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Sequential Improvement Method in Probabilistic Criteria Optimization Problems for Linear-in-State Jump Diffusion Systems

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Abstract—Here we study the problems of probabilistic and quantile optimization of multidimensional controllable jump diffusion. As the main tool we use Chebyshev-type probability estimates. With them the problems under consideration are reduced to one auxiliary deterministic optimal control problem in terms of the moment characteristics of the process. To find its solution, we use Krotov's global improvement method.

Keywords: probabilistic criterion, quantile optimization, jump diffusion, Krotov method

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1. INTRODUCTION

From the applied and theoretical points of view, the following similar problems of probabilistic optimization are both well known: to maximize the probability of a controllable dynamical object getting into a certain area of a given size and to minimize the size of the area that this object hits with a given probability. Quality functionals in such problems are usually called the probability functional and the quantile criterion, respectively [1]. Problems of this kind are naturally arise for the objects that operate under the influence of random external perturbations, the dynamics of which is described by one or another stochastic equation. In this paper, we consider a class of dynamical objects described by a continuous in time mathematical model of a centered jump diffusion. As the target area, we take a neighborhood of the expected (zero) terminal state of the dynamical system. The control is supposed to be a deterministic function of only one argument — time (open-loop control).

The problems of optimizing probabilistic criteria have numerous applications in economics and technology. An extensive review of such applications is given in [1], where the authors also discuss the possibility of using Chebyshev-type deterministic probability estimates as an analytical tool for studying these problems. Some solving algorithms are specified in [1] for finite-dimensional static mathematical models. Discrete-time dynamical models and probabilistic criteria optimization problems for them studied in [2–4]. In the infinite-dimensional case, there are well known sufficient conditions for the optimality of feedback control in diffusion stochastic systems with respect to the probability functional [5]. Similar results for controllable jump diffusions are given in [6]. As for the quantile criterion, some sufficient conditions in the optimization problem were obtained in [7] for the diffusion type model. These papers also present various numerical schemes for finding an approximate solution to the indicated problems of optimizing feedback control. The general necessary conditions for open-loop control optimality (Stochastic Maximum Principle) with respect to the probability functional for the jump diffusion model are also well known. Subject to the distribution density existence they are given as a special case, for example, in [8]. However, very few results are known about constructive algorithms for exact or approximate search for an optimal control functions in such problems. To some extent, these include an approach based on approximation to the density by partial sums of semi-invariant series [9].

In [10, 11], there are proposed some constructive algorithms for solving the optimal open-loop control problems for state-linear diffusion and jump diffusion systems with respect to the linear-quadratic quality functional. Such a functional can be written explicitly in terms of the moment characteristics of the controllable process. The problems considered here differ in that the quality functionals do not have such an explicit expression in the general case. In this regard, the solution to problems in this paper is proposed to be found approximately, in several steps. First, we write out estimates of the probabilistic quality functionals explicitly using the moment characteristics of the controllable random process. Further, based on these estimates, we construct an auxiliary deterministic optimal control problem. The solution to this problem provides an approximation to the solution to the original problems. Then we use the iterative global improvement method sugested by V.F. Krotov [12]. At the last step, the result is analyzed using the mentioned estimates.

The solving scheme formulated above mainly determines the structure of the paper. In the next two sections, the mathematical statement of the problem is formulated and discussed; in the last two sections, a number of meaningful theoretical and practical examples are studied.

2. STATEMENT OF THE OPTIMIZATION PROBLEMS

Consider a controllable dynamical system

$$d\xi(t) = A(t,u(t))\xi(t)dt + \sum_{l=1}^{\nu_1}\Big(B_l(t,u(t))\xi(t) + C_l(t,u(t))\Big)dw_l(t) + \sum_{r=1}^{\nu_2} D_r(t,u(t))\xi(t^-)dp_r(t), \quad \xi(0)=\xi_0, \quad (1)$$

where $t \in [0;T]$ is time; $\xi(t)$ is n-dimensional vector characterizing the state of the system at time t; ξ_0 is a given centered random vector with finite second moment; $u(t)$ is m-dimensional vector of the control function at time t, where $u(t) \in U$ and U is a compact set in $\mathbb{R}^m$, and $t \mapsto u(t)$ is a piecewise continuous function (by $\mathcal{U}$ we denote the set of all such control functions); $w_l(\cdot)$ are standard Wiener processes; $p_r(\cdot)$ are Poisson processes with controllable inhomogeneous jump intensities $\lambda_r(t,u(t))$; the mappings A, B_l, C_l, D_r and λ_r are given and continuous on $[0;T]\times U$, and $\lambda_r(t,u) \geqslant 0$ $\forall (t,u) \in [0;T]\times U$; hereinafter, we use the notation $\xi(t^-) := \lim_{s\to t-0}\xi(s)$, $t \in (0;T]$, $\xi(0^-) := \xi_0$. Initial point ξ_0, Wiener processes $w_l(\cdot)$ and Poisson processes $p_r(\cdot)$ are assumed to be mutually independent.

We will simultaneously study the following two problems with respect to classical probabilistic criteria (see [1, Chapter 2]). Let a vector $\kappa \in \mathbb{R}^n_+$ and a number $\varphi > 0$ be given. Denote by Π_φ the closed parallelepiped in $\mathbb{R}^n$ with sides $2\varphi\kappa_i$, $i=\overline{1,n}$, and center at zero, i.e.,

$$\Pi_\varphi := \left\{x \in \mathbb{R}^n : |x_i| \leqslant \varphi\kappa_i,\ i=\overline{1,n}\right\}.$$

In the first problem, for a given number $\varphi > 0$, it is required to choose a control $u \in \mathcal{U}$ so as to maximize the probability of the random vector $\xi(T)$ hitting the set Π_φ. In other words, we solve the problem of probabilistic optimization

$$P_\varphi(u) := \mathbf{P}\{\xi(T) \in \Pi_\varphi\} \to \sup_{u\in\mathcal{U}}. \quad (2)$$

In the second problem, for a given number $\alpha \in (0;1)$, it is required to choose $u \in \mathcal{U}$ so as to minimize the size φ of the parallelepiped Π_φ that the random vector $\xi(T)$ hits with a probability no less than α. In other words, we solve the quantile optimization problem

$$\varphi_\alpha(u) := \inf\left\{\varphi > 0 : \mathbf{P}\left\{\xi(T) \in \Pi_\varphi\right\} \geqslant \alpha\right\} \to \inf_{u \in \mathcal{U}}. \tag{3}$$

3. LINEAR STOCHASTIC SYSTEMS OF DIFFUSION TYPE

Suppose first that in (1) $B_l = D_r = 0$, i.e., the controllable process is given by the linear Itô equation

$$d\xi(t) = A(t, u(t))\xi(t)dt + C(t, u(t))dw(t), \quad \xi(0) = \xi_0,$$

where $C : [0;T] \times U \to \mathbb{R}^{n\times\nu_1}$, $w(\cdot)$ is ν_1-dimensional standard Wiener process, and suppose, in addition, that the random vector ξ_0 has a normal distribution with zero mean and a positive-definite covariance matrix N_0. As known [13, Theorem 11.7], in this case, for any $t \in [0;T]$, the vector $\xi(t)$ is also normally distributed, has zero mean and positive-definite covariance matrix $N(t)$. It is important that the known distribution allows us for any fixed control $u \in \mathcal{U}$ to explicitely calculate the value of the probability $P_\varphi(u)$. To do this, we can use the distribution density of the vector $\xi(T)$ [13, p. 300], so we obtain

$$P_\varphi(u) = (2\pi)^{-n/2}\left(\det[N(T)]\right)^{-1/2} \int\limits_{\Pi_\varphi} \exp\left\{-\frac{1}{2}\langle y, N(T)^{-1}y\rangle\right\} dy. \tag{4}$$

This way is very attractive for studying linear stochastic systems of diffusion type, since it allows one to solve the extremal problem (2) directly, but it is not suitable for the problems of general form (1)–(3).

