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Optimizing the Placement and Number of Measurement Points in Heating Process Control

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Abstract—In this paper, a heating process control law with steam supply is designed for a fluid in a heat exchanger. The process is described by a linear hyperbolic equation of the first order with a nonlocal boundary condition with a time-delayed argument. The temperature of the supplied steam is found as a linear dependence on fluid temperature values at measurement points in the heat exchanger. Explicit formulas are obtained for the gradient of the objective functional of the control problem in the space of the feedback coefficients (parameters) of this dependence. A numerical scheme is developed for determining the feedback parameters based on these formulas. Finally, an algorithm is proposed for determining the rational (optimal) number of measurement points.

Keywords: distributed parameter system, heating process control, feedback, measurement point, gradient of the functional, feedback parameters

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1. INTRODUCTION

In this paper, an optimal heating process control law is designed for a fluid in a tubular heat exchanger described by a hyperbolic equation [1]. The fluid is heated by supplying steam to the heat exchanger and the steam temperature is the control action. The heated fluid circulates in a closed heating system. Hence, there exists a cyclic boundary condition relating fluid temperature values when leaving the heat exchanger and returning to it after passing through the heating system. The circulation time of the fluid in the heating system is given and is determined by fluid velocity and the length of the pipeline network.

The goal of control is to maintain a required value of fluid temperature at the heat exchanger outlet. In the problem under consideration, the temperature of the steam supplied to the heat exchanger is found as a linear dependence on the fluid temperature values measured at separate points of the tubular heat exchanger. In other words, a linear dependence on the measured temperature values is used for the designed control law. The linear dependence coefficients (the feedback parameters to be designed) are obtained by minimizing an objective functional that determines the deviation of the desired fluid temperature at the heat exchanger outlet from the mathematical model-based value under given feedback parameters. A similar problem was considered in [1] and later in [2–5].

In contrast to the problem statements presented previously, besides determining the values of control parameters, we introduce an approach to optimizing the placement of temperature measurement points for the fluid in the tubular heat exchanger. Moreover, we investigate an efficient (in some sense, optimal) number of temperature measurement points.

The process control design approach described below reduces the original hyperbolic differential equation to a loaded differential equation in which the loading points are the optimized locations of the measurement points. Such loaded problems were examined in [6, 7], and numerical methods for their solution were proposed in [8, 9].

Generally speaking, control design problems for distributed parameter systems described by partial differential equations [1, 10] are underinvestigated compared to those described by ordinary differential equations [11]. First of all, the reasons are the complexity of building adequate mathematical models and their parametric identification. This is due to missed or insufficiently accurate operational information about the current state of the processes. The lack of effective numerical methods and computational tools for solving initial boundary-value problems and reliable remote control equipment is of no small importance.

In recent years, interest in these problems has increased significantly [12–18] following the growing capabilities of computing and measuring devices as well as the development of numerical methods of computational mathematics, optimization, and optimal control.

Relatively few authors have proposed solutions to regulation and optimal control problems for systems with lumped or distributed parameters and feedback, in contrast to control problems without feedback; for example, see [19–24]. The history and current state of research on control design problems were rather comprehensively analyzed in [11].

The approach presented below differs from the known optimal feedback control methods mainly as follows: the original problem is reduced to an optimal parametric control problem of a relative loaded differential equation. In this case, the parameters to be optimized are the feedback parameters, and the loading points are the measurement points (the locations where the current state of the process is observed). In particular, this approach was applied in our earlier studies [4, 16, 17] to design control laws within other problem statements (with different types of differential equations, boundary conditions, and objective functionals).

As an illustrative example, the control design method and the feedback parameter formulas are applied to a test problem. Numerical experiments are carried out and their results are discussed.

2. PROBLEM STATEMENT

Consider an optimal heating process control problem for a fluid in a tubular heat exchanger. The fluid then enters a heating system. Heating is carried out by supplying hot steam with a controlled temperature into the heat exchanger; see Fig. 1. According to [1], the fluid heating process in a tubular heat exchanger of a given length L can be described by a linear hyperbolic differential

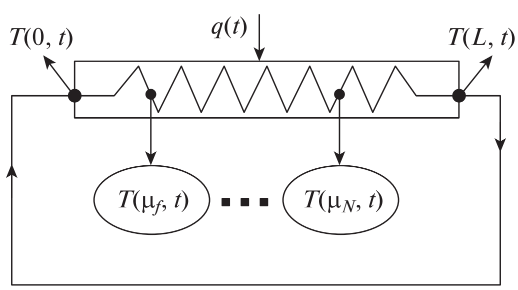

Fig. 1. The block diagram of a heating system.

transfer equation of the form

$$\frac{\partial T(x,t)}{\partial t} + \vartheta \frac{\partial T(x,t)}{\partial x} = \lambda \left[q(t) - T(x,t) \right], \quad x \in (0, L), \quad t \in [0, t_f], \tag{2.1}$$

with the following notations: $T(x,t)$ is a continuous and almost everywhere differentiable function that determines the fluid temperature at a point $x \in (0, L)$ at a time instant $t \in [0, t_f]$; t_f is the duration of the heating process; ϑ is the steady-state fluid velocity, a constant value at all points; $q(t)$, $t \in [0, t_f]$, is a piecewise continuous control function (law) that determines the temperature of steam supplied to the heat exchanger; finally, λ is a given coefficient of heat exchange between the fluid in the heat exchanger and the steam supplied to it.

The temperature of the steam supplied must satisfy the technological conditions

$$\underline{q} \leqslant q(t) \leqslant \bar{q}, \quad t \in [0, t_f]. \tag{2.2}$$

Let the fluid heated in the heat exchanger return from the heating system back to the exchanger in a given time τ. This time is determined by the length l of the pipeline network under the assumption that $l >> L$, i.e., $\tau = (l + L)/\vartheta$. (In other words, the pipeline network is much longer than the tubular heat exchanger.) The flowing fluid cools down due to heat exchange with the heating medium, and the range of possible temperature losses is known. As a result, the fluid temperature at the heat exchanger inlet and outlet satisfies the condition

$$T(0,t) = (1-\gamma) T(L, t-\tau), \quad t \in [0, t_f], \tag{2.3}$$

$$\gamma \in \Gamma = [1 - \delta,\, 1]. \tag{2.4}$$

The value $\delta > 0$ determining the range of temperature losses of the fluid passing through the heated medium is given.

