *there exists at least one impulse solution with at most m pulses:*

$$\Phi^0(n) = \sum_{i=1}^{m} \Phi^{(i)}\delta(n - n^{(i)}). \tag{15}$$

2. *The solution λ^0 of the dual problem* (13), (14) *does exist.*
3. *The values of the primal and dual problems coincide: $I_0 = I^0$.*
4. *If the optimal estimator is an impulse function given by* (15) *with all nonzero coefficients, then*

$$|H^{\mathsf{T}}(n^{(i)})\lambda^0| = 1, \quad i = 1, \ldots, m, \quad \textit{where } \operatorname{sgn}\Phi^{(i)} = \operatorname{sgn}\left(H^{\mathsf{T}}(n^{(i)})\lambda^0\right). \tag{16}$$

5. *The optimal estimator is zero on the set of orientation vectors n such that $|H^{\mathsf{T}}(n)\lambda^0| < 1$.*
6. *If $\Phi(n)$ and λ are admissible elements of the primal and dual problems, respectively, and the objective functionals of both problems take the same value on these elements, then they are the solutions of the corresponding problems.*

These assertions follow from general theorems of convex analysis. For n spanning through the entire sphere S, their proofs were presented in [18]. In the case $n \in S^+$, the proofs are similar.

Consider the set of nine orientation vectors

$$n^{(1)} = \mathrm{col}(1,0,0), \quad n^{(2)} = \mathrm{col}(0,1,0), \quad n^{(3)} = \mathrm{col}(0,0,1), \tag{17}$$
$$n^{(4)} = \mathrm{col}\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right), \quad n^{(5)} = \mathrm{col}\left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right), \quad n^{(6)} = \mathrm{col}\left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$
$$n^{(7)} = \mathrm{col}\left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right), \quad n^{(8)} = \mathrm{col}\left(\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right), \quad n^{(9)} = \mathrm{col}\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right).$$

Let $\Phi_{(\nu)}$ be the solution to the corresponding system of equations derived from the unbiasedness conditions (12) by substituting the impulse solution (15) concentrated on the orientation vectors (17):

$$\mathcal{H}\Phi_{(\nu)} = a_{(\nu)}, \tag{18}$$

where $\mathcal{H} = \left(H(n^{(1)}), \ldots, H(n^{(9)})\right)$ and $\Phi_{(\nu)} = \mathrm{col}\left(\Phi_{(\nu)}^{(1)}, \ldots, \Phi_{(\nu)}^{(9)}\right)$, $\nu = 1, 4, 7$.
Theorem 1 can be used to establish the following result.

Theorem 2. *The estimators*

$$\Phi_{(\nu)}^0(n) = \sum_{i=1}^{9} \Phi_{(\nu)}^{(i)} \delta(n - n^{(i)}), \qquad \nu = 1, 4, 7, \tag{19}$$

where $n^{(i)}$ *and* $\Phi_{(\nu)}^{(i)}$ *are given by* (17) *and* (18), *provide the solution of the moment problem* (11), (12).
The proof of Theorem 2 forms the main content of this paper; see below. Note that the solutions (19) are not unique. The solutions of the moment problem for the remaining values ν are obtained from the previous ones by a cyclic change of coordinates.

3.2. Numerical Results

Let us first analyze the numerical solutions. Consider the discrete analogs of the primal and dual problems for $a = a_{(1)} = \mathrm{col}(1, 0, \ldots, 0) \in \mathbb{R}^9$, i.e., when estimating $q_1 = \Gamma_{11}$. (For simplicity, the subscript (1) will be omitted.) We construct an approximately uniform grid of $M \sim 10^5$ points on S^+ and take all possible impulse functions concentrated on this grid. As a result, the following problems are obtained:

$$\min_{\Phi \in \mathbb{R}^M} \sum_{k=1}^{M} |\Phi^{[k]}|$$

subject to the unbiasedness conditions

$$\sum_{k=1}^{M} H(n^{[k]})\, \Phi^{[k]} = a = \mathrm{col}(1, 0, \ldots, 0), \qquad H(n^{([k]}), a \in \mathbb{R}^m \qquad (m = 9)$$

(the analog of the primal problem (11), (12)) and

$$\max_{\lambda \in \mathbb{R}^m} a^{\mathsf{T}} \lambda$$

subject to the conditions

$$|H(n^{[k]})\lambda| \leqslant 1, \qquad k = 1, \ldots, M, \qquad n^{[k]} \in S^+$$

(the analog of the dual problem (13), (14)).

These problems are naturally reduced to linear programming problems and solved by standard numerical procedures (for quite large $M \sim 10^5$). As is well known, the solution of the primal problem contains at most $m = 9$ nonzero components. Based on the obtained numerical results, it can be observed that the numerical solutions have the following properties:

a) The vectors $n^{[k]} \in \mathbb{R}^3$ corresponding to the nonzero components of $\Phi^{[k]}$ consist of the four vectors

$$\mathrm{col}(1,0,0), \qquad \mathrm{col}(0,1,0), \qquad \mathrm{col}(0,0,1), \qquad \mathrm{col}\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$$

and five more vectors from S^+.

b) The solution of the dual problem possesses the structure

$$\lambda_1 = \lambda_2 = \lambda_3, \qquad \lambda_4 = \lambda_5 = \lambda_6, \qquad \lambda_7 = \lambda_8 = \lambda_9.$$

To construct the precise solutions of the continuous problems (11), (12), (13), (14), we hypothesize that *the precise solutions possess properties* a) *and* b) *as well*. While property a) seems partially blurred, property b) is mathematically rich and indicates the importance of analyzing the dual problem.