4. ESTIMATES OF PROBABILISTIC CRITERIA

Let $u \in \mathcal{U}$ be a control function. Then (1) has the unique strong solution [14, p. 517] on the interval $[0;T]$ with zero expectation and finite second moment. In particular, the covariance matrix $N(T)$ of random vector $\xi(T)$ is well-defined.

Choose some $\varkappa \in \mathbb{R}^n_+$. Assume that all diagonal elements of the matrix $N(T)$ are strictly positive. Then the following estimate is valid, first obtained by Olkin and Pratt in [15]:

$$\mathbf{P}\left\{\max\left\{\frac{|\xi_i(T)|}{\varkappa_i\sqrt{N_{ii}(T)}},\ i = \overline{1,n}\right\} \geqslant 1\right\} \leqslant \frac{\left(\sqrt{\eta} + \sqrt{(n\theta - \eta)(n-1)}\right)^2}{n^2},$$

where

$$\eta = \sum_{i,j=1}^{n} \frac{N_{ij}(T)}{\varkappa_i\varkappa_j\sqrt{N_{ii}(T)}\sqrt{N_{jj}(T)}}, \quad \theta = \sum_{i=1}^{n}\frac{1}{\varkappa_i^2}.$$

Let $\varkappa_i = \varphi\kappa_i/\sqrt{N_{ii}(T)}$, then

$$\mathbf{P}\left\{\max\left\{\frac{|\xi_i(T)|}{\kappa_i},\ i = \overline{1,n}\right\} \geqslant \varphi\right\} \leqslant \frac{\left(\sqrt{\eta} + \sqrt{(n\theta - \eta)(n-1)}\right)^2}{n^2},$$

$$\eta = \frac{1}{\varphi^2}\sum_{i,j=1}^{n}\frac{N_{ij}(T)}{\kappa_i\kappa_j}, \quad \theta = \frac{1}{\varphi^2}\sum_{i=1}^{n}\frac{N_{ii}(T)}{\kappa_i^2}.$$

Note that the latter formula is also meaningful in the case when some of the diagonal elements of $N(T)$ are zero.

From the obtained formula we can derive the following two estimates for the values of $P_\varphi(u)$ and $\varphi_\alpha(u)$:

$$P_\varphi(u) \geqslant P^*_\varphi(u) := \frac{n^2 - \varphi^{-2} f(u)}{n^2}, \tag{5}$$

$$\varphi_\alpha(u) \leqslant \varphi^*_\alpha(u) := \inf\left\{\varphi > 0 : \varphi^{-2} f(u) \leqslant n^2(1-\alpha)\right\} = n^{-1}\sqrt{(1-\alpha)^{-1} f(u)}, \tag{6}$$

where

$$f(u) = \left(\sqrt{\mathrm{tr}\,[QEQN(T)]} + \sqrt{\mathrm{tr}\,[Q\Lambda QN(T)]}\right)^2,$$

$$Q = \mathrm{diag}(\kappa_1^{-1}, \ldots, \kappa_n^{-1}), \quad E = \begin{pmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} (n-1)^2 & 1-n & \ldots & 1-n \\ 1-n & (n-1)^2 & \ddots & 1-n \\ \vdots & \vdots & \ddots & \vdots \\ 1-n & 1-n & \ldots & (n-1)^2 \end{pmatrix}.$$

Consider now two new optimization problems:

$$P^*_\varphi(u) = 1 - n^{-2}\varphi^{-1} f(u) \to \sup_{u \in \mathcal{U}}, \qquad \varphi^*_\alpha(u) = n^{-1}\sqrt{(1-\alpha)^{-1} f(u)} \to \inf_{u \in \mathcal{U}}.$$

Thanks to the estimates (5) and (6), by solving them we can get an approximation to the solution to the original problems (2) and (3). At the same time, as is easy to see, these two problems are equivalent to one problem

$$f(u) = \left(\sqrt{\mathrm{tr}\,[QEQN(T)]} + \sqrt{\mathrm{tr}\,[Q\Lambda QN(T)]}\right)^2 \to \inf_{u \in \mathcal{U}},$$

which lacks φ and α parameters, i.e., its solution, if it exists, is an approximation to the solution to problems (2) and (3) for all $\varphi > 0$ and $\alpha \in (0; 1)$ simultaneously. This is our problem for the further studying. We will discuss one more possible and rather natural approach after the research procedure has been outlined.

Remark 1. As is known [15, Theorem 3.7], the equality in Olkin–Pratt estimate can only be reached in the case of distribution of a very special kind, which the random vector $\xi(T)$ cannot have. It follows that the values $P^*_\varphi(u)$ and $\varphi^*_\alpha(u)$ found as a result of studying the problem $f(u) \to \inf$ may be too rough estimates of the desired values of $P(u)$ and $\varphi_\alpha(u)$. However, it should be kept in mind that the latter does not directly correlate with the quality of the results obtained by the method proposed below. The idea of this paper is based on the assumption that the exact values of the functionals to be optimized and their estimates change simultaneously by changing the argument $u \in \mathcal{U}$. This heuristic hypothesis is tested in the Section 9 with various examples.

5. AUXILIARY DETERMINISTIC OPTIMIZATION PROBLEM

For any $u \in \mathcal{U}$ the covariance matrix function $N(t)$ of the corresponding random process $\xi(t)$ can be found as the solution to a Cauchy problem for some ordinary matrix differential equation.