In addition, the distribution of the values γ on the set Γ has a known density $\rho_\Gamma(\gamma)$ such that

$$\rho_\Gamma(\gamma) \geqslant 0, \quad \int_\Gamma \rho_\Gamma(\gamma) d\gamma = 1.$$

The fluid temperature before the heating process is constant over time and over the length of the heat exchanger. It has the following range of possible values:

$$T(x,t) = \varphi = \text{const} \in \Phi = [\underline{\Phi}_0, \bar{\Phi}^0], \quad x \in [0, L], \quad t \leqslant 0, \tag{2.5}$$

where $\underline{\Phi}_0$ and $\bar{\Phi}^0$ are given. Also, the distribution density $\rho_{\rm T}(\varphi)$ is known and

$$\rho_\Phi(\varphi) \geqslant 0, \quad \int_\Phi \rho_\Phi(\varphi)\, d\varphi = 1.$$

The heating process control problem for the fluid in the heat exchanger is to find an appropriate control function $q(t)$ (the temperature of the steam supplied to the exchanger) that minimizes the objective functional

$$J(q) = \int_\Phi \int_\Gamma \int_{t_b}^{t_f} \left[T(L,t;q,\varphi,\gamma) - V \right]^2 \rho_\Gamma(\gamma)\, \rho_\Phi(\varphi)\, dt\, d\gamma\, d\varphi + \varepsilon \| q(t) - \tilde{q}(t) \|^2_{L^2[0,t_f]}. \tag{2.6}$$

Here, $T(L,t;q,\varphi,\gamma)$ is the temperature at the heat exchanger outlet $x = L$ obtained by solving the initial boundary-value problem (2.1), (2.3), and (2.5) under given values of the steam temperature $q = q(t)$, $t \in [0, t_f]$, the initial fluid temperature φ, and the heat loss coefficient γ_0.

Note that the objective functional (2.6) assesses the quality of the control function $q(t)$, $t \in [0, t_f]$, with a bundle of trajectories $T(x, t; q, \varphi, \gamma)$ for $\varphi \in \Phi$ and $\gamma \in \Gamma$. Thus, the control problem is to determine the control law $q(t)$ for which this bundle maximizes the objective functional. In other words, we seek the control function that is optimal on average over the sets of possible initial conditions Φ and heat loss coefficients Γ.

The given value V is the desired fluid temperature at the heat exchanger outlet under all possible values of the initial temperature $\varphi \in \Phi$ and the heat loss coefficient $\gamma \in \Gamma$. It must be maintained on the time interval $[t_b, t_f]$. The given value t_b, $0 \leqslant t_b \leqslant t_f$, determines the time instant after which the fluid temperature at the heat exchanger outlet must be in the neighborhood of the desired temperature V; ε and $\tilde{q}(t)$ are given regularization parameters.

The objective functional (2.6) assesses the behavior of the controlled heating process on average over all possible values of the initial fluid temperature $\varphi \in \Phi$ and heat loss coefficient $\gamma \in \Gamma$. This functional assesses the quality of control with the bundle of state-space trajectories defined by the value sets of the initial conditions Φ and heat loss coefficients.

Suppose that the current temperature values are measured at the points $\mu_i \in [0, L]$, $i = 1, \ldots, N$, of the tubular heat exchanger: in continuous time, i.e.,

$$T_i(t) = T(\mu_i, t), \quad t \in [0, t_f], \quad i = 1, \ldots, N, \tag{2.7}$$

or at given discrete instants $t_j = j\Delta t$, $j = 1, \ldots, M$, $\Delta t = t_f/M$, i.e.,

$$T_{ij} = T(\mu_i, t_j), \quad i = 1, \ldots, N, \quad j = 1, \ldots, M. \tag{2.8}$$

Generally, the measurement points μ_i, $i = 1, \ldots, N$, $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$, and their number can be the optimized parameters. Note that they must satisfy the obvious constraints

$$0 \leqslant \mu_i \leqslant L, \quad i = 1, \ldots, N. \tag{2.9}$$

In the continuous-time case (2.7), the current temperature values for the supplied steam are assigned using the linear feedback law (dependence)

$$q(t; \mathbf{P}) = \sum_{i=1}^{N} \alpha_i \left[T(\mu_i, t) - T_i^{nom}\right] = \sum_{i=1}^{N} [\alpha_i T(\mu_i, t) - \beta_i], \quad t \in [0, t_f], \tag{2.10}$$

where

$$\beta_i = \alpha_i T_i^{nom}, \quad i = 1, \ldots, N;$$
$$\mathbf{P} = (\mu_1, \alpha_1, \beta_1, \ldots, \mu_N, \alpha_N, \beta_N) = (\mu, \alpha, \beta) \in \mathbb{R}^{3N}.$$

The feedback parameter vector $\mathbf{P}$ determines the current temperature value of the steam supplied to the heat exchanger, thereby determining the operation of the entire heating process. By analogy with control design in lumped parameter systems, the coefficients α_i, $i = 1, \ldots, N$, will be called gains.

The optimized parameters T_i^{nom}, $i = 1, \ldots, N$, determine the nominal temperature values that must be maintained at the measurement points μ_i, $i = 1, \ldots, N$, during heating.

In the case of discrete-time measurements (2.8), we use the following piecewise constant dependence for the temperature of steam supplied to the heat exchanger:

$$q(t; \mathbf{P}) = \sum_{i=1}^{N} [\alpha_i T_{ij} - \beta_i], \quad t \in [t_{j-1}, t_j), \quad j = 1, \ldots, M. \tag{2.11}$$

The parameters $\alpha_i,\ \beta_i,\ i = 1, \ldots, N$, have the same meaning as above. Substituting the control function (2.6) into Eq. (2.1) gives the loaded differential equation

$$\frac{\partial T(x,t)}{\partial t} + \vartheta \frac{\partial T(x,t)}{\partial x} = \lambda \left[\sum_{i=1}^{N} \alpha_i T(\mu_i, t) - \beta_i \right], \quad x \in (0, L), \quad t \in [0, t_f], \tag{2.12}$$

where loading points are temperature measurement points; for details, see [6, 8].

Substituting the dependence (2.10) into (2.1), we arrive at the differential equation

$$\frac{\partial T(x,t)}{\partial t} + \vartheta \frac{\partial T(x,t)}{\partial x} = \lambda \left[\sum_{i=1}^{N} \alpha_i T(\mu_i, t_{j-1}) - \beta_i \right], \qquad x \in (0, L), \quad t \in [t_{j-1}, t_j], \quad j = 1, \ldots, M, \tag{2.13}$$

with the natural condition

$$T\left(x, t_j + 0\right) = T\left(x, t_j\right), \quad x \in [0, L], \quad j = 1, \ldots, M-1. \tag{2.14}$$

It expresses the continuous heating process at measurement instants.

For continuous (2.7) and discrete (2.8) measurements, the objective functional (2.6) of control performance takes the form

$$J\left(\mathbf{P}\right) = \int_{\Phi} \int_{\Gamma} \int_{t_b}^{t_f} \left[T(L, t; \mathbf{P}, \varphi, \gamma) - V\right]^2 \rho_\Gamma\left(\gamma\right)\, \rho_\Phi\left(\varphi\right)\, dt\, d\gamma\, d\varphi + \varepsilon \left\| \mathbf{P} - \tilde{\mathbf{P}} \right\|_{\mathbb{R}^{3N}}. \tag{2.15}$$

Here, ε and $\tilde{\mathbf{P}}$ are the regularization parameters, and $T(x, t; \mathbf{P}, \varphi, \gamma)$ is the solution of the initial boundary-value problem in the case of continuous measurements (2.7) or the solution of problem (2.13), (2.3), and (2.5) in the case of discrete measurements (2.9) under given values of the feedback parameters $\mathbf{P}$, the initial temperature $\varphi \in \Phi$, and the temperature loss coefficient $\gamma \in \Gamma$.