3.3. Constructing a Candidate Solution of the Dual Problem

Under the hypothesis accepted above, we construct a candidate solution of the dual problem. As will be proved below, it is the true solution. Let $a = a_{(1)} = \mathrm{col}(1, 0, \ldots, 0)$. Consider the function from the constraints of the dual problem:

$$S(n;\lambda) = H^{\mathsf{T}}(n)\,\lambda = \lambda_1 + \lambda_4(n_1n_2 + n_1n_3 + n_2n_3) + \lambda_7(n_1 + n_2 + n_3), \quad n \in S^+.$$

It can be compactly written as

$$S(n;\lambda) = \lambda_1 + \lambda_4\frac{(n_1+n_2+n_3)^2 - 1}{2} + \lambda_7(n_1+n_2+n_3) = s(t(n);\lambda_1,\lambda_4,\lambda_7), \tag{20}$$
$$t = n_1 + n_2 + n_3, \qquad 1 \leqslant t \leqslant \sqrt{3},$$

where

$$s(t;\lambda_1,\lambda_4,\lambda_7) = \lambda_1 + \lambda_4\frac{(t^2-1)}{2} + \lambda_7 t.$$

Then the dual problem takes an appreciably simplified form:

$$\max_{\lambda_1,\lambda_4,\lambda_7} \lambda_1 \qquad \text{subject to the condition} \qquad |s(t;\lambda_1,\lambda_4,\lambda_7)| \leqslant 1, \qquad 1 \leqslant t \leqslant \sqrt{3}. \tag{21}$$

Let $\lambda^0 = \mathrm{col}(\lambda_1^0, \ldots, \lambda_9^0)$ denote the solution of problem (13), (14). According to property a), the optimal estimator Φ^0 is concentrated on the orientation vectors n such that $t(n) = 1$, $t(n) = \sqrt{3}$, and $t(n) = t^{(i)}$ for some set $i = 1, \ldots, i^0, i^0 \leqslant 5$. By assertion 4 of Theorem 1, for these orientations, the function $S(n;\lambda^0)$ takes the extremum values: $|S(n;\lambda^0)| = 1$.

Now we demonstrate that $\lambda_4^0 \neq 0$. Assume on the contrary that $\lambda_4^0 = 0$. Then property a) holds only if $\lambda_7^0 = 0$; in this case, $|\lambda_1| \leqslant 1$. But the optimal value of the objective functional in problem (21) exceeds 1: for example, obviously, the element $\lambda = (1 + \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}})$ is admissible.

The function $s(t(n); \lambda_1^0, \lambda_4^0, \lambda_7^0) \triangleq s^0(t(n))$ with $\lambda_4^0 \neq 0$ is a parabola in t, taking its extremum values at most at three points: at the two ends and the parabola vertex. Therefore,

$$|s^0(1)| = 1, \qquad \left| s^0\left(\frac{1+\sqrt{3}}{2}\right)\right| = 1, \qquad |s^0(\sqrt{3})| = 1, \tag{22}$$

and the strict inequality

$$|s^0(t)| < 1, \quad t \neq 1, \quad t \neq \frac{1+\sqrt{3}}{2}, \quad t \neq \sqrt{3}, \tag{23}$$

holds for all other t. Furthermore, due to the parabola shape,

$$\operatorname{sgn} s^0(1) = -\operatorname{sgn} s^0\left(\frac{1+\sqrt{3}}{2}\right) = \operatorname{sgn} s^0(\sqrt{3}). \tag{24}$$

The relations (22), (24) generate a system of three linear equations for λ_i^0, $i = 1, 4, 7$, and determine λ^0 within its sign. Clearly, it should be chosen to make $a^\top \lambda^0 = \lambda_1^0$ positive. (Otherwise, $-\lambda^0$ would provide a larger value to the objective functional.) Solving this system of equations, we obtain an admissible element of the dual problem in the form

$$\begin{aligned} \lambda_1^0 &= \lambda_2^0 = \lambda_3^0 = 3(7+4\sqrt{3}), \\ \lambda_4^0 &= \lambda_5^0 = \lambda_6^0 = 8(2+\sqrt{3}), \\ \lambda_7^0 &= \lambda_8^0 = \lambda_9^0 = -4(5+3\sqrt{3}), \end{aligned} \tag{25}$$

where

$$\operatorname{sgn} s^0(1) = -\operatorname{sgn} s^0\left(\frac{1+\sqrt{3}}{2}\right) = \operatorname{sgn} s^0(\sqrt{3}) = 1. \tag{26}$$

Presumably, this element is the solution of the dual problem.

3.4. Constructing a Candidate Solution of the Primal Problem

Let a candidate solution of the primal problem have the form (15). By assertion 4 of Theorem 1 and (22), (23), it is concentrated on the orientation vectors such that either $t(n) = 1$ or $t(n) = \frac{1+\sqrt{3}}{2}$ or $t(n) = \sqrt{3}$. According to property a), without loss of generality, it can be assumed that

$$n^{(1)} = \mathrm{col}(1,0,0), \quad n^{(2)} = \mathrm{col}(0,1,0), \quad n^{(3)} = \mathrm{col}(0,0,1) \quad \to t(n) = 1, \tag{27}$$

$$n^{(4)},\ n^{(5)},\ n^{(6)}, n^{(7)},\ n^{(8)} \in S^+ \quad \to\ t(n) = \frac{1+\sqrt{3}}{2}, \tag{28}$$

$$n^{(9)} = \mathrm{col}\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right) \quad \to\ t(n) = \sqrt{3}. \tag{29}$$

Due to (26)–(29), the signature

$$e = \mathrm{col}\Big(\operatorname{sgn}\left(H(n^{(1)})^\top \lambda^0\right), \ldots, \operatorname{sgn}\left(H(n^{(9)})^\top \lambda^0\right)\Big)$$

of the dual problem is

$$e = \mathrm{col}(+1, +1, +1, -1, -1, -1, -1, -1, +1). \tag{30}$$

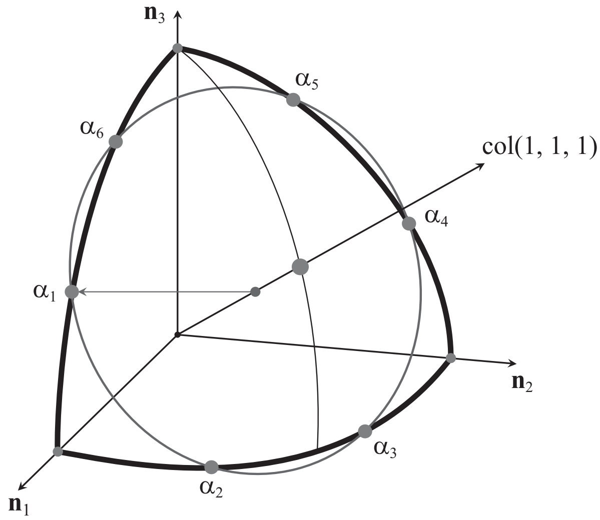

Fig. 1. The surface of admissible orientations.