This equation can be written explicitly, which is done, for example, in [11, 16]. Together with the initial condition, it has the form

$$\begin{aligned}\dot{N}(t) &= A(t,u(t))N(t) + N(t)A(t,u(t))^{\mathrm{T}} \\ &+ \sum_{l=1}^{\nu_1}\left(B_l(t,u(t))N(t)B_l(t,u(t))^{\mathrm{T}} + C_l(t,u(t))C_l(t,u(t))^{\mathrm{T}}\right) \\ &+ \sum_{r=1}^{\nu_2}\lambda_r(t,u(t))\left(D_r(t,u(t))N(t) + N(t)D_r(t,u(t))^{\mathrm{T}}\right. \\ &\qquad\qquad \left. + D_r(t,u(t))N(t)D_r(t,u(t))^{\mathrm{T}}\right), \\ &\qquad\qquad N(0) = \mathbb{E}[\xi_0\xi_0^{\mathrm{T}}].\end{aligned} \tag{7}$$

Let us complete it with the problem obtained in the previous section:

$$J(N(T)) = \mathrm{tr}\left[(QEQ + Q\Lambda Q)N(T)\right] + 2\sqrt{\mathrm{tr}\left[QEQN(T)\right]\mathrm{tr}\left[Q\Lambda QN(T)\right]} \to \inf_{u\in\mathcal{U}}. \tag{8}$$

Note that the problem (7)–(8) is deterministic, given with respect to the terminal (and, generally speaking, non-convex) control quality functional and a dynamical system (7), which is linear in state $N(t)$. It allows natural vector representation

$$\dot{x}(t) = \tilde{A}(t,u(t))x(t) + \tilde{B}(t,u(t)), \quad x(0) = x_0,$$
$$\tilde{J}(x(T)) = \langle q_e + q_\lambda, x(T)\rangle + 2\sqrt{\langle q_e, x(T)\rangle\langle q_\lambda, x(T)\rangle} \to \inf_{u\in\mathcal{U}},$$

where the vectors x and x_0 are obtained by symmetric vectorization of the matrices N and N_0 (symmetric elements outside the main diagonal are included in the components of the corresponding vector only once), and the vectors q_e and q_λ are obtained in a similar way from the matrices QEQ and $Q\Lambda Q$, but the elements outside the main diagonal are doubled, i.e., for example,

$$q_e = \left((QEQ)_{11}, 2(QEQ)_{12}, 2(QEQ)_{13}, \ldots, (QEQ)_{22}, 2(QEQ)_{23}, \ldots, (QEQ)_{nn}\right).$$

The matrix $\tilde{A}$ and the vector $\tilde{B}$ are formed according to the Eq. (7).

To solve such a problem, one can apply V.F. Krotov's global improvement method [12]. In addition to the iterative procedure, it contains some necessary optimality conditions. Let us start with their formulation.

6. NECESSARY CONDITIONS FOR OPTIMALITY

For convenience, we rewrite the problem obtained in the previous sections in more standard notation. Assume that we are considering a controllable dynamical system

$$\dot{x}(t) = A(t,u(t))x(t) + B(t,u(t)), \quad x(0) = x_0 \in C_q \subset \mathbb{R}^n, \tag{9}$$

where $A : [0;T]\times U \to \mathbb{R}^{n\times n}$ and $B : [0;T]\times U \to \mathbb{R}^n$ are known continuous mappings, state vector dimension $n \in \{1, 3, 6, \ldots, k(k+1)/2, \ldots\}$, the initial condition x_0 belongs to the set C_q, which is defined in the following way: an element $x \in C_q$ iff the numbers $\langle q_1, x\rangle$ and $\langle q_2, x\rangle$ are strictly positive unless $|q_1||q_2| = 0$ or $q_1 = -q_2$, where $q_1, q_2 \in \mathbb{R}^n$ are known vectors. In the case $|q_1||q_2| = 0$ we define $C_q = \{x \in \mathbb{R}^n : \langle q_1 + q_2, x\rangle > 0\}$. The case $q_1 = -q_2$ is excluded from the consideration. In accordance with the problem (7)–(8) it is additionally assumed that the mappings A and B

are such that for any $u \in \mathcal{U}$ and for any $t \in [0;T]$ the vector $x(t)$ belongs to the set C_q. The optimization problem has the form

$$J(x(T)) = \langle q_1 + q_2, x(T)\rangle + 2\sqrt{\langle q_1, x(T)\rangle\langle q_2, x(T)\rangle} \to \inf_{u \in \mathcal{U}}. \tag{10}$$

Remark 2. In fact, based on the relations in the problem (7)–(8), the vector $x(t)$ belongs to some fixed set, which is the image of the set of all possible covariance matrices under the vectorization mapping described in the previous section. At the same time, the vectors q_1 and q_2 are by definition such that the inequalities $\langle q_1, x\rangle > 0$, $\langle q_2, x\rangle \geqslant 0$ hold, and $\langle q_2, x\rangle = 0 \Leftrightarrow q_2 = 0 \Leftrightarrow n = 1$. However, for what follows, it will be convenient to expand the domain of the function (10), and consider the vectors q_i to be arbitrarily chosen (but not opposite directed to each other).

Thus, by construction, for any $q_1, q_2 \in \mathbb{R}^n$, $q_1 \neq -q_2$, the function J of the form (10) is well-defined on the set C_q and takes strictly positive values.

Lemma 1. *For any $q_1 \neq -q_2$ the set C_q is a non-empty open convex cone in $\mathbb{R}^n$, and the function $J : C_q \to \mathbb{R}_+$ is differentiable and concave on C_q.*

Proof of Lemma 1. The properties of the set C_q are obvious by construction. It also follows from the definitions and Chain Rule Theorem that the function J is differentiable. Finally, we write the Weierstrass function for J:

$$\mathcal{E}(x, y) = J(x) - J(y) - \langle J'(y), x - y\rangle.$$

Concavity means that $\forall x, y \in C_q \quad \mathcal{E}(x, y) \leqslant 0$. By direct calculation we find

$$\mathcal{E}(x, y) = -\frac{\left(\sqrt{\langle q_1, x\rangle\langle q_2, y\rangle} + \sqrt{\langle q_1, y\rangle\langle q_2, x\rangle}\right)^2}{\sqrt{\langle q_1, y\rangle\langle q_2, y\rangle}},$$

if both vectors q_i are nonzero, and $\mathcal{E}(x, y) \equiv 0$ if one of them is zero. ■

Thanks to this statement, the classical linear implementation of the Krotov method [17, 18] can be applied to the problem (9)–(10). The method is as follows.

Let $\mathcal{AC}^n([0;T])$ denote the space of absolutely continuous n-dimensional vector functions on the segment $[0;T]$ and let $\hat{u} \in \mathcal{U}$ be some arbitrary control function, $\hat{x} \in \mathcal{AC}^n([0;T])$ be the corresponding (unique) solution to the linear Cauchy problem

$$\dot{x}(t) = A(t, \hat{u}(t))x(t) + B(t, \hat{u}(t)), \quad x(0) = x_0. \tag{11}$$

Further, let $\hat{\psi} \in \mathcal{AC}^n([0;T])$ be the solution to the Cauchy problem for the adjoint system

$$\dot{\psi}(t) = -A(t, \hat{u}(t))^{\mathrm{T}}\psi(t), \quad \psi(T) = -J'(\hat{x}(T)). \tag{12}$$

Consider the functions

$$\hat{R}(t, x, u) = \langle \dot{\hat{\psi}}(t), x\rangle + \langle \hat{\psi}(t), A(t, u)x + B(t, u)\rangle, \tag{13}$$

$$\hat{G}(x) = \langle \hat{\psi}(T), x\rangle - \langle \hat{\psi}(0), x_0\rangle + J(x). \tag{14}$$

Here the notations $\hat{R}$ and $\hat{G}$ are related to the fact that these functions are defined by the element $\hat{\psi} \in \mathcal{AC}^n([0;T])$, i.e., eventually, by an arbitrarily chosen control $\hat{u} \in \mathcal{U}$.