By assumption, fluid temperature values in the tubular heat exchanger under all possible initial temperatures (2.5) and temperature loss coefficients $\gamma \in \Gamma$ belong to a known range:

$$\underline{T} \leqslant T(x,t) \leqslant \overline{T}, \quad x \in (0, L), \quad t \in (0, t_f). \tag{2.16}$$

Obviously, condition (2.16) must hold for all measured values:

$$\underline{T} \leqslant T(\mu_i, t) \leqslant \overline{T}, \quad x \in (0, L), \quad t \in (0, t_f), \quad i = 1, \ldots, N.$$

In other words, the N-dimensional vector

$$\breve{T}(t) = (T(\mu_1, t), \ldots, T(\mu_N, t)), \quad t \in (0, t_f),$$

belongs to the N-dimensional cube $\overset{s}{\breve{T}} = (\overset{s}{\breve{T}}_1, \ldots, \overset{s}{\breve{T}}_N)$, $s = 1, \ldots, 2^N$, with the vertices $\underline{T}$ or $\bar{T}$, i.e.,

$$\overset{s}{\breve{T}}_j = \underline{T} \wedge \overline{T}, \quad x \in (0, L), \quad j = 1, \ldots, N, \quad s = 1, \ldots, 2^N.$$

In view of the dependence (2.9), the control conditions (2.2) lead to the following constraints on the feedback parameters:

$$\underline{q} \leqslant \left[\sum_{i=1}^{N} \alpha_i \overline{T} - \beta_i \right] \leqslant \bar{q}, \quad \underline{q} \leqslant \left[\sum_{i=1}^{N} \alpha_i \underline{T} - \beta_i \right] \leqslant \overline{q} \tag{2.17}$$

(in both the continuous and discrete cases).

Thus, the original steam temperature control problem with the continuous feedback law (2.7) for heating the fluid and optimizing the placement of measurement points has been reduced to the parametric optimal control problem (2.12), (2.3), (2.5), (2.9), (2.16), and (2.17). In the case of the discrete feedback law (2.8), Eq. (2.12) must be replaced with Eq. (2.13). The optimized feedback parameter vector $\mathbf{P}$ consists of $3N$ elements, and the linear constraints (2.9) and (2.17) are imposed on them.

According to the aforesaid, the resulting problem has the following peculiarities: the differential Eq. (2.1) is loaded, and the objective functional (2.6) assesses the behavior of a bundle of trajectories under given feedback parameters. Despite the convexity of the objective functional in the control variable $q(t)$ in the original problem (2.1)–(2.6), the objective functional (2.15) is nonconvex in the parameters $\mathbf{P}$ for both the continuous and discrete feedback laws. (This property follows from the differential Eqs. (2.12) and (2.13).) Note also the small dimension of the feedback parameter vector, equal to $3N$. In real applications, N does not exceed 5–8, and consequently, the number of constant feedback parameters does not exceed 20–30. The numerical solution of such parametric optimal control problems causes no particular difficulties: real-time calculations are not required.

3. NECESSARY OPTIMALITY CONDITIONS FOR FEEDBACK PARAMETERS

To investigate differentiability and obtain first-order necessary optimality conditions, we prove the following result using the well-known estimation method for the increment of functionals.

Theorem 1. *Consider the solution of the initial boundary-value problem* (2.12)*,* (2.3)*, and* (2.5) *with the continuous feedback law* (2.7)*. The objective functional* (2.15) *on this solution is differentiable with respect to the parameters* μ_i*,* α_i*, and* β_i*,* $i = 1, \ldots, N$*, and the components of its gradient have the form*

$$\operatorname{grad}_{\mu_i} J(\mathbf{P}) = \int\limits_{\Phi} \left[\int\limits_{\Gamma} \left\{ -\lambda\alpha_i \int\limits_0^{t_f} \left(\int\limits_0^L \psi(x,t;\mathbf{P},\varphi,\gamma)dx \right) T_x(\mu_i,t;\mathbf{P},\varphi,\gamma)dt + 2\sigma(\mu_i - \tilde{\mu}_i) \right\} \rho_\Gamma(\gamma)d\gamma \right] \rho_\Phi(\varphi)\,d\varphi, \tag{3.1}$$

$$\operatorname{grad}_{\alpha_i} J(\mathbf{P}) = \int\limits_{\Phi} \left[\int\limits_{\Gamma} \left\{ -\lambda \int\limits_0^{t_f} (T(\mu_i,t;\mathbf{P},\varphi,\gamma) - \beta_i) \left(\int\limits_0^L \psi(x,t;\mathbf{P},\varphi,\gamma)dx \right) dt + 2\sigma(\alpha_i - \tilde{\alpha}_i) \right\} \rho_\Gamma(\gamma)d\gamma \right] \rho_\Phi(\varphi)\,d\varphi, \tag{3.2}$$

$$\operatorname{grad}_{\beta_i} J(\mathbf{P}) = \int\limits_{\Phi} \left[\int\limits_{\Gamma} \left\{ \lambda\alpha_i \int\limits_0^L \psi(x,t;\mathbf{P},\varphi,\gamma)dx + 2\sigma(\beta_i - \tilde{\beta}_i) \right\} \rho_\Gamma(\gamma)d\gamma \right] \rho_\Phi(\varphi)d\varphi, \tag{3.3}$$

where $i = 1, \ldots, N$.

Under the given feedback parameters $\mathbf{P}$*, initial temperature, and heat loss coefficient* φ*, the function* $\psi(x,t;\mathbf{P},\varphi,\gamma)$ *is the solution of the adjoint boundary-value problem*

$$\psi_t(x,t)+\vartheta\psi_x(x,t)=\lambda\psi(x,t), \quad (x,t)\in\Omega, \tag{3.4}$$

$$\psi(x,t_f)=0, \quad x\in[0,L], \tag{3.5}$$

$$\psi(L,t)=-\frac{2}{\vartheta}(T(L,t)-V), \quad t\in(t_f-\tau,t_f], \tag{3.6}$$

$$\psi(L,t)=-\frac{\lambda}{\vartheta}(1-\gamma)\psi(0,t+\tau)-\frac{2}{a}(T(L,t)-V), \quad t\in(t_b,t_f-\tau], \tag{3.7}$$

$$\psi(L,t)=-\frac{\lambda}{\vartheta}(1-\gamma)\psi(0,t+\tau), \quad t\in(0,t_b], \tag{3.8}$$

with the conditions

$$\psi(\mu_i^-,t)=\psi(\mu_i^+,t)+\frac{\lambda}{\vartheta}\alpha_i\int\limits_0^L\psi(x,t)dx, \quad i=1,2,\ldots,N, \tag{3.9}$$