The set

$$C = \left\{ n \in S^+ \middle| \, t(n) = n_1 + n_2 + n_3 = \frac{1+\sqrt{3}}{2} \right\} \tag{31}$$

is generated by the intersection of the plane $n_1 + n_2 + n_3 = \frac{1+\sqrt{3}}{2}$ (with the normal vector $\mathrm{col}(1,1,1)$) with the unit sphere S (representing a circle) with the additional component non-negativity constraint. That is, this set consists of the three arcs $[\alpha_1, \alpha_2]$, $[\alpha_3, \alpha_4]$, and $[\alpha_5, \alpha_6]$ of the circle. Figure 1 shows the surface $S^+ = S \cap \mathbb{R}^3_+$ (the thickened line), this circle (the closed line of regular thickness), the intersection points of S^+ with the coordinate axes (the small-sized points), the intersection point of the octant diagonal with the surface S^+ (the large-sized point), and the intersection points of the circle with S^+ (the medium-sized points). As is easily verified, these intersection points of the circle with S^+ divide the corresponding arcs of the big circle lying in the coordinate planes into three equal parts (30° each).

A candidate solution of the primal problem is the one for which the orientation vectors from the set C lie at the extreme points of C, i.e., at the ends of the arcs $[\alpha_1, \alpha_2]$, $[\alpha_3, \alpha_4]$, and $[\alpha_5, \alpha_6]$. There are six such points, and we need to determine five. For example, let us discard the point corresponding to the orientation vector $n = \mathrm{col}(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0)$ (the point α_2 in Fig. 1).

Then the missing orientation vectors take the form

$$n^{(4)} = \mathrm{col}\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right), \quad n^{(5)} = \mathrm{col}\left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right), \quad n^{(6)} = \mathrm{col}\left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$
$$n^{(7)} = \mathrm{col}\left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right), \qquad n^{(8)} = \mathrm{col}\left(\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right). \tag{32}$$

The system of unbiasedness conditions corresponding to these orientation vectors is obtained by substituting into (12) the impulse function generated by the vectors (27), (29) and (32). This system has the form

$$\mathcal{H}\Phi = a, \quad \text{where} \quad \mathcal{H} = \Big(H(n^{(1)}), \ldots, H(n^{(9)})\Big), \quad \Phi = \mathrm{col}(\Phi^{(1)}, \ldots, \Phi^{(9)}); \tag{33}$$

as mentioned above, the subscript (1) is omitted for simplicity. Note that $\det \mathcal{H} \neq 0$. This can be verified by reducing the matrix $\mathcal{H}$ to an upper-triangular form, e.g., by Gauss elimination, with ignoring common multipliers in rows or columns. Such a method is rather computationally-intensive. In addition, according to numerical calculations, the absolute values of the diagonal elements of the upper-triangular matrix yielded by the QR-decomposition lie in the interval (0.07, 1.5); with the SVD-procedure, all the nine singular values lie in the interval (0.02, 3).

We construct a candidate solution of the direct problem $\Phi^0(n)$ in the form (15) with the weight coefficients determined by Φ from (33). Obviously, $\Phi = \mathcal{H}^{-1}a$. The explicit calculation of the vector Φ from (33), which would be sufficient to justify the optimality of the solution, seems almost impracticable due to the high order of the system. A subtle approach can be adopted here. Formulas (16), (26), and (30) imply the equality

$$\mathcal{H}^{\mathsf{T}}\lambda^0 = e. \tag{34}$$

From (34) it follows that $a^{\mathsf{T}}\lambda^0 = a^{\mathsf{T}}\left(\mathcal{H}^{\mathsf{T}}\right)^{-1}e$. Therefore,

$$e^{\mathsf{T}}\Phi = e^{\mathsf{T}}\mathcal{H}^{-1}a = a^{\mathsf{T}}\lambda^0. \tag{35}$$

Since $\det \mathcal{H} \neq 0$ and the condition number of $\mathcal{H}$ is quite moderate (~ 150), the unbiasedness Eqs. (33) can be easily solved numerically. The resulting weight coefficients are such that $\max_{i=1,\ldots,9}|\Phi^{(i)}| > 1.8$; hence, the numerical information about the signs of $\Phi^{(i)}$ is reliable. (For a more formal justification, see the Appendix.) Furthermore, it turns out that

$$\operatorname{col}(\operatorname{sgn}\Phi^{(1)},\ldots,\operatorname{sgn}\Phi^{(9)}) = e. \tag{36}$$

Due to (35) and (36), for the constructed admissible elements $\Phi^0(n)$ and λ^0 of the primal and dual problems, respectively,

$$\int\limits_{S^+}|\Phi^0(n)|\,dS = \sum_{i=1}^{9}|\Phi^{(i)}| = e^{\mathsf{T}}\Phi = a^{\mathsf{T}}\lambda^0. \tag{37}$$

Thus, by assertion 6 of Theorem 1, these elements $\Phi^0(n)$ and λ^0 are the solutions of the primal and dual problems, respectively. The optimal value of the objective functional is $I_{0,(1)} = 3\left(7+4\sqrt{3}\right)$ (up to the multiplier $\sqrt{3}\sigma$, which coincides with the optimal guaranteed estimation error). Note that despite the numerical reasoning of (36), the final result (37) is strictly proved. The cases $\nu = 4$ and $\nu = 7$ are studied similarly. The corresponding solutions of the dual problem coincide with λ^0 from (25) up to the sign, and the optimal values of the objective functional are $I_{0,(4)} = 8\left(2+\sqrt{3}\right)$ and $I_{0,(7)} = 4\left(5+3\sqrt{3}\right)$, respectively. Theorem 2 is proved.

Remark 2. Consider estimation, e.g., for the parameters Γ_{11}, Γ_{12}, Γ_{22}, ε_1, and ε_2. In the problem without angular position constraints (when the orientation vector of the instrumental frame spans through the entire unit sphere S), the optimal calibration plan for the planar problem ($n_3 = 0$) is also optimal for the 3D problem [17]. However, this is not true for the case considered here, with constraints imposed on the admissible angular positions of the 3D sensor. In the set of "planar" plans ($n_3 = 0$), the guaranteed estimation error is almost three times higher than the optimal one. This is an unobvious peculiarity of the problem with angular position constraints.

3.5. Other Solutions

The proof of Theorem 2 leads to the following fact: if some set of orientation vectors from C that is augmented by the vectors (27) and (29) to a nonsingular matrix preserves the signature e

of the dual problem of the weight coefficients, then this set generates a new optimal estimator. This sufficient condition can be used to verify another important property: with the orientation vector $n = \mathrm{col}(\frac{\sqrt{3}}{2}, 0, \frac{1}{2})$ being excluded from the six points of the boundary of C (the point α_1 in Fig. 1), the remaining five points (with (27) and (29)) generate another solution for the case $\nu = 1$ and $\nu = 7$. On the other hand, eliminating the vector $n = \mathrm{col}(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ from the six extreme points of C (the point α_3 in Fig. 1) gives a new solution for the case $\nu = 4$.