Lemma 2. *Let $\hat{u} \in \mathcal{U}$, $\hat{x} \in \mathcal{AC}^n([0;T])$ be the solution to (11), $\hat{\psi} \in \mathcal{AC}^n([0;T])$ be the solution to (12). Then*

$$\hat{R}(t, \hat{x}(t), \hat{u}(t)) = \min_{x \in \mathbb{R}^n} \hat{R}(t, x, \hat{u}(t)) \quad \dot{\forall} t \in [0;T], \tag{15}$$

$$\hat{G}(\hat{x}(T)) = \max_{x \in C_q} \hat{G}(x). \tag{16}$$

Proof of Lemma 2. Due to (12) we have

$$\hat{R}(t, x, \hat{u}(t)) = \langle \hat{\psi}(t), B(t, \hat{u}(t)) \rangle \quad \dot{\forall} t \in [0; T],$$

so the condition (15) is trivially satisfied, while the condition (16) is equivalent to the relation

$$\mathcal{E}(x, \hat{x}(T)) = J(x) - J(\hat{x}(T)) - \langle J'(\hat{x}(T)), x - \hat{x}(T) \rangle \leqslant 0 \quad \forall x \in C_q,$$

which is true for all functions concave on C_q. ■

Theorem 1. *Let $\hat{u} \in \mathcal{U}$, $\hat{x} \in \mathcal{AC}^n([0;T])$ be the solution to* (11), *$\hat{\psi} \in \mathcal{AC}^n([0;T])$ be the solution to* (12) *and $\pi : [0;T] \times \mathbb{R}^n \to \mathbb{R}^m$ be an arbitrary mapping satisfying the conditions:*

1) $\hat{R}(t, x, \pi(t, x)) = \max_{v \in U} \hat{R}(t, x, v)$ $\forall x \in C_q$ $\dot{\forall} t \in [0; T]$;

2) *there exists a solution $\tilde{x} \in \mathcal{AC}^n([0;T])$ to the nonlinear Cauchy problem*

$$\dot{x}(t) = A(t, \pi(t, x(t)))x(t) + B(t, \pi(t, x(t))), \quad x(0) = x_0;$$

3) *there exists $\tilde{u} \in \mathcal{U}$ such that $\tilde{u}(t) = \pi(t, \tilde{x}(t))$ $\dot{\forall} t \in [0; T]$.*

Then $J(\tilde{x}(T)) \leqslant J(\hat{x}(T))$.

Proof of Theorem 1. By Newton–Leibniz formula and the definitions (13), (14) we have

$$J(\tilde{x}(T)) = \hat{G}(\tilde{x}(T)) - \int_0^T \hat{R}(t, \tilde{x}(t), \tilde{u}(t))dt = \hat{G}(\tilde{x}(T)) - \int_0^T \hat{R}(t, \tilde{x}(t), \pi(t, \tilde{x}(t)))dt$$

due to 2) and 3). Then from condition 1) and relations (15), (16)

$$J(\tilde{x}(T)) \leqslant \hat{G}(\tilde{x}(T)) - \int_0^T \hat{R}(t, \tilde{x}(t), \hat{u}(t))dt \leqslant \hat{G}(\hat{x}(T)) - \int_0^T \hat{R}(t, \hat{x}(t), \hat{u}(t)) = J(\hat{x}(T)).$$ ■

Theorem 1 naturally contains necessary optimality conditions for the problem (9)–(10).

Corollary 1. *Let $\hat{u} \in \mathcal{U}$ be an optimal control in the problem* (10), *$\hat{x} \in \mathcal{AC}^n([0;T])$ be the solution to* (11), *$\hat{\psi} \in \mathcal{AC}^n([0;T])$ be the solution to* (12). *Then for any mapping π satisfying conditions* 1)–3) *of Theorem 1, and for $\tilde{x} \in \mathcal{AC}^n([0;T])$ corresponding to π in the sense of these conditions the equality $J(\tilde{x}(T)) = J(\hat{x}(T))$ holds.*

The following simple statement establishes a connection between the obtained result and Maximum Principle.

Corollary 2. *Let $\hat{u}$, $\hat{x}$, $\hat{\psi}$ be taken from Theorem 1. If there exists a mapping π for which conditions* 1)–3) *hold for some $\tilde{u}$, and $\tilde{u}(t) = \hat{u}(t)$ for almost all t, then $\hat{u}$ is a Pontryagin extremal control.*

Remark 3. Non-improvability of a control $\hat{u} \in \mathcal{U}$ in terms of the value J with any $\tilde{u} \in \mathcal{U}$ constructed by Theorem 1, generally speaking, does not mean that the pair $(\hat{x}(\cdot), \hat{u}(\cdot))$ satisfies Maximum Principle [19]. At the same time, the Pontryagin extremal control can turn out to be improvable in the same sense [20].

Remark 4. Under some additional assumptions (for example, when the mapping π and the trajectory $\tilde{x}$ are defined by conditions 1) and 2) of the theorem uniquely), it can be shown that the non-improvability of a control $\hat{u}$ implies that the pair $(\hat{x}(\cdot), \hat{u}(\cdot))$ satisfies Maximum Principle. We will not dwell on this issue in detail here.

The authors thank the reviewer of this paper for the following useful remark.

Remark 5. Within the framework of the problem (7)–(8) both vectors q_i in (10) are nonzero if and only if $n > 1$ (see Remark 2). In accordance with the proof of Lemma 1, in this case the function J is strictly concave on C_q. This implies that the last inequality in the proof of Theorem 1 will be strict for $\tilde{x}(T) \neq \hat{x}(T)$. Thus, when we study the problem (7)–(8) in the case $n > 1$, conditions 1)–3) of Theorem 1 and the additional requirement $\tilde{x}(T) \neq \hat{x}(T)$ guarantees an improvement of the control $\hat{u}$.

7. ITERATIVE GLOBAL IMPROVEMENT METHOD

For applications, the most important corollary of Theorem 1 is the following method of sequential global improvement of a control function $\hat{u} \in \mathcal{U}$.

1) Set $u^{(0)} = \hat{u}$, $k = 0$.

2) Find the solution $x^{(k)}(t)$ to the Cauchy problem

$$\dot{x}(t) = A(t, u^{(k)}(t))x(t) + B(t, u^{(k)}(t)), \quad x(0) = x_0.$$

3) Find the solution $\psi^{(k)}(t)$ to the Cauchy problem

$$\dot{\psi}(t) = -A(t, u^{(k)}(t))^{\mathrm{T}}\psi(t),$$
$$\psi(T) = -\left(1 + \sqrt{\frac{\langle q_2, x^{(k)}(T)\rangle}{\langle q_1, x^{(k)}(T)\rangle}}\right) q_1 - \left(1 + \sqrt{\frac{\langle q_1, x^{(k)}(T)\rangle}{\langle q_2, x^{(k)}(T)\rangle}}\right) q_2.$$

4) Find a feedback control function $\pi^{(k)}(t, x)$ satisfying for all $x \in C_q$ and almost all $t \in [0; T]$ the equation

$$\langle \psi^{(k)}(t), A(t, \pi^{(k)}(t, x))x + B(t, \pi^{(k)}(t, x))\rangle = \max_{v \in U} \langle \psi^{(k)}(t), A(t, v)x + B(t, v)\rangle.$$

5) Find a solution $x^{(k+1)}(t)$ to the nonlinear Cauchy problem

$$\dot{x}(t) = A(t, \pi^{(k)}(t, x(t)))x(t) + B(t, \pi^{(k)}(t, x(t))), \quad x(0) = x_0.$$

6) Check the improvement condition

$$J(x^{(k+1)}(T)) < J(x^{(k)}(T));$$

if there is no improvement, set $\tilde{u} = u^{(k)}$ and finish the calculations.