for $t\in[0,t_f]$ *at the points* μ_i*,* $i=1,2,\ldots,N$*, and the conditions*

$$\psi(x,t_b^-)=\psi(x,t_b^+), \quad x\in[0,L], \tag{3.10}$$

at the time instant t_b*.*

Theorem 2. *Consider the solution of the initial boundary-value problem* (2.13)*,* (2.3)*, and* (2.5) *with the discrete feedback law* (2.8)*. The objective functional* (2.15) *on this solution is differentiable with respect to the feedback parameters and the components of its gradient have the form*

$$\begin{aligned}\operatorname{grad}_{\mu_i}J(\mathbf{P})=\int\limits_\Phi\Bigg[\int\limits_\Gamma\Bigg\{-\lambda\alpha_i\sum_{j=1}^M\int\limits_{t_{j-1}}^{t_j}\left(\int\limits_0^L\psi(x,t;\mathbf{P},\varphi,\gamma)dx\right)T_x(\mu_i,t;\mathbf{P},\varphi,\gamma)\,dt\\ +2\sigma(\mu_i-\tilde{\mu}_i)\Bigg\}\rho_\Gamma(\gamma)d\gamma\Bigg]\rho_\Phi(\varphi)\,d\varphi, \quad t_{j-1}\leqslant t\leqslant t_j,\end{aligned} \tag{3.11}$$

$$\begin{aligned}\operatorname{grad}_{\alpha_i}J(\mathbf{P})=\int\limits_\Phi\Bigg[\int\limits_\Gamma\Bigg\{-\lambda\sum_{j=1}^M\int\limits_{t_{j-1}}^{t_j}(T(\mu_i,t;\mathbf{P},\varphi,\gamma)-\beta_i)\left(\int\limits_0^L\psi(x,t;\mathbf{P},\varphi,\gamma)dx\right)dt\\ +2\sigma(\alpha_i-\tilde{\alpha}_i)\Bigg\}\rho_\Gamma(\gamma)d\gamma\Bigg]\rho_\Phi(\varphi)d\varphi, \quad t_{j-1}\leqslant t\leqslant t_j,\end{aligned} \tag{3.12}$$

$$\operatorname{grad}_{\beta_i}J(\mathbf{P})=\int\limits_\Phi\left[\int\limits_\Gamma\left\{\lambda\mu_i\int\limits_0^L\psi(x,t;\mathbf{P},\varphi,\gamma)dx+2\sigma(\beta_i-\tilde{\beta})_i\right\}\rho_\Gamma(\gamma)d\gamma\right]\rho_\Phi(\varphi)d\varphi, \tag{3.13}$$

where $i=1,\ldots,N$*.*

Under the given values $\mathbf{P}$*,* φ*, and* γ*, the function* $\psi(x,t;\mathbf{P},\varphi,\gamma)$ *is the solution of the initial boundary-value problem*

$$\psi_t(x,t)+\vartheta\psi_x(x,t)=\lambda\psi(x,t),\quad t_{j-1}\leqslant t\leqslant t_j,\quad j=1,\ldots,M, \tag{3.14}$$

$$\psi(x,t_f)=0,\quad x\in[0,L], \tag{3.15}$$

$$\psi(L,t)=-\frac{2}{\vartheta}(T(L,t)-V),\quad t\in(t_M-\tau,t_M], \tag{3.16}$$

$$\psi(L,t)=-\frac{\lambda}{\vartheta}(1-\gamma)\psi(0,t+\tau)-\frac{2}{a}(T(L,t)-V),\quad t\in(t_b,t_M-\tau], \tag{3.17}$$

$$\psi(L,t)=-\frac{\lambda}{\vartheta}(1-\gamma)\psi(0,t+\tau),\quad t\in(0,t_k], \tag{3.18}$$

with the conditions

$$\psi(\mu_i^-,t)=\psi(\mu_i^+,t)+\frac{\lambda}{\vartheta}\alpha_i\int\limits_0^L\psi(x,t)dx,\quad i=1,2,\ldots,N, \tag{3.19}$$

for $t_{j-1}\leqslant t\leqslant t_j$*,* $j=1,\ldots,M$*, at the points* μ_i*,* $i=1,2,\ldots,N$*, and the conditions*

$$\psi(x,t_j^-)=\psi(x,t_j^+)+\frac{\lambda}{\vartheta}\sum_{i=1}^N\alpha_i\delta(x-\mu_i)\int\limits_{t_{j-1}}^{t_j}\int\limits_{\mu_i}^{\mu_{i+1}}\psi(x,t)\,dx\,dt,\quad x\in[0,L],\quad j=1,2,\ldots,M, \tag{3.20}$$

at the time instants t_j*,* $j=1,\ldots,M$*.*

Now we present necessary optimality conditions in the variational form [25] with respect to the feedback parameters in both cases (continuous and discrete measurements). As has been mentioned in Section 2, these conditions consider the nonconvexity of the objective functional (2.15).

Theorem 3. *Assume that the* $3N$*-dimensional feedback parameter vector* $\mathbf{P}^*=(\mu^*,\alpha^*,\beta^*)$ *satisfying conditions* (2.9)*,* (2.16)*, and* (2.17) *locally minimizes the objective functional* (2.15)*. Then the inequalities*

$$\left\langle\frac{\partial J(\mathbf{P}^*)}{\partial\mu},\mu^*-\mu\right\rangle\leqslant 0,$$

$$\left\langle\frac{\partial J(\mathbf{P}^*)}{\partial\alpha},\alpha^*-\alpha\right\rangle\leqslant 0,$$

$$\left\langle\frac{\partial J(\mathbf{P}^*)}{\partial\beta},\beta^*-\beta\right\rangle\leqslant 0$$

hold for an arbitrary vector $\mathbf{P}=(\mu,\alpha,\beta)\in R^{3N}$ *satisfying conditions* (2.9)*,* (2.16)*, and* (2.17)*. Here,* $\langle\,,\rangle$ *stands for the scalar product in the* N*-dimensional space.*

4. A NUMERICAL SOLUTION SCHEME FOR THE DESIGN PROBLEM

According to the previous sections, the optimal values of both continuous (2.7) and discrete (2.8) feedback parameters can be determined by solving parametric optimal control problems. For this purpose, we employ first-order optimization methods [25, 26]. In view of the linear constraints (2.16) and (2.17) on the optimized parameters and the explicit formulas for the gradients of the objective functionals (Theorems 1 and 2), it is efficient to choose the gradient projection method

$$\mathbf{P}^{\gamma+1}=\Pr[\mathbf{P}^{\gamma}-\eta_{\gamma}\mathrm{grad}\,J(\mathbf{P}^{\gamma})]. \tag{4.1}$$

Here, Pr[.] denotes the projection operator on the constraints (2.16) and (2.17). Due to their linearity, it has a constructive character [24, 25]. The step η_γ can be found using any one-dimensional optimization method:

$$\eta_\gamma = \arg\min_{\eta \geqslant 0} J(\mathbf{P}^\gamma - \eta \operatorname{grad} J(\mathbf{P}^\gamma)).$$

Conditions (2.3) for the initial boundary-value problem, as well as conditions (3.7) for the adjoint one, have a delay. Therefore, the well-known method of steps with the natural step τ [27] can be applied. In this method, the time interval $[0, t_f]$ is partitioned into subintervals of length τ: $[t_s, t_{s+1}]$, $\tau = t_{s+1} - t_s$, $s = 1, \ldots, m$, $m = [t_f/\tau]$, $t_0 = 0$, $t_m = t_f$. (The symbol $[a]$ means the integer part of a number a; if t_f/τ is not an integer, then $t_m = [t_f/\tau] + 1$.) The initial boundary-value problem is sequentially solved from $s = 0$ to $s = m$, whereas the adjoint one (3.4)–(3.9) is solved backwards, from $s = m$ to $s = 0$. In this case, the boundary conditions of the corresponding boundary-value problem on each subinterval contain no delay.