There is another class of solutions whose optimality also follows from this sufficient condition. Consider a symmetric set of orientation vectors $n^{(5)}$, $n^{(6)}$, and $n^{(7)}$ from C (see (31)) in which the ends of the vectors lie at the midpoints of the arcs $[\alpha_1, \alpha_2]$, $[\alpha_3, \alpha_4]$, and $[\alpha_5, \alpha_6]$, respectively:

$$n^{(5)} = \mathrm{col}\left(\frac{\sqrt{3}+1+2\sqrt{4-\sqrt{3}}}{6}, \frac{\sqrt{3}+1-\sqrt{4-\sqrt{3}}}{6}, \frac{\sqrt{3}+1-\sqrt{4-\sqrt{3}}}{6}\right),$$

$$n^{(6)} = \mathrm{col}\left(\frac{\sqrt{3}+1-\sqrt{4-\sqrt{3}}}{6}, \frac{\sqrt{3}+1+2\sqrt{4-\sqrt{3}}}{6}, \frac{\sqrt{3}+1-\sqrt{4-\sqrt{3}}}{6}\right),$$

$$n^{(7)} = \mathrm{col}\left(\frac{\sqrt{3}+1-\sqrt{4-\sqrt{3}}}{6}, \frac{\sqrt{3}+1-\sqrt{4-\sqrt{3}}}{6}, \frac{\sqrt{3}+1+2\sqrt{4-\sqrt{3}}}{6}\right).$$

(In other words, these ends are on the large circles, i.e., the bisectors of the spherical triangle S^+.) We supplement it with the vectors $n^{(1)}$, $n^{(2)}$, $n^{(3)}$, and $n^{(9)}$ from (27), (29), which always present in the solution. Two more vectors should be added to form a nonsingular matrix $\mathcal{H}$ under the unbiasedness conditions. The two facts below can be established similarly to proving the optimality of the solution $\Phi^0_{(1)}(n)$. With one of the three pairs of vectors $n^{(4)}$, $n^{(8)}$ (the extreme points of C, clearly indicated on the right) of the form

$$\begin{cases} \mathrm{col}\left(\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right) \alpha_1 \\ \mathrm{col}\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right) \alpha_3, \end{cases} \quad \begin{cases} \mathrm{col}\left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right) \alpha_6 \\ \mathrm{col}\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right) \alpha_3, \end{cases} \quad \begin{cases} \mathrm{col}\left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right) \alpha_6 \\ \mathrm{col}\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right) \alpha_2, \end{cases} \tag{38}$$

being added, the resulting nondegenerate set of nine vectors forms the optimal solution for the cases $\nu = 1$ and $\nu = 7$. On the other hand, adding one of the four pairs $n^{(4)}$, $n^{(8)}$ of the form

$$\begin{cases} \mathrm{col}\left(\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right) \alpha_1 \\ \mathrm{col}\left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right) \alpha_5, \end{cases} \quad \begin{cases} \mathrm{col}\left(\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right) \alpha_1 \\ \mathrm{col}\left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right) \alpha_4, \end{cases} \quad \begin{cases} \mathrm{col}\left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right) \alpha_6 \\ \mathrm{col}\left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right) \alpha_5, \end{cases} \quad \begin{cases} \mathrm{col}\left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right) \alpha_6 \\ \mathrm{col}\left(\frac{0, \sqrt{3}}{2}, \frac{1}{2}\right) \alpha_4, \end{cases} \tag{39}$$

gives a nondegenerate set of nine vectors representing the optimal solution for the case $\nu = 4$.

Thus, the solution of the primal problem is not unique. Moreover, sufficiently small deformations (within C) of the set of orientation vectors generating the solution of the primal problem will not change the signature of the solutions of the unbiasedness equations. Therefore, they will also produce a new solution of the primal problem. A complete description of all solutions goes beyond the scope of this paper.

4. THE SIMPLIFIED FORM OF THE MOMENT PROBLEM

Assertion 5 of Theorem 1 and the results obtained above imply that the optimal estimator is nonzero only on three cutsets of S^+ by three planes. The first plane is $n_1 + n_2 + n_3 = 1$; due to

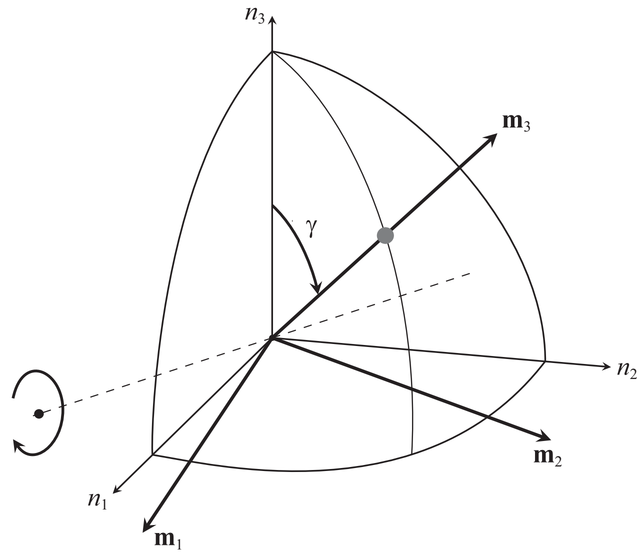

Fig. 2. The rotated coordinate system.

the obvious inequality $1 = n_1^2 + n_2^2 + n_3^2 \leqslant n_1 + n_2 + n_3$, it is possible only for three vectors of (27). The second plane is defined by the equality $n_1 + n_2 + n_3 = \frac{1+\sqrt{3}}{2}$. The third plane is described by the equation $n_1 + n_2 + n_3 = \sqrt{3}$; by the Cauchy–Bunyakovsky–Schwarz inequality, the intersection consists of only one element of (29). Thus, the moment problem is "concentrated" only on four vectors and the set C given by (31).

To consider this circumstance explicitly, we introduce an orthogonal coordinate system $0m_1m_2m_3$ rotated relative to $0n_1n_2n_3$ as follows (see Fig. 2). The axis of rotation lies in the plane $0n_1n_2$ and, in this plane, the equation of the axis of rotation has the form $n_1 + n_2 = 0$; obviously, this is the diagonal of the second and fourth quadrants (the dashed line in Fig. 2). Let us rotate the initial coordinate system $0n_1n_2n_3$ around this rotation axis by the angle γ so that the third axis $0m_1m_2m_3$ coincides with the vector $\mathrm{col}(1, 1, 1)$ (in the original coordinate system $0n_1n_2n_3$), i.e., so that the third axis lies along the diagonal of the octant $\mathbb{R}_+^3$.