7) Set $u^{(k+1)}(t) = \pi^{(k)}(t, x^{(k+1)}(t))$.

8) Increment k by one and go to step 3.

Integration of the nonlinear differential equation at step 5 can be done numerically, in parallel with the execution of step 4; in this case, at step 4, each time a finite-dimensional minimization problem is solved and the values of the vector $u^{(k+1)}(t)$ are directly determined; step 7 is not required. For practical purposes, it is natural to replace the improvement condition at step 6 with a stopping condition, for example,

$$|J(x^{(k+1)}(T)) - J(x^{(k)}(T))| < \varepsilon,$$

where the number $\varepsilon > 0$ is chosen at step 1.

8. ON MINIMIZATION OF THE TERMINAL STATE NORM

Let us go back to the original problems (2)–(3). It is quite natural to assume that a convenient approximation to the solution to these problems can be obtained by minimizing the functional

$$\mathcal{J}(N(T)) = \mathrm{tr}[QN(T)]. \tag{17}$$

In particular, if Q is identity matrix (i.e., if in problems (2)–(3) the area Π_φ is square), then the latter means minimizing standard norm of the random vector $\xi(T)$, because in this case

$$\mathcal{J}(N(T)) = \mathrm{tr}N(T) = \mathbb{E}[|\xi(T)|^2] =: ||\xi(T)||^2.$$

It is clear that the optimization problem for the system (7) with the functional (17) is simpler than the problem (7)–(8), since the first of them is completely linear in state. For this problem, the writing and verification of the necessary optimality conditions are correspondingly simplified, albeit slightly, as well as the iterative procedure of Krotov's global improvement. The simplification is that at step 3 of the improvement method, the dual Cauchy problem is solved with the condition $\psi(T) = \mathrm{const}$ known in advance. Moreover, the linear structure of the problem allows us to formulate several alternative iterative global improvement procedures, of dual and "shuttle" type [20], which, generally speaking, are not equivalent to the direct procedure formulated above (see examples in [20, 21]).

However, not always a Krotov improvement in terms of the functional (17) is an improvement in terms of the functional (8). Indeed, consider on the time interval $[0; 1]$ the controllable system

$$d\xi_1(t) = 0, \quad d\xi_2(t) = -u(t)\xi_1(t)dt + u(t)dw(t),$$

for which geometric constraints on the control are given as $0 \leqslant u(t) \leqslant \varepsilon$, where $\varepsilon \in (0; 1)$. Let the vector $\xi(0)$ have a normal distribution with zero expectation and covariance matrix

$$N_0 = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}.$$

We choose the target area Π_φ to be square, i.e., set $\kappa = (1, 1)$.

In terms of the auxiliary problem (9)–(10) we have the following data: $t \in [0; 1]$, $x(t) \in \mathbb{R}^3$, $u(t) \in [0; \varepsilon] \subset \mathbb{R}$,

$$A(t,u) = \begin{pmatrix} 0 & 0 & 0 \\ -u & 0 & 0 \\ 0 & -2u & 0 \end{pmatrix}, \quad B(t,u) = \begin{pmatrix} 0 \\ 0 \\ u^2 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 1 \\ \varepsilon \\ 1 \end{pmatrix},$$

vectors q_i are given as $q_1 = (1, 2, 1)$, $q_2 = (1, -2, 1)$.

Let the control $\hat{u}(t) \equiv 0$ be given. Now we use the Krotov method to improve this control with respect to the functional $\mathcal{J}(x(1)) = x_1(1) + x_3(1)$. To do this, in the formula (14) we replace J with $\mathcal{J}$ and use the constructions from Theorem 1. First of all, we have

$$\hat{x}_1(t) = \hat{x}_3(t) \equiv 1, \quad \hat{x}_2(t) \equiv \varepsilon,$$

therefore,

$$\mathcal{J}(\hat{x}(1)) = 2.$$

Further, since in this case $\hat{\psi}(1) = (-1, 0, -1)$, then

$$\hat{\psi}_1(t) = \hat{\psi}_3(t) \equiv -1, \quad \hat{\psi}_2(t) \equiv 0,$$
$$\langle \hat{\psi}(t), A(t,v)x + B(t,v)\rangle = -v^2 + 2x_2 v.$$

Let us apply Theorem 1. From condition 1) we find

$$\pi(t,x) = \begin{cases} 0, & x_2 \leqslant 0 \\ x_2, & 0 < x_2 \leqslant \varepsilon \\ \varepsilon, & x_2 > \varepsilon, \end{cases}$$

from condition 2)

$$\tilde{x}_1(t) \equiv 1, \quad \tilde{x}_2(t) = \varepsilon e^{-t}, \quad \tilde{x}_3(t) = \frac{\varepsilon^2}{2}\left(e^{-2t} - 1\right) + 1,$$

from condition 3)

$$\tilde{u}(t) = \pi(t, \tilde{x}(t)) = \tilde{x}_2(t) = \varepsilon e^{-t}.$$

Accordingly,

$$\mathcal{J}(\tilde{x}(1)) = 2 - \frac{\varepsilon^2}{2}\left(1 - e^{-2}\right).$$

At the same time

$$J(\hat{x}(1)) = 2\mathcal{J}(\hat{x}(1)) + 2\sqrt{\mathcal{J}(\hat{x}(1))^2 - 4\hat{x}_2(1)^2} = 4 + 4\sqrt{1-\varepsilon^2},$$

$$J(\tilde{x}(1)) = 2 - \varepsilon^2\left(1 - e^{-2}\right) + 2\sqrt{\left(2 - \frac{\varepsilon^2}{2}\left(1 - e^{-2}\right)\right)^2 - 4\varepsilon^2 e^{-2}}.$$

Note that for any $\varepsilon \in (0;1)$, the pair of inequalities

$$\mathcal{J}(\hat{x}(1)) > \mathcal{J}(\tilde{x}(1)), \quad J(\hat{x}(1)) < J(\tilde{x}(1))$$

holds. In particular, this is true for values of ε close to 1. Moreover, for such values of the parameter ε, numerical experiments show that both inequalities remain valid for an arbitrarily large number of repeated iterations.