Another peculiarity of the problem is the loaded differential Eq. (2.12). Such loaded problems for different types of differential equations were investigated in [6, 8]. Their numerical solution methods [8] involve grids and a special representation for the resulting reduced finite-difference boundary-value problems. This approach can be easily applied to the problems under consideration.

The algorithms proposed in [25, 26] were used to regularize the problem, particularly to select the regularization parameters ε and $\tilde{\mathbf{P}}$ in the objective functional (2.15).

The initial boundary-value problems were solved using the implicit grid method scheme and the grid steps h_x in x and h_t in time were selected through numerical experiments.

5. OPTIMIZING THE NUMBER OF MEASUREMENT POINTS

In some cases, the number of measurement points for the controlled process can be not fixed. Then it is necessary to optimize their number and locations. Therefore, we consider the following approach to select an optimal number of measurement points. Obviously, the optimal number of measurement points must satisfy, to some extent, the minimality condition.

We denote by $J_N^* = J^*(\mathbf{P}^N; N)$ the minimum value of the objective functional in problem (2.3), (2.4)–(2.15) given N measurement points and by $\mathbf{P}^N$ the optimal values of the designed feedback control parameters. It is clear that $J_N^* = J^*(\mathbf{P}^N; N)$ is nonincreasing as a complex function of the variable N. In other words,

$$J^*(\mathbf{P}^*; \cdot) \leqslant J^*(\mathbf{P}^{N_1}; N_1) \leqslant J^*(\mathbf{P}^{N_2}; N_2), \quad N_2 < N_1. \tag{5.1}$$

Here, $J_N^* = J^*(\mathbf{P}^N; N)$ is the optimal value of the objective functional in the original problem (2.1)–(2.6) given N measurement points; $J^* = J^*(\mathbf{P}^*; \cdot)$ is the optimal value of the objective functional in the problem with the feedback distributed along the entire rod, which corresponds to measurements at almost all points of the rod, i.e.,

$$J^*(\mathbf{P}^*; \cdot) = \lim_{N \to \infty} J^*(\mathbf{P}^N; N).$$

Due to (5.1), as the number of measurement points increases, the optimal values of the objective functional can only decrease and approach J^* infinitely close (Fig. 2). Consequently, for an arbitrary number $\delta > 0$, it is possible to determine a number N_δ of measurement points such that

$$J^*(\mathbf{P}^N; N) \leqslant J^* + \delta \ \text{ for } \ N > N_\delta.$$

In some problems, there may exist a finite number N^* such that

$$J^*(\mathbf{P}^N; N) = J^* \ \text{ for } \ N > N^*.$$

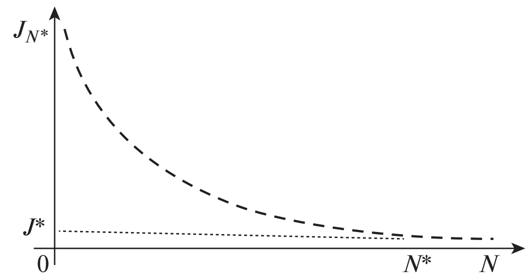

Fig. 2. The optimal value of the objective functional depending on the number of measurement points.

We select an optimal number of measurement points as a minimum value N^* under which one of the following inequalities holds for the first time:

$$\Delta J^*(\mathbf{P}^{N^*}; N^*) = \left| J^*(\mathbf{P}^{N^*+1}; N^*+1) - J^*(\mathbf{P}^{N^*}; N^*) \right| \leqslant \delta,$$
$$\Delta J^*(\mathbf{P}^{N^*}; N^*)/J^*(\mathbf{P}^{N^*}; N^*) \leqslant \delta. \tag{5.2}$$

Here, δ is a given positive value determined by the required accuracy of optimizing the number of measurement points.

Assume that the optimal heating control parameters for the rod with a given number N of measurement points have been obtained as described above. The number N can be reduced if in the optimal vector μ^N, two neighbor measurement points satisfy the inequality

$$|\mu_{j+1}^N - \mu_j^N| \leqslant \delta_1, \quad j = 1, 2, \ldots, N-1, \tag{5.3}$$

with a given sufficiently small value $\delta_1 > 0$. Under condition (3.2), one of the two neighbor measurement points can be eliminated. (In this case, the total number of measurement points is decreased by 1.) Obviously, reducing the number of measurement points improves the reliability of the control system as well as cuts the related system costs (design and maintenance).

6. THE RESULTS OF NUMERICAL EXPERIMENTS

Now we present the results of numerical experiments for the original problem (2.1)–(2.6).

The problem was solved under the following initial data: $L = 1$, $\vartheta = 1$, $\lambda = 0.1$, $\tau = 0.2$, $t_f = 5$, $V = 70$, $\Phi = [0,\ 0.2]$, $\underline{q} = 55$, $\bar{q} = 75$, $\bar{\alpha}_1 = \bar{\alpha}_2 = 8$, $\underline{\alpha}_1 = \underline{\alpha}_2 = 1$, $\bar{\beta}_1 = \bar{\beta}_2 = 75$, and $\underline{\beta}_1 = \underline{\beta}_2 = 57$. The distribution density $\rho_\Gamma(\gamma)$ was set uniform on $[0,\ 0.2]$, and the integral over Γ was approximated using the method of rectangles with a step of 0.05.

Note that the values $\underline{\alpha}_i$, $\bar{\alpha}_i$, $i = 1, 2$, were selected using the results of test calculations under the technological condition (2.2) with the given values $\underline{q}$ and $\bar{q}$.

The numerical experiments were carried out under different initial parameter values $(\mathbf{P}^0)^j = (\alpha_1^0,\ \alpha_2^0, \beta_1^0,\ \beta_2^0, \mu_1^0,\ \mu_2^0)^j$, $\quad j = 1, 2, \ldots, 5$, used for the iterative optimization procedure (4.1). Table 1 shows these values and the corresponding values of the objective functional at these points.