As is easily checked,

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} \dfrac{\sqrt{3}+3}{6} & \dfrac{\sqrt{3}-3}{6} & -\dfrac{\sqrt{3}}{3} \\ \dfrac{\sqrt{3}-3}{6} & \dfrac{\sqrt{3}+3}{6} & -\dfrac{\sqrt{3}}{3} \\ \dfrac{\sqrt{3}}{3} & \dfrac{\sqrt{3}}{3} & \dfrac{\sqrt{3}}{3} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}. \tag{40}$$

In the frame $0m_1m_2m_3$, we introduce the spherical coordinates

$$m_1 = \cos\alpha\cos\theta, \qquad m_2 = \sin\alpha\cos\theta, \qquad m_3 = \sin\theta, \qquad 0 \leqslant \alpha, \ \ \theta \leqslant \frac{\pi}{2}.$$

In the cutset by the second plane,

$$m_3 = \sin\theta = \frac{\sqrt{3}}{3}(n_1 + n_2 + n_3) = \frac{\sqrt{3}+3}{6}, \qquad \text{hence} \qquad \cos\theta = \sqrt{\frac{4-\sqrt{3}}{6}};$$

therefore,

$$m_1 = \sqrt{\frac{4-\sqrt{3}}{6}}\cos\alpha, \qquad m_2 = \sqrt{\frac{4-\sqrt{3}}{6}}\sin\alpha.$$

After some calculations, the expressions for the boundary angles of the set C take the form

$$\alpha_1 = -\arcsin\frac{5\sqrt{3}-3}{2\sqrt{6(4-\sqrt{3})}}, \quad \alpha_2 = \arcsin\frac{3-\sqrt{3}}{\sqrt{6(4-\sqrt{3})}},$$
$$\alpha_3 = \arcsin\frac{\sqrt{6}}{\sqrt{3(4-\sqrt{3})}}, \quad \alpha_4 = \pi - \arcsin\frac{3+\sqrt{3}}{2\sqrt{6(4-\sqrt{3})}}, \tag{41}$$
$$\alpha_5 = \pi + \arcsin\frac{3-\sqrt{3}}{2\sqrt{6(4-\sqrt{3})}}, \quad \alpha_6 = \pi + \arcsin\frac{9-\sqrt{3}}{2\sqrt{6(4-\sqrt{3})}}.$$

(Here, the same symbol denotes both the points on C and the corresponding angles in the spherical coordinate system.) In view of (40), we have

$$n_1(\alpha) = \frac{1}{6}\sqrt{\frac{4-\sqrt{3}}{6}}\Big[(\sqrt{3}+3)\cos\alpha + (\sqrt{3}-3)\sin\alpha\Big] + \frac{\sqrt{3}+1}{6},$$
$$n_2(\alpha) = \frac{1}{6}\sqrt{\frac{4-\sqrt{3}}{6}}\Big[(\sqrt{3}-3)\cos\alpha + (\sqrt{3}+3)\sin\alpha\Big] + \frac{\sqrt{3}+1}{6},$$
$$n_3(\alpha) = -\frac{\sqrt{3}}{3}\sqrt{\frac{4-\sqrt{3}}{6}}\,[\cos\alpha + \sin\alpha] + \frac{\sqrt{3}+1}{6}$$

on the set C given by (31) and (41).

Then the original moment problem (11), (12) defined on the surface S^+ turns into the moment problem on the circle arcs:

$$\min_{\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi(\alpha)}\left(|\varphi_1| + |\varphi_2| + |\varphi_3| + |\varphi_4| + \int_C |\varphi(\alpha)|\,d\alpha\right),$$
$$C = [\alpha_1,\alpha_2]\cup[\alpha_3,\alpha_4]\cup[\alpha_5,\alpha_6]$$

subject to the unbiasedness condition

$$H(n^{(1)})\varphi_1 + H(n^{(2)})\varphi_2 + H(n^{(3)})\varphi_3 + H(n^{(9)})\varphi_4 + \int_C H(n(\alpha))\varphi(\alpha)\,d\alpha = a,$$

where the orientation vectors $n^{(1)}, n^{(2)}, n^{(3)}$, and $n^{(9)}$ are given by (27) and (29) whereas the boundary angles α_i by (41). Such a reduction with decreasing the grid dimension significantly facilitates numerical calculations.

5. THE REFINED MOMENT PROBLEM

The moment problem (11), (12) discussed above proceeds from a simple but somewhat high estimate of the noise $\varrho(n)$ (5). An achievable noise estimate can be constructed:

$$|\,\varrho(n)| = |n^{\mathsf{T}}\varrho'(n)| \leqslant (n_1+n_2+n_3)\,\sigma, \qquad n\in S^+.$$

In this case, the moment problem has a more complex formulation:

$$I_0 = \inf_{\Phi(n)\in\mathcal{F}}\int_{S^+}(n_1+n_2+n_3)|\,\Phi(n)|\,dS \tag{42}$$

subject to the unbiasedness condition

$$\int\limits_{S^+} H(n)\Phi(n)\,dS = a, \qquad H(n) \in \mathbb{R}^m \qquad (m = 9). \tag{43}$$

(For simplicity, the constant factor σ is omitted and the notations I_0 and I^0 for the problem values are retained.) Then the dual problem takes the following form:

$$I^0 = \sup_{\lambda \in \mathbb{R}^m} a^\top \lambda \qquad (m = 9)$$

subject to the condition

$$|H^\top(n)\lambda| \leqslant n_1 + n_2 + n_3, \qquad n \in S^+.$$

Theorem 1 will remain valid if the equality $|H^\top(n^{(i)})\lambda^0| = 1$ (16) in assertion 4 and the inequality $|H^\top(n)\lambda^0| < 1$ in assertion 5 are replaced by $|H^\top(n^{(i)})\lambda^0| = n_1^{(i)} + n_2^{(i)} + n_3^{(i)}$ and $|H^\top(n)\lambda^0| < n_1 + n_2 + n_3$, respectively. For simplicity, let us analyze only the case $a = a_{(1)} = \mathrm{col}(1, 0, \ldots, 0)$. The numerical solutions of the refined primal and dual problems again give the basis for introducing the hypothesis from Section 3.2. As a result, the reduced dual problem takes the form

$$\max_{\lambda_1,\lambda_4,\lambda_7} \lambda_1 \qquad \text{subject to the condition} \qquad \left|\frac{s(t;\lambda_1,\lambda_4,\lambda_7)}{t}\right| \leqslant 1, \qquad 1 \leqslant t \leqslant \sqrt{3},$$

where the function $s(t;\lambda_1,\lambda_4,\lambda_7)$ is given by (20). As in Section 3.3, this hypothesis implies the following result: the optimal estimator is nonzero for those orientation vectors for which t is at least at three points of the segment $[1, \sqrt{3}]$, two of which are the ends of this segment and the others lie inside it. By assertion 4 of Theorem 1, for these values of t, we have

$$\left|\frac{s^0(t)}{t}\right| = 1, \qquad \text{where} \quad s^0(t) = s(t;\lambda_1^0,\lambda_4^0,\lambda_7^0), \tag{44}$$

and λ^0 is the solution of the refined dual problem.