Observe that the stochastic system under consideration is linear, so the random vectors $\hat{\xi}(1)$ and $\tilde{\xi}(1)$ corresponding to the controls $\hat{u}$ and $\tilde{u}$ have a normal distribution together with $\xi(0)$, and their covariance matrices are, respectively,

$$\hat{N}(1) = \begin{pmatrix} \hat{x}_1(1) & \hat{x}_2(1) \\ \hat{x}_2(1) & \hat{x}_3(1) \end{pmatrix}, \quad \tilde{N}(1) = \begin{pmatrix} \tilde{x}_1(1) & \tilde{x}_2(1) \\ \tilde{x}_2(1) & \tilde{x}_3(1) \end{pmatrix}.$$

For $\varepsilon = 0.9999$ using the formula (4) we find $P_1(\hat{u}) \approx 0.68$ and $P_1(\tilde{u}) \approx 0.58$. So, despite the fact of decreasing of the terminal state norm, the probability of hitting the target area is also decreased by as much as 10%.

Thus, there are problems for which a Krotov improvement of a control function in terms of the terminal state norm is a disimprovement not only in terms of estimates (5)–(6), but also in terms of solving the original problems (2)–(3). At the same time, the approach proposed in this paper works correctly in such situation: estimates (5)–(6) cannot be worsened. Nevertheless, when it is used in the general case, the quality of control can also be decreased with respect to the desired probability. But due to estimates (5)–(6) we can expect that if this happens, the disimprovement will not be so significant (see Example 2 below).

9. EXAMPLES

Example 1. On the time interval $[0;T]$ consider the system

$$d\xi_1(t) = (u(t)\xi_1(t) - \xi_2(t))dt + (\xi_2(t) - \xi_1(t))dw(t), \quad \xi_1(0) \sim \mathcal{N}(0,1),$$
$$d\xi_2(t) = (\xi_1(t) + u(t)\xi_2(t))dt + (\xi_1(t) + \xi_2(t))dw(t), \quad \xi_2(0) \sim \mathcal{N}(0,1),$$

the random variables $\xi_1(0)$ and $\xi_2(0)$ are independent. The geometric constraints on the control are given as $|u(t)| \leqslant u_{\max}$. We choose the target area Π_φ to be square, i.e., set $\kappa = (1, 1)$.

In terms of the auxiliary problem (9)–(10) we have the following data: $t \in [0; T]$, $x(t) \in \mathbb{R}^3$, $u(t) \in [-u_{\max}; u_{\max}] \subset \mathbb{R}$, $B(t, u) \equiv 0$,

$$A(t,u) = \begin{pmatrix} 2u+1 & -4 & 1 \\ 0 & 2u & 0 \\ 1 & 4 & 2u+1 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

vectors q_i are given as $q_1 = (1, 2, 1)$, $q_2 = (1, -2, 1)$.

We note at once that, by virtue of the Eq. (9), for any $u \in \mathcal{U}$ the second component of the vector $x(t)$ (the mixed covariance of the components of the vector $\xi(t)$) is identically equal to zero. Hence it follows that

$$\langle q_1, x(T)\rangle = \langle q_2, x(T)\rangle = \langle q_0, x(T)\rangle, \quad q_0 = (1, 0, 1),$$

that is

$$J(x(T)) = 4\langle q_0, x(T)\rangle = 4\mathcal{J}(x(T))$$

and the problem (9)–(10) is completely equivalent to the problem (9), (17) of minimizing the norm of the random vector $\xi(T)$. For convenience, we will further work with the functional $\mathcal{J}$.

Let the control $\hat{u}(t) \equiv 0$ be given. We have

$$\hat{x}_1(t) = \hat{x}_3(t) = e^{2t}, \quad \hat{x}_2(t) \equiv 0 \quad \Rightarrow \quad \mathcal{J}(\hat{x}(T)) = 2e^{2T}.$$

Since the functional $\mathcal{J}$ is linear and $\hat{\psi}(T) = -q_0$, we have

$$\hat{\psi}_1(t) = \hat{\psi}_3(t) = -e^{2(T-t)}, \quad \hat{\psi}_2(t) \equiv 0,$$
$$\langle \hat{\psi}(t), A(t,v)x + B(t,v)\rangle = -2e^{2(T-t)}(v+1)(x_1 + x_3), \quad x_1 + x_3 = \langle q_0, x\rangle > 0.$$

Therefore, from condition 1) of Theorem 1 $\pi(t, x) \equiv -u_{\max}$. In fact, as it is easy to check, in the problem under consideration for any $u \in \mathcal{U}$

$$\mathcal{J}(x(T)) = 2e^{2T} \exp\left\{2\int_0^T u(t)dt\right\},$$

therefore $\tilde{u}(t) \equiv -u_{\max}$ delivers an absolute optimum to the functional $\mathcal{J}$. We have shown that this optimum is found by Krotov global improvement method in exactly one iteration.

Let, for definiteness, $u_{\max} = 1$, $T = 1$, then $\tilde{u}(t) \equiv -1$ and $J(\tilde{x}(T)) = 4\mathcal{J}(\tilde{x}(T)) = 8$. Substituting this value instead of $f(u)$ into the formulas (5)–(6) for $n = 2$, we find the following estimates for the values of the functionals (2) and (3):

$$P_\varphi(\tilde{u}) \geqslant 1 - 2\varphi^{-2}, \quad \varphi_\alpha(\tilde{u}) \leqslant \sqrt{2(1-\alpha)^{-1}}.$$

Example 2. Let us now complete the study of the special example constructed in the previous section. Recall the statement of the problem: on the time interval $[0; 1]$ the controllable system is given as

$$d\xi_1(t) = 0, \quad d\xi_2(t) = -u(t)\xi_1(t)dt + u(t)dw(t),$$
$$\xi(0) \sim \mathcal{N}(0, N_0), \quad N_0 = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix},$$

geometric control constraints are $0 \leqslant u(t) \leqslant \varepsilon$, $\varepsilon \in (0;1)$, target area Π_φ is square, $\kappa = (1,1)$. Data for the auxiliary problem (9)–(10): $t \in [0;1]$, $x(t) \in \mathbb{R}^3$, $u(t) \in [0;\varepsilon] \subset \mathbb{R}$,

$$A(t,u) = \begin{pmatrix} 0 & 0 & 0 \\ -u & 0 & 0 \\ 0 & -2u & 0 \end{pmatrix}, \quad B(t,u) = \begin{pmatrix} 0 \\ 0 \\ u^2 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 1 \\ \varepsilon \\ 1 \end{pmatrix},$$

the vectors q_i are the same as in Example 1.