Table 1. The initial values of the optimized parameters μ, α, and β and the corresponding values of the objective functional

no.	μ_1^0	μ_2^0	α_1^0	α_2^0	β_1^0	β_2^0	$J(\mathbf{P}^0)$
1	0.1	0.8	4	6	61	63	363.210004
2	0.2	0.9	3	5	65	60	357.150011
3	0.4	0.8	1	8	62	63	257.310003
4	0.5	0.7	5	2	63	66	165.150016
5	0.2	0.7	6	4	66	62	205.190007

Table 2. The resulting values of the optimized parameters and objective functional

no.	$\mu_1^{(6)}$	$\mu_2^{(6)}$	$\alpha_1^{(6)}$	$\alpha_2^{(6)}$	$\beta_1^{(6)}$	$\beta_2^{(6)}$	$J(\mathbf{P}^{(6)})$
1	0.2994	0.5994	5.9956	3.9952	66.9945	68.9949	0.3422
2	0.3000	0.6000	5.9977	3.9983	66.9978	68.9954	0.3259
3	0.2971	0.5971	5.9962	3.9988	66.9951	68.9948	0.3538
4	0.3000	0.6000	5.9978	3.9971	66.9991	68.9975	0.3145
5	0.3000	0.6000	5.9991	3.9961	66.9964	68.9973	0.3062

Table 3. The problem solutions under different numbers of measurement points

no.	(μ^0); (α^0); (β^0)	$J(\mathbf{P}^0)$	(μ^*); (α^*); (β^*)	$J(\mathbf{P}^*)$
3	(0.1; 0.4; 0.7); (3; 4; 8); (61; 65; 67)	336.46	(0.300; 0.600; 0.899); (5.002; 4.201; 4.002); (66.998; 67.998; 68.998)	0.3456
4	(0.1; 0.5; 0.7; 0.8); (1; 4; 8; 2); (60; 63; 66; 67)	323.64	(0.150; 0.300; 0.600; 0.849); (5.001; 4.102; 4.006; 3.999); (66.996; 67.999; 68.001; 68.999)	0.3549
5	(0.1; 0.2; 0.5; 0.7; 0.8); (3; 5; 7; 8; 3); (61; 63; 64; 66; 67)	368.54	(0.250; 0.300; 0.610; 0.800; 0.896); (5.101; 4.126; 4.106; 4.012; 3.9982); (66.987; 67.979; 68.201; 68.571; 68.989)	0.3436
6	(0.1; 0.2; 0.5; 0.6; 0.7; 0.8); (3; 5; 6; 7; 8; 3); (58; 61; 64; 65; 66; 68)	408.37	(0.208; 0.305; 0.481; 0.605; 0.805; 0.900); (5.003; 4.086; 4.015; 4.013; 3.906; 3.999); (66.997; 67.999; 68.121; 68.571; 68.989; 68.999)	0.3234
7	(0.1; 0.2; 0.3; 0.5; 0.6; 0.7; 0.8); (3; 4; 5; 6; 7; 8; 3); (58; 60; 63; 64; 66; 67; 70)	217.23	(0.198; 0.303; 0.307; 0.491; 0.62; 0.791; 0.901); (0.162; 0.202; 0.198; 0.51; 0.303; 0.363; 0.371); (5.003; 4.086; 4.015; 4.013; 3.906; 3.912; 3.998); (66.998; 68.003; 68.323; 68.772; 68.979; 69.002; 69.012)	0.3023

Table 4. The values of the objective functional and the relative deviations from the desired temperature at the heat exchanger outlet under different noise levels

	$\chi = 0.00$	$\chi = 0.01$	$\chi = 0.03$	$\chi = 0.05$
$\max\limits_{t\in[0,1]} \lvert T(L.t) - V\rvert / \lvert V\rvert$	0.021941	0.033052	0.038311	0.064574
$J^*(\mathbf{P}^*)$	0.3023	0.3543	0.3762	0.3916

Next, Table 2 combines the values of the parameters $(\mathbf{P}^{(6)})^j = (\mu_1^{(6)}, \mu_2^{(6)}, \alpha_1^{(6)}, \alpha_2^{(6)}, \beta_1^{(6)}, \beta_2^{(6)})^j$ and the objective functional $J(\mathbf{P}^6)^j$ obtained at the sixth iteration of the gradient projection method from the initial points $(\mathbf{P}^0)^j$, $j = 1, 2, \ldots, 5$ (Table 1).

According to Table 3, for $N = 6$ and $N = 7$, the minimum values of the objective functionals satisfy condition (5.2); for $N = 7$, the optimal values of the second and third components of the vector μ satisfy condition (5.3) with $\delta = \delta_1 = 0.01$. Hence, the optimal number of measurement points is $N^* = 6$.

In the numerical experiments, the exact measurements $T(\mu_1, t)$, $T(\mu_2, t)$ (the process states observed at the measurement points) were corrupted by random noises:

$$T(\mu_i, t) = T(\mu_i, t)\left(1 + \chi(2\theta_i - 1)\right), \quad i = 1, 2,$$

where θ_i is a random variable with the uniform distribution on $[0, 1]$ and χ specifies the noise level.

Table 4 presents the resulting values of the objective functional and the relative deviations from the desired temperature at the heat exchanger outlet under the noise levels $\chi = 0$ (no noises), 0.01, 0.03, and 0.05 (0%, 1%, 3%, and 5%, respectively).

According to Table 4, the feedback heating process control law for the fluid in the heat exchanger is stable to measurement errors.

7. CONCLUSIONS

This paper has investigated the temperature control design problem for steam supplied to a heat exchanger to heat a fluid in a closed heating system. The problem is described by a linear hyperbolic equation of the first order, and the boundary conditions incorporate a delay due to the circulation time of the fluid in the system.

The temperature of steam supplied to the heat exchanger has been assigned depending on the current temperature values at the measurement points. The original control design problem has been reduced to a parametric optimal control problem with respect to the loaded differential equation. The finite-dimensional feedback parameter vector, consisting of the coefficients of the above dependence, has been optimized. Optimality conditions have been established for the feedback parameters. They contain explicit formulas for the gradient of an objective functional. A test problem has been solved using these formulas, and the results of numerical experiments have been presented.

The proposed approach can be applied to design control laws for other distributed parameter systems.

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APPENDIX

Proof of Theorem 1. Explicit formulas expressing the increment of the objective functional through the increments of its optimized arguments can be obtained using the well-known method [25]. The linear part of the increment of the functional in each argument is the required component of its gradient with respect to the corresponding argument.