Suppose that $\lambda_4^0 = 0$. Then it follows from the accepted hypothesis that $\lambda_1 = 0$ as well. (Otherwise, (44) would not hold for more than two points t.) Obviously, the vector $(\lambda_1, \lambda_4, \lambda_7) = (1, 0, 0)$ provides a greater value for the objective functional of the dual problem. Therefore, $\lambda_4^0 \neq 0$.

The derivative of the function figuring in the constraint of the simplified dual problem is

$$\frac{d}{dt}\left(\frac{s^0(t)}{t}\right) = \frac{\lambda_4^0 - 2\lambda_1^0}{t^2} - \lambda_4^0, \tag{45}$$

where $\lambda_4^0 - 2\lambda_1^0 \neq 0$. (Otherwise, property a) would fail.) Hence, this derivative is monotonic and has a single zero $t^0 \in (1, \sqrt{3})$. (Otherwise, property a) would be violated.) This means that $t^{-1}s^0(t)$ is either convex or concave, having a near-parabolic shape. Therefore, the admissibility of the element λ can be checked by checking admissibility for the extreme values of $s^0(t)$ at its three extreme points: for the other points t, admissibility will hold automatically. Using the same considerations as in Section 3.3, we easily arrive at the following relations similar to (22) and (24):

$$s^0(1) = -\left(\frac{s^0(t^0)}{t^0}\right) = \left(\frac{s^0(\sqrt{3})}{\sqrt{3}}\right), \quad \text{where} \quad t^0 = \sqrt{\frac{2\lambda_1^0}{\lambda_4^0} - 1} \quad \text{due to (45).} \tag{46}$$

In addition, $\operatorname{sgn} s^0(1) = \pm 1 : \lambda_1^0 > 0$ when choosing a correct sign and $\lambda_1^0 < 0$ otherwise. In view of (44) and the signature $\operatorname{sgn} s^0(1) = 1$, equalities (46) take the explicit form

$$\lambda_1^0 + \lambda_7^0 = 1, \qquad \frac{2\lambda_1^0 - \lambda_4^0}{\sqrt{\frac{2\lambda_1^0}{\lambda_4^0} - 1}} + \lambda_7^0 = -1, \qquad \frac{\lambda_1^0 + \lambda_4^0}{\sqrt{3}} + \lambda_7^0 = 1. \tag{47}$$

From the first and third equations in (47) it follows that

$$\sqrt{3}\lambda_1^0 = \lambda_1^0 + \lambda_4^0 \quad \text{and, consequently,} \quad t^0 = \sqrt[4]{3} \quad \text{by (46).}$$

Then the candidate solution of the refined dual problem has the form

$$\begin{aligned} \lambda_1^0 = \lambda_2^0 = \lambda_3^0 &= \frac{(1+\sqrt[4]{3})^2(1+\sqrt{3})^3}{2}, \\ \lambda_4^0 = \lambda_5^0 = \lambda_6^0 &= (1+\sqrt[4]{3})^2(1+\sqrt{3})^2, \\ \lambda_7^0 = \lambda_8^0 = \lambda_9^0 &= -\frac{(1+\sqrt[4]{3})^4(1+\sqrt{3})^2}{4}. \end{aligned} \tag{48}$$

The correct signature choice in (47) is confirmed by the positivity of λ_1^0 in (48).

Let us analyze the primal problem. We introduce the set C_M analogous to (31) but with $n_1 + n_2 + n_3 = \sqrt[4]{3}$. Resembling Fig. 1, the corresponding figure is not given here. Then the angles similar to $\{\alpha_i\}_1^6$ (41) also change to

$$\begin{aligned} \beta_1 &= -\arcsin \frac{\sqrt{3}+1-(\sqrt{3}-1)\sqrt{2\sqrt{3}-3}}{4\sqrt{\sqrt{3}-1}}, \\ \beta_2 &= \arcsin \frac{1-\sqrt{2\sqrt{3}-3}}{2\sqrt{\sqrt{3}-1}}, \\ \beta_3 &= \arcsin \frac{1+\sqrt{2\sqrt{3}-3}}{2\sqrt{\sqrt{3}-1}}, \\ \beta_4 &= \pi - \arcsin \frac{\sqrt{3}-1+(\sqrt{3}+1)\sqrt{2\sqrt{3}-3}}{4\sqrt{\sqrt{3}-1}}, \\ \beta_5 &= \pi + \arcsin \frac{(\sqrt{3}+1)\sqrt{2\sqrt{3}-3}-\sqrt{3}+1}{4\sqrt{\sqrt{3}-1}}, \\ \beta_6 &= \pi + \arcsin \frac{\sqrt{3}+1+(\sqrt{3}-1)\sqrt{2\sqrt{3}-3}}{4\sqrt{\sqrt{3}-1}}. \end{aligned} \tag{49}$$

By analogy, a candidate solution of the primal problem is the one for which the orientation vectors lie on the edges of the set C_M, i.e., at the ends of the arcs $[\beta_1, \beta_2]$, $[\beta_3, \beta_4]$, and $[\beta_5, \beta_6]$. (For simplicity, the same symbol β denotes both the points on C_M and the angles: no confusion occurs.) There exist six such vectors, but only five are needed. We discard the point β_2 corresponding to the orientation vector $n = \operatorname{col}\Big(\frac{\sqrt[4]{3}+\sqrt{2-\sqrt{3}}}{2}, \frac{\sqrt[4]{3}-\sqrt{2-\sqrt{3}}}{2}, 0\Big)$.