Let the control $\hat{u}(t) \equiv 0$ be given. We have

$$\hat{x}_1(t) = \hat{x}_3(t) \equiv 1, \quad \hat{x}_2(t) \equiv \varepsilon \quad \Rightarrow \quad J(\hat{x}(1)) = 4 + 4\sqrt{1-\varepsilon^2}.$$

By virtue of

$$\hat{\psi}(1) = -J'(\hat{x}(T)) = -\left(1 + \sqrt{\frac{\langle q_2, \hat{x}(T)\rangle}{\langle q_1, \hat{x}(T)\rangle}}\right) q_1 - \left(1 + \sqrt{\frac{\langle q_1, \hat{x}(T)\rangle}{\langle q_2, \hat{x}(T)\rangle}}\right) q_2,$$

we get

$$\hat{\psi}_1(t) = \hat{\psi}_3(t) \equiv -\beta_1, \quad \hat{\psi}_2(t) \equiv -\beta_2,$$
$$\langle \hat{\psi}(t), A(t,v)x + B(t,v)\rangle = -\beta_1 v^2 + (\beta_2 x_1 + 2\beta_1 x_2)v,$$

where

$$\beta_1 = 2\left(1 + \frac{1}{\sqrt{1-\varepsilon^2}}\right), \quad \beta_2 = -\frac{4\varepsilon}{\sqrt{1-\varepsilon^2}}.$$

From condition 1) of Theorem 1 we find

$$\pi(t,x) = \begin{cases} 0, & \beta_2 x_1 + 2\beta_1 x_2 \leqslant 0 \\ \beta_2 x_1 + 2\beta_1 x_2, & 0 < \beta_2 x_1 + 2\beta_1 x_2 \leqslant \varepsilon \\ \varepsilon, & \beta_2 x_1 + 2\beta_1 x_2 > \varepsilon. \end{cases}$$

Since for any $\varepsilon \in (0;1)$

$$\pi(0,x_0) = \beta_2 + 2\beta_1\varepsilon = 4\varepsilon > \varepsilon$$

and

$$\beta_2 + 2\beta_1\varepsilon(1-t^*) = \varepsilon$$

for

$$t^* = \frac{3}{4}\left(1 + \frac{1}{\sqrt{1-\varepsilon^2}}\right)^{-1} \in (0;1),$$

the new control $\tilde{u}$ will have switches starting at time t^*.

Let $\varepsilon = 0.9999$. Then $J(\hat{x}(1)) \approx 4.057$, and for the new control $\tilde{u}$ and the corresponding trajectory $\tilde{x}$, based on the numerical calculation, we find $J(\tilde{x}(1)) \approx 4.041$. Due to the estimate (5), this only means that $P_\varphi(\tilde{u}) \geqslant P_\varphi^*(\tilde{u}) \approx 1 - \varphi^{-1}$, so $P_1(\tilde{u}) \geqslant 0$. But in reality, the covariance matrix for the new solution has the form

$$\tilde{N}(1) = \begin{pmatrix} \tilde{x}_1(1) & \tilde{x}_2(1) \\ \tilde{x}_2(1) & \tilde{x}_3(1) \end{pmatrix} \approx \begin{pmatrix} 1 & 0.991 \\ 0.991 & 0.983 \end{pmatrix},$$

whence, thanks to the formula (4),

$$P_1(\tilde{u}) \approx 0.681 > 0.68 \approx P_1(\hat{u}).$$

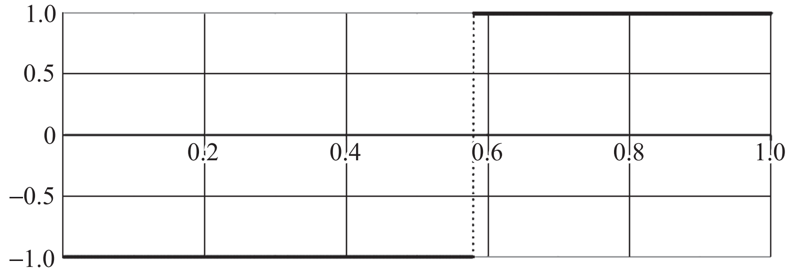

Fig. 1. Function $\tilde{u}_1(t)$ in Example 3.

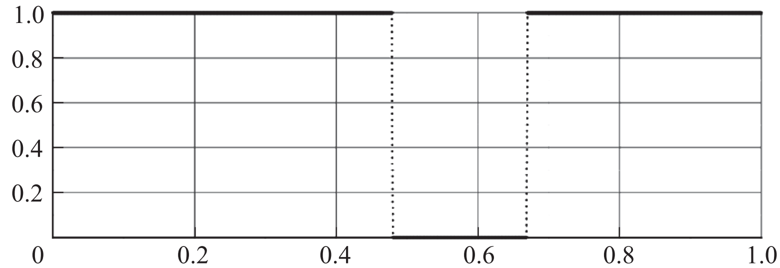

Fig. 2. Function $\tilde{u}_2(t)$ in Example 3.

Thus, the Krotov method, built on the basis of estimates (5)–(6), allows even in such a specially selected "bad" example to increase, albeit slightly, the probability of hitting the target area, with that the estimates themselves do not contain any useful information. The effect of repeated iterations on the result turns out to be insignificant here.

Example 3. On the time interval $[0;1]$ consider the system

$$d\xi_1(t) = \xi_2(t)dt + \xi_2(t^-)dp(t), \quad \xi_1(0) \sim \mathcal{N}(0,1),$$
$$d\xi_2(t) = u_1(t)\xi_1(t)dt, \quad \xi_2(0) = 0,$$

where the Poisson process $p(\cdot)$ has intensity $\lambda(t,u(t)) = u_2(t)$. Geometric control constraints are given as $|u_1(t)| \leqslant 1$, $0 \leqslant u_2(t) \leqslant 1$. Target area Π_φ is square, $\kappa = (1,1)$.

In terms of the auxiliary problem (9)–(10) we have the following data: $t \in [0;1]$, $x(t) \in \mathbb{R}^3$, $u(t) \in [-1;1] \times [0;1] \subset \mathbb{R}^2$, $B(t,u) \equiv 0$,

$$A(t,u) = \begin{pmatrix} 0 & 2+2u_2 & u_2 \\ u_1 & 0 & 1+u_2 \\ 0 & 2u_1 & 0 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The vectors q_i are the same as in Examples 1 and 2.

Let the control $\hat{u}_1(t) = \hat{u}_2(t) \equiv 0$ be given. The value of the auxiliary quality functional (10) on it is 4. Let us perform further calculations numerically, following the iterative procedure described in Section 7. As a result, after ten iterations, a new control $\tilde{u}(t)$ will be found (see Figs. 1 and 2), on which the functional is approximately equal to 1.28. Now, using the estimates (5) and (6), we can state that with the control $\tilde{u}(t)$ the probability of the random vector $\tilde{\xi}(T)$ hitting the square Π_1 is not less than 68%, and if it is required to guarantee the probability of hitting the given square area not less than $\alpha = 0.85$, then the square Π_1 should be increased in size by at least $\varphi = 1.46$ times. If we replace the functional (10) with a functional (17) and apply the method, then another control program $u^*(t)$ will be found, only slightly different (upward) from $\tilde{u}(t)$ by value (10). At the same

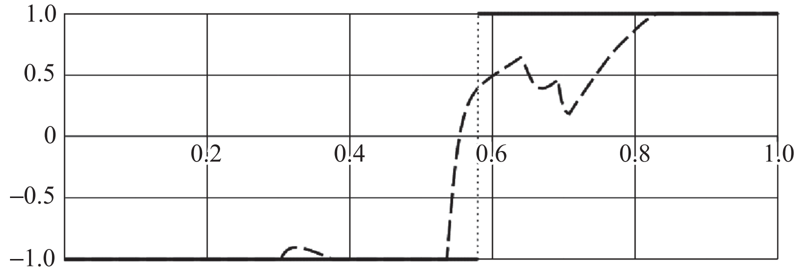

Fig. 3. Comparison of the first components of the vector function $\tilde{u}(t)$ in Examples 3 and 4.

time, at each iteration, along with the decrease in the values (17), the values (10) are also decrease. In order to more accurately analyze the quality of the programs found, we numerically simulate a certain number of implementations of the random process $\xi(t)$ and estimate the frequency of $\xi(1)$ hits the target area Π_φ for its different sizes. The results are presented in the table. For comparison, the second column shows the exact values of $P_\varphi(\hat{u})$, calculated by the formula (4).