First of all, note the following. The initial fluid temperature $\varphi \subset \Phi$ and the heat loss coefficient $\gamma \in \Gamma$ are mutually independent and do not depend on the heating process in the heat exchanger. Hence, from (2.11) and (2.12) it follows that

$$\begin{aligned}\operatorname{grad} J(\mathbf{P}) &= \operatorname{grad} \int\limits_{\Phi} \int\limits_{\Gamma} I(\mathbf{P};\varphi,\gamma)\rho_\Gamma(\gamma)\rho_\Phi(\varphi)d\gamma d\varphi \\ &= \int\limits_{\Phi} \int\limits_{\Gamma} \operatorname{grad} I(\mathbf{P};\varphi,\gamma)\rho_\Gamma(\gamma)\rho_\Phi(\varphi)d\gamma d\varphi,\end{aligned} \tag{A.1}$$

where

$$I(\mathbf{P};\varphi,\gamma) = \int\limits_{t_b}^{t_f} [T(L,t;\mathbf{P},\varphi,\gamma) - V]^{\,2} dt + \varepsilon \|\mathbf{P} - \tilde{\mathbf{P}}\|_{\mathbb{R}^{3N}}.$$

Therefore, the formula for grad $I(\mathbf{P};\varphi,\gamma)$ will be derived under arbitrary admissible feedback parameters $\mathbf{P}$, heat loss coefficient $\gamma\in\Gamma$, and initial condition $T(x,t)=\varphi$, $t\leqslant 0$.

Let $T(x,t;\mathbf{P},\varphi,\gamma)$ be the solution of the loaded initial boundary-value problem (2.12), (2.3), and (2.5) under arbitrarily chosen optimized parameter vector $\mathbf{P}=(\mu,\alpha,\beta)'$, initial condition $\varphi\in\Phi$, and heat loss coefficient $\gamma\in\Gamma$. For brevity, whenever no confusion occurs, the parameters $\mathbf{P}$, φ, γ of the solution $T(x,t;\mathbf{P},\varphi,\gamma)$ will be omitted.

Consider an admissible increment $\Delta\mathbf{P}=(\Delta\mu,\Delta\alpha,\Delta\beta)'$ of the parameters $\mathbf{P}=(\mu,\alpha,\beta)'$ and let $\tilde{T}(x,t)=\tilde{T}(x,t;\tilde{\mathbf{P}},\varphi,\gamma)=T(x,t)+\Delta T(x,t)$ be the solution of problem (2.12), (2.3), and (2.5) that corresponds to the incremental argument vector $\tilde{\mathbf{P}}=\mathbf{P}+\Delta\mathbf{P}$.

Substituting the function $\tilde{T}(x,t)$ into conditions (2.12), (2.3), and (2.5) gives the initial boundary-value problem

$$\begin{aligned}\Delta T_t(x,t)+\vartheta\,\Delta T_x(x,t)=\lambda\sum_{i=1}^{N}\Big[\alpha_i\Delta T(\mu_i,t)+\alpha_i T_x(\mu_i,t)\Delta\mu_i\\+\left(T(\mu_i,t)-\beta_i\right)\Delta\alpha_i-\alpha_i\Delta\beta_i\Big]-\lambda\Delta T(x,t),\quad (x,t)\in\Omega,\end{aligned}\tag{A.2}$$

$$\Delta T(x,0)=0,\quad x\in[0,l],\tag{A.3}$$

$$\Delta T(0,t)=\begin{cases}0, & t\leqslant\tau,\\ (1-\gamma)\Delta T(L,t-\tau), & t\geqslant\tau,\end{cases}\tag{A.4}$$

where the accuracy is within the terms of the first order of smallness with respect to the increment $\Delta T(x,t)$ of the state variable. Formula (A.2) involves the relation

$$T(\mu_i+\Delta\mu_i,t)=T(\mu_i,t)+T_x(\mu_i,t)\Delta\mu_i+o(|\Delta\mu_i|).$$

The increment of the functional (2.12) can be easily represented as

$$\begin{aligned}\Delta I(\mathbf{P};\varphi,\gamma)&=I(\tilde{\mathbf{P}};\varphi,\gamma)-I(\mathbf{P};\varphi,\gamma)=I(\mathbf{P}+\Delta\mathbf{P};\varphi,\gamma)-I(\mathbf{P};\varphi,\gamma)\\&=2\int_{t_b}^{t_f}\left[T(L,t\,;\mathbf{P},\varphi,\gamma)-V\right]\Delta T(L,t)dt+2\sigma\sum_{i=1}^{3N}(\mathbf{P}_i-\tilde{\mathbf{P}}_i)\Delta\mathbf{P}_i,\end{aligned}$$

$$\sum_{i=1}^{3N}(\mathbf{P}_i-\tilde{\mathbf{P}}_i)\Delta\mathbf{P}_i=\sum_{i=1}^{3N}\Big[(\mu_i-\tilde{\mu}_i)\Delta\mu_i+(\alpha_i-\tilde{\alpha}_i)\Delta\alpha_i+(\beta_i-\tilde{\beta}_i)\Delta\beta_i\Big].$$

Let $\psi(x,t;\mathbf{P},\varphi,\gamma)$ be an arbitrary (so far) function that is continuous everywhere in Ω except the points $x=\mu_i$, $i=1,2,\ldots,N$, differentiable with respect to x for $x\in(\mu_i,\mu_{i+1})$, $i=0,1,\ldots,N$, $\mu_0=0$, $\mu_{N+1}=L$, and differentiable with respect to t for $t\in(0,T)$. The arguments $\mathbf{P}$, φ, and γ of the function $\psi(x,t;\mathbf{P},\varphi,\gamma)$ indicate its possible change when varying the feedback parameter vector $\mathbf{P}$, the initial temperature φ, and the heat loss coefficient γ. Whenever possible, $\mathbf{P}$, φ, and γ will be omitted for the function $\psi(x,t;\mathbf{P},\varphi,\gamma)$. Under the accepted assumptions and conditions (A.3) and (A.4), integrating Eq. (A.2) with the factor $\psi(x,t)$ along the rectangle Ω gives

$$\begin{gathered}\int_0^{t_f}\int_0^L \psi(x,t)\Delta T_t(x,t)dxdt + \vartheta\sum_{i=0}^{N}\int_{\mu_i}^{\mu_{i+1}}\int_0^{t_f}\psi(x,t)\Delta T_x(x,t)dtdx \\ -\lambda\int_0^{t_f}\int_0^L \psi(x,t)\sum_{i=1}^{N}\Big[\alpha_i\Delta T(\mu_i,t)+\alpha_i T_x(\mu_i,t)\Delta\mu_i \\ +\left(T(\mu_i,t)-\beta_i\right)\Delta\alpha_i-\alpha_i\Delta\beta_i\Big]\,dx\,dt+\lambda\int_0^{t_f}\int_0^L\psi(x,t)\Delta T(x,t)dx\,dt=0.\end{gathered} \tag{A.5}$$