Then the orientation vectors lying on the edges of C_M take the form

$$\begin{aligned}
n^{(4)} &= \operatorname{col}\left(\frac{\sqrt[4]{3}-\sqrt{2-\sqrt{3}}}{2}, \frac{\sqrt[4]{3}+\sqrt{2-\sqrt{3}}}{2}, 0\right),\\
n^{(5)} &= \operatorname{col}\left(0, \frac{\sqrt[4]{3}+\sqrt{2-\sqrt{3}}}{2}, \frac{\sqrt[4]{3}-\sqrt{2-\sqrt{3}}}{2}\right),\\
n^{(6)} &= \operatorname{col}\left(0, \frac{\sqrt[4]{3}-\sqrt{2-\sqrt{3}}}{2}, \frac{\sqrt[4]{3}+\sqrt{2-\sqrt{3}}}{2}\right),\\
n^{(7)} &= \operatorname{col}\left(\frac{\sqrt[4]{3}-\sqrt{2-\sqrt{3}}}{2}, 0, \frac{\sqrt[4]{3}+\sqrt{2-\sqrt{3}}}{2}\right),\\
n^{(8)} &= \operatorname{col}\left(\frac{\sqrt[4]{3}+\sqrt{2-\sqrt{3}}}{2}, 0, \frac{\sqrt[4]{3}-\sqrt{2-\sqrt{3}}}{2}\right).
\end{aligned} \tag{50}$$

Similarly to (33), we compile the system of unbiasedness conditions corresponding to the orientation vectors (50). As has been described above, the QR-decomposition or the SVD-procedure can be used to verify the nonsingular property of the unbiasedness condition matrix. Furthermore, the signs of the coefficients of the candidate solution of the primal problem coincide with the signature of the solution of the refined dual problem. Then, similarly to (37), it turns out that

$$\int\limits_{S^+} (n_1+n_2+n_3)|\Phi^0(n)|\,dS = a^{\mathsf{T}}\lambda^0$$

for the constructed admissible elements $\Phi^0(n)$ and λ^0 of the primal and dual problems, respectively, in the refined moment problem. Hence, by assertion 6 of Theorem 1, these elements $\Phi^0(n)$ and λ^0 are the solutions of the refined primal and dual problems, respectively. The optimal value of the objective functional is $I_{0,(1)} = (1+\sqrt[4]{3})^2(1+\sqrt{3})^3/2$ (up to the factor σ coinciding with the optimal guaranteed estimation error). The cases $\nu = 4$ and $\nu = 7$ are investigated using the scheme described above. The corresponding solutions of the refined dual problem coincide within the sign, and $I_{0,(4)} = (1+\sqrt[4]{3})^2(1+\sqrt{3})^2$ and $I_{0,(7)} = (1+\sqrt[4]{3})^4(1+\sqrt{3})^2/4$.

As above, it is easy to observe the following property: with the orientation vector $\operatorname{col}\left(\frac{\sqrt[4]{3}+\sqrt{2-\sqrt{3}}}{2}, 0, \frac{\sqrt[4]{3}-\sqrt{2-\sqrt{3}}}{2}\right)$ corresponding to β_1 being excluded, the remaining five points (with (27) and (29)) generate another solution for the case $\nu = 1$ and $\nu = 7$. On the other hand, eliminating the vector $\operatorname{col}\left(\frac{\sqrt[4]{3}-\sqrt{2-\sqrt{3}}}{2}, \frac{\sqrt[4]{3}+\sqrt{2-\sqrt{3}}}{2}, 0\right)$ corresponding to β_3 from the six extreme points of C_M gives a new solution for the case $\nu = 4$.

By analogy, we take a symmetric set of orientation vectors $n^{(5)}$, $n^{(6)}$, and $n^{(7)}$ from C_M in which the ends of vectors lie at the midpoints of the arcs $[\beta_1, \beta_2]$, $[\beta_3, \beta_4]$, and $[\beta_5, \beta_6]$, respectively:

$$\begin{aligned}
n^{(5)} &= \operatorname{col}\left(\frac{\sqrt[4]{3}+\sqrt{6-2\sqrt{3}}}{3}, \frac{2\sqrt[4]{3}-\sqrt{6-2\sqrt{3}}}{6}, \frac{2\sqrt[4]{3}-\sqrt{6-2\sqrt{3}}}{6}\right),\\
n^{(6)} &= \operatorname{col}\left(\frac{2\sqrt[4]{3}-\sqrt{6-2\sqrt{3}}}{6}, \frac{\sqrt[4]{3}+\sqrt{6-2\sqrt{3}}}{3}, \frac{2\sqrt[4]{3}-\sqrt{6-2\sqrt{3}}}{6}\right),\\
n^{(7)} &= \operatorname{col}\left(\frac{2\sqrt[4]{3}-\sqrt{6-2\sqrt{3}}}{6}, \frac{2\sqrt[4]{3}-\sqrt{6-2\sqrt{3}}}{6}, \frac{\sqrt[4]{3}+\sqrt{6-2\sqrt{3}}}{3}\right).
\end{aligned}$$

Let this set be supplemented by the vectors from (27), (29). To form a nonsingular matrix $\mathcal{H}$, it is necessary to add two more vectors to them. Here, the following result can be established by analogy: with one of the three pairs of vectors (the extreme points of C_M similar to (38) being added to these vectors, the resulting nondegenerate set of nine vectors forms the optimal solution for the cases $\nu = 1$ and $\nu = 7$ simultaneously. On the other hand, attaching one of the four pairs similar to (39) gives a nondegenerate set of nine vectors representing the optimal solution for the case $\nu = 4$.

On the set C_M, we have

$$\begin{aligned} n_1(\alpha) &= \frac{1}{6}\sqrt{\frac{3-\sqrt{3}}{3}}\Big[(\sqrt{3}+3)\cos\alpha + (\sqrt{3}-3)\sin\alpha\Big] + \frac{\sqrt[4]{3}}{3}, \\ n_2(\alpha) &= \frac{1}{6}\sqrt{\frac{3-\sqrt{3}}{3}}\Big[(\sqrt{3}-3)\cos\alpha + (\sqrt{3}+3)\sin\alpha\Big] + \frac{\sqrt[4]{3}}{3}, \\ n_3(\alpha) &= -\frac{\sqrt{3}}{3}\sqrt{\frac{3-\sqrt{3}}{3}}\left[\cos\alpha + \sin\alpha\right] + \frac{\sqrt[4]{3}}{3}. \end{aligned} \tag{51}$$

Then the moment problem (42), (43) defined on the surface S^+ turns into the moment problem on the arcs of the circle C_M:

$$\min_{\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi(\alpha)} \left(|\varphi_1| + |\varphi_2| + |\varphi_3| + \sqrt{3}|\varphi_4| + \int\limits_{C_M} (n_1(\alpha) + n_2(\alpha) + n_3(\alpha))|\varphi(\alpha)|\, d\alpha \right),$$
$$C_M = [\beta_1, \beta_2] \cup [\beta_3, \beta_4] \cup [\beta_5, \beta_6]$$

subject to the unbiasedness condition

$$H(n^{(1)})\varphi_1 + H(n^{(2)})\varphi_2 + H(n^{(3)})\varphi_3 + H(n^{(9)})\varphi_4 + \int\limits_{C_M} H(n(\alpha))\varphi(\alpha)\, d\alpha = a,$$

where the orientation vectors $n^{(1)}, n^{(2)}, n^{(3)}$, and $n^{(9)}$ are given by (27) and (29), $n(\alpha)$ is given by (51), and the boundary nodes β_i are given by (49).

6. CONCLUSIONS

This paper has considered the problem of calibrating a 3D sensor in the field of a constant calibration signal under significant constraints on its admissible angular positions. Such constraints arise, e.g., in the asymmetric models of sensor readings, which depend on the sign of the input signal. This problem has been reduced to the moment problem on the nonnegative octant. The solutions of the moment problem have been obtained in an explicit (analytically closed) form. Thus, the optimal plan of the angular positions of a test bench, the corresponding optimal estimates of the error parameters of the 3D sensor, and the accuracy of these estimates have been determined.

APPENDIX

We begin with proving the inequality $\det \mathcal{H} \neq 0$ in (33). Let $\mathcal{H}_\delta$ be the numerical image of the matrix $\mathcal{H}$: $\mathcal{H}_\delta = \mathcal{H} + \delta\mathcal{H}$. Also, let B denote the inverse of the matrix $\mathcal{H}$ calculated approximately; the matrix B is precisely known. Then $B\mathcal{H}_\delta + \Delta m = I + \Delta\mathcal{H}$, where Δm is the error matrix when multiplying the matrices B and $\mathcal{H}$, and $\Delta\mathcal{H}$ is the known error characterizing the inversion accuracy. From these equalities it follows that

$$B\mathcal{H} = I + \Delta\mathcal{H} - \Delta m - B\delta\mathcal{H}. \tag{A.1}$$

Direct calculations of the matrix $\Delta\mathcal{H}$ show that its elements satisfy the inequality $|\,(\Delta\mathcal{H})_{ij}\,| \leqslant 10^{-14}$, $i, j = 1, \ldots, 9$. Let the elements Δm and $\delta\mathcal{H}$ obey the constraints

$$|\Delta m_{ij}| \leqslant \epsilon, \qquad |\delta\mathcal{H}_{ij}| \leqslant \epsilon, \qquad i, j = 1, \ldots, 9, \qquad \text{where } \epsilon \ll 1.$$

Then $|\,(B\delta\mathcal{H})_{ij}\,| \leqslant \epsilon \sum_{s=1}^{9} |B_{is}|$ and, consequently,

$$|\,(\Delta\mathcal{H} - \Delta m - B\delta\mathcal{H})_{ij}\,| \leqslant 10^{-14} + \epsilon\left[1 + \sum_{s=1}^{9} |B_{is}|\right] \qquad i, j = 1, \ldots, 9.$$

The elements of the known matrix B belong to the intervals $0.1 < |B_{ij}| < 12$. Therefore, $|\,(\Delta\mathcal{H} - \Delta m - B\delta\mathcal{H})_{ij}\,| \leqslant 109\epsilon + 10^{-14}$. Assume that $\epsilon \leqslant 10^{-5}$; this can be ensured by modern computing means. Then the matrix $I + \Delta\mathcal{H} - \Delta m - B\delta\mathcal{H}$ in (A.1) is diagonally dominant and, hence, nonsingular by the Levy–Desplanques theorem [31]. As a result, the same property applies to the matrices $\mathcal{H}$ and B.

Consider the case $a = a_{(1)}$. According to (A.1),

$$\Phi = \mathcal{H}^{-1}a = (I + \Delta\mathcal{H} - \Delta m - B\delta\mathcal{H})^{-1}\,Ba = (I + W)\,Ba = \Phi_{\text{calc}} + \Delta\Phi,$$

where $\Phi_{\text{calc}} = Ba$ is the calculated value of Φ, $\Delta\Phi = WBa$ is the error of calculations, and

$$\|W\| \leqslant \frac{\|\Delta\mathcal{H} - \Delta m - B\delta\mathcal{H}\|}{1 - \|\Delta\mathcal{H} - \Delta m - B\delta\mathcal{H}\|},$$

with $\|W\|$ standing for the spectral norm of W. Then $\|\Delta\Phi\| \leqslant \|W\|\|Ba\|$, where $\|\Delta\Phi\|$ and $\|Ba\|$ are the Euclidean norms of the corresponding vectors. Obviously,

$$\|W\| \leqslant \frac{\|\Delta\mathcal{H}\|_{\text{F}} + \|\Delta m + B\delta\mathcal{H}\|_{\text{F}}}{1 - \|\Delta\mathcal{H}\|_{\text{F}} - \|\Delta m + B\delta\mathcal{H}\|_{\text{F}}};$$

here, the subscript F indicates the Frobenius norm as a majorant for the spectral norm. (It is more difficult to estimate the error of calculations for the spectral norm.) Therefore,

$$\|W\| \leqslant \frac{3\epsilon R + 10^{-13}}{1 - 3\epsilon R - 10^{-13}}, \qquad R = \sqrt{\sum_{i=1}^{9}\left[1 + \sum_{s=1}^{9}|B_{is}|\right]^2}.$$

Let the accuracy of calculating the value R not exceed ϵ. In view of $\|Ba\| \leqslant 16$, we have

$$\|W\| \leqslant \frac{429\epsilon + 3\epsilon^2 + 10^{-13}}{1 - (429\epsilon + 3\epsilon^2 + 10^{-13})} \text{ and } \|\Delta\Phi\| \leqslant \frac{0.069 \times 10^5\epsilon + 3\epsilon^2 + 2 \times 10^{-12}}{1 - (0.005 \times 10^5\epsilon + 3\epsilon^2 + 10^{-12})}.$$

Thus, $\|\Delta\Phi\| \leqslant 0.07$ for $\epsilon = 10^{-5}$ and $\|\Delta\Phi\| \leqslant 0.007$ for $\epsilon = 10^{-6}$.

Since the known elements of the vector $Ba = Ba_{(1)}$ are not less than 1.8 in absolute value, we have proved that approximate calculations surely establish the signs of the components of the vector Φ. For $\nu = 4$ and $\nu = 7$, the considerations are similar.

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