In view of (A.3)–(A.5), we integrate by parts the first and second terms in (A.5) separately to get

$$\begin{gathered}\int_0^{t_f}\int_0^L \psi(x,t)\Delta T_t(x,t)\,dx\,dt = \int_0^L \psi(x,t_f)\Delta T(x,t_f)dx \\ +\int_0^L\Big[\psi(x,t_b^-)-\psi(x,t_b^+)\Big]\Delta T(x,t_b)\,dx-\int_0^{t_f}\int_0^L\psi_t(x,t)\Delta T(x,t)\,dx\,dt,\end{gathered} \tag{A.6}$$

$$\begin{gathered}\vartheta\sum_{i=0}^{N}\int_{\mu_i}^{\mu_{i+1}}\int_0^{t_f}\psi(x,t)\Delta T_x(x,t)dtdx=\vartheta\int_0^{t_f}\left[\psi(l,t)\Delta T(L,t)-\psi(0,t)\Delta T(0,t)\right]dt \\ +\vartheta\sum_{i=1}^{N}\int_0^{t_f}\Big[\psi(\mu_i^-,t)-\psi(\mu_i^+,t)\Big]\,\Delta T(\mu_i,t)\,dt-\vartheta\int_0^{t_f}\int_0^L\psi_x(x,t)\Delta T(x,t)dxdt \\ =\vartheta\int_0^{t_f}\psi(L,t)\Delta T(L,t)dt-\vartheta(1-\gamma)\int_\tau^{t_f}\psi(0,t)\Delta T(L,t-\tau)dt \\ +a\sum_{i=1}^{N}\int_0^{t_f}\Big[\psi(\mu_i^-,t)-\psi(\mu_i^+,t)\Big]\,\Delta T(\mu_i,t)\,dt-\vartheta\int_0^{t_f}\int_0^L\psi_x(x,t)\Delta T(x,t)dxdt \\ =\vartheta\int_0^{t_f}\psi(L,t)\Delta T(L,t)dt-\vartheta(1-\gamma)\int_0^{t_f-\tau}\psi(0,t+\tau)\Delta T(L,t)dt \\ +\vartheta\sum_{i=1}^{N}\int_0^{t_f}\Big[\psi(\mu_i^-,t)-\psi(\mu_i^+,t)\Big]\,\Delta T(\mu_i,t)\,dt-\vartheta\int_0^{t_f}\int_0^L\psi_x(x,t)\Delta T(x,t)dxdt.\end{gathered} \tag{A.7}$$

In these formulas,

$$\psi(\mu_i^-,t)=\psi(\mu_i-0,t),\quad \psi(\mu_i^+,t)=\psi(\mu_i+0,t).$$

Considering (A.5)–(A.7), the increment of the objective functional takes the form

$$\Delta I = \int_0^L \psi(x,t_f)\Delta T(x,t_f)dx + \int_{t_f-\tau}^{t_f} \left[\vartheta\psi(L,t) + 2(T(L,t)-V)\right]\Delta T(L,t)dt$$
$$+ \int_{t_k}^{t_f-\tau} \left[\vartheta\psi(L,t) + \lambda(1-\gamma)\psi(0,t+\tau) + 2(T(L,t)-V)\right]\Delta T(L,t)dt$$
$$+ \int_0^{t_b} \left[\vartheta\psi(L,t) + \lambda(1-\gamma)\psi(0,t+\tau)\right]\Delta T(L,t)dt$$
$$+ \int_0^{t_f}\int_0^L [-\psi_t(x,t) - \vartheta\psi_x(x,t) + \lambda\psi(x,t)]\Delta T(x,t)dxdt \tag{A.8}$$
$$+ a\sum_{i=1}^N \int_0^{t_f} \left[\psi(\mu_i^-,t) - \psi(\mu_i^+,t) - \frac{\lambda}{\vartheta}\alpha_i \int_0^L \psi(x,t)dx\right] \Delta T(\mu_i,t)\,dt$$
$$- \lambda \int_0^{t_f}\int_0^L \psi(x,t)\sum_{i=1}^N [\alpha_i T_x(\mu_i,t)\Delta\mu_i + (T(\mu_i,t)-\beta_i)\,\Delta\alpha_i - \alpha_i\Delta\beta_i]\,dx\,dt$$
$$+ 2\sigma\sum_{i=1}^N \Big[(\mu_i-\tilde{\mu}_i)\Delta\xi_i + (\alpha_i-\tilde{\alpha}_i)\Delta\alpha_i + (\beta_i-\tilde{\beta}_i)\Delta\beta_i\Big].$$

Since the function $\psi(x,t)$ is arbitrary, let it be the solution of the initial boundary-value problem (3.5)–(3.9) almost everywhere.

Recall that the components of the gradient of the functional are determined by the linear part of its increment with respect to the increments of the corresponding arguments. Consequently,

$$\operatorname{grad}_{\mu_i} I = -\lambda\,\alpha_i \int_0^{t_f} \left(\int_0^L \psi(x,t)dx\right) T_x(\mu_i,t)dt + 2\sigma(\mu_i-\tilde{\mu}_i), \quad i=1,2,\ldots,N, \tag{A.9}$$

$$\operatorname{grad}_{\alpha_i} I = -\lambda \int_0^{t_b} (T(\mu_i,t)-\beta_i)\left(\int_0^L \psi(x,t)dx\right)dt + 2\sigma(\alpha_i-\tilde{\alpha}_i), \quad i=1,2,\ldots,N, \tag{A.10}$$

$$\operatorname{grad}_{\beta_i} I = \lambda\alpha_i \int_0^L \psi(x,t)dx + 2\sigma(\beta_i-\tilde{\beta}_i), \qquad i=1,2,\ldots,N. \tag{A.11}$$

The proof of this theorem is complete.

It remains to obtain the adjoint initial boundary-value problem in the form (3.10) equivalent to (3.5)–(3.9) but without the jump conditions (3.9). Based on the property of the δ-function, the third term in (A.5) can be written as

$$\lambda \int_0^{t_f} \int_0^L \psi(x,t) \sum_{i=1}^{N} \left(\alpha_i \Delta T(\mu_i,t) + \alpha_i T_x(\mu_i,t)\Delta\mu_i + \left(T(\mu_i,t) - \beta_i\right)\Delta\alpha_i - \alpha_i \Delta\beta_i\right) dxdt$$

$$= \lambda \sum_{i=1}^{N} \alpha_i \int_0^{t_f} \int_0^L \int_0^L \psi(\zeta,t)\,\delta(\zeta-\mu_i)\Delta T(\zeta,t)d\zeta dxdt$$

$$+ \lambda \int_0^{t_f} \int_0^L \psi(x,t) \sum_{i=1}^{N} (\alpha_i T_x(\mu_i,t)\Delta\mu_i + (T(\mu_i,t)-\beta_i)\Delta\alpha_i - \alpha_i\Delta\beta_i) dxdt.$$

Now we change the order of integration over ζ and x in the first triple integral and swap the names of these variables to obtain

$$\int_0^{t_f} \int_0^L \int_0^L \psi(\zeta,t)\,\delta(\zeta-\mu_i)\Delta T(\zeta,t)d\zeta dxdt = \int_0^{t_f} \int_0^L \left(\int_0^L \psi(\zeta,t)d\zeta \right) \delta(x-\mu_i)\Delta T(x,t)dxdt. \quad \text{(A.12)}$$

In formula (A.5), we regroup the terms and, using (A.12), finally arrive from (3.5) and (3.8) at the integro-differential Eq. (3.12) for the adjoint problem.

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