Statistics of hits in the Π_φ region based on simulation results

φ	$P_\varphi(\hat{u})$	$\hat{u}(t) \equiv 0$	$u(t) = \tilde{u}(t)$	$u(t) = u^*(t)$
1	0.68	74 out of 100	90 out of 100	95 out of 100
1.5	0.87	90 out of 100	99 out of 100	100 out of 100
2.6	0.99	99 out of 100	100 out of 100	100 out of 100

Thus, within the framework of the considered example (as well as in the case of Example 1), the problem (9)–(10) can be changed to the simpler linear problem (9), (17) without losing the quality of the solution to the original pair of problems (1)–(3).

Example 4. Suppose that the control $u_1(t)$ in the system from the previous example is implemented with a multiplicative type error. To do this, on the time interval $[0;1]$ consider the system

$$d\xi_1(t) = \xi_2(t)dt + \xi_2(t^-)dp(t), \quad \xi_1(0) \sim \mathcal{N}(0,1),$$
$$d\xi_2(t) = u_1(t)\xi_1(t)dt + \varepsilon u_1(t)\xi_1(t)dw(t), \quad \xi_2(0) = 0,$$

where the Poisson process $p(\cdot)$ has the same intensity $\lambda(t, u(t)) = u_2(t)$. The geometric constraints are not changed and are given as $|u_1(t)| \leqslant 1$, $0 \leqslant u_2(t) \leqslant 1$. The target area Π_φ is square, $\kappa = (1,1)$. The data for the auxiliary problem (9)–(10) differ from Example 3 only by one element of the matrix $A(t,u)$, namely, $A_{31}(t,u) = \varepsilon^2 u_1^2$, where we take ε equal to 0.1.

Repeating the calculations of Example 3 for the same initial approximation $\hat{u}_1(t) = \hat{u}_2(t) \equiv 0$, at the tenth iteration we obtain a new control $\tilde{u}(t)$, the first component of which, in contrast to that shown in Fig. 1, becomes continuous, and the second does not differ from that shown in Fig. 2. At the same time, substituting the control $\tilde{u}$ from Example 3 into the perturbed system considered here gives the value of the estimated functional (10) approximately 1.4, while on the new $\tilde{u}$ it has a smaller value 1.39. The comparison of the $\tilde{u}_1(t)$ functions obtained in this and previous examples is shown in Fig. 3.

Example 5. Consider the problem of stabilizing the height of an aircraft from [22]: on the interval $[0;T]$ we have the system

$$dH(t) = V(t)dt,$$
$$dV(t) = (u_1(t)H(t) + u_2(t)V(t))\,dt + \epsilon\,(u_1(t)H(t) + u_2(t)V(t))\,dw_1(t) + c\,dw_2(t)$$

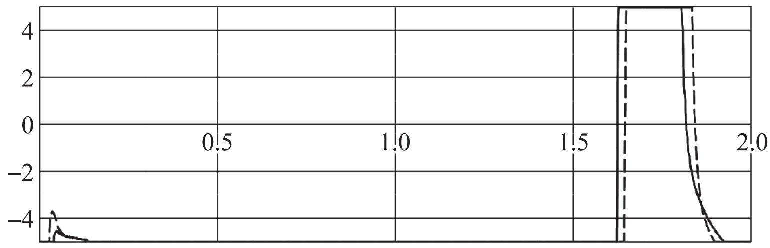

Fig. 4. Comparison of the first components of found controls in Example 5.

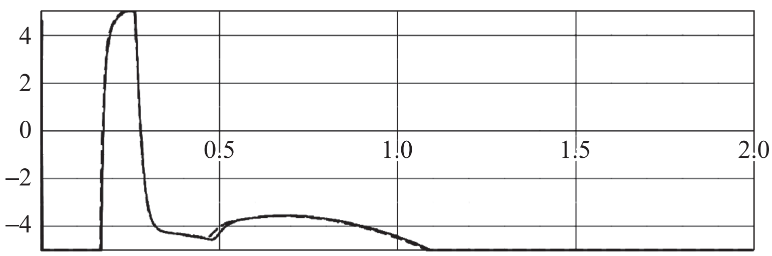

Fig. 5. Comparison of the second components of found controls in Example 5.

with initial conditions

$$H(0) \sim \mathcal{N}(0,1), \quad V(0) = 0,$$

where H and V are deviations of height and vertical speed from the given values, ϵ is an error factor of control implementation, c is the parameter of wind force. The geometric constraints on the control characterize the technical possibilities for amplification in the linear feedback controller and are given in form $|u_i(t)| \leqslant u_{\max}$. At the terminal time T, it is required to maximize the probability of finding deviations in height and vertical speed within the specified errors. We will assume that the exact speed is twice as important as the exact height. In other words, we solve the problem of getting into the rectangular target area Π_φ for $\kappa = (2,1)$ with the highest probability. Problems of this kind naturally arise in practice, for example, when refueling aircraft in the air or when docking spacecraft.

In terms of the auxiliary problem (9)–(10) we have the following data: $t \in [0;T]$, $x(t) \in \mathbb{R}^3$, $u(t) \in [-u_{\max}; u_{\max}]^2 \subset \mathbb{R}^2$,

$$A(t,u) = \begin{pmatrix} 0 & 2 & 0 \\ u_1 & u_2 & 1 \\ \epsilon^2 u_1^2 & 2u_1 + 2\epsilon^2 u_1 u_2 & 2u_2 + \epsilon^2 u_2^2 \end{pmatrix}, \quad B(t,u) = \begin{pmatrix} 0 \\ 0 \\ c^2 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

vectors q_i are given as $q_1 = (0.25, 1, 1)$, $q_2 = (0.25, -1, 1)$. For definiteness, we set $\epsilon = 0.1$, $c = 1$, $u_{\max} = 5$, $T = 2$.

Let the control $\hat{u}_1(t) = \hat{u}_2(t) \equiv 0$ be given. The value of the auxiliary quality functional (10) on it is approximately equal to 10. Using the global improvement method numerically, we construct a new control $\tilde{u}$ with the value of the functional (10) approximately 0.43. As before, for comparison, we construct another control u^* by applying the method to the functional (17). The value of the functional (10) on it is approximately the same. Plots of both components are presented for comparison in Figs. 4 and 5 (correspondence of plots to variants is not essential).

As last thing, we note that in the analyzed examples the Krotov method already in one iteration finds a control funtion with a value of the functional J that is significantly less than the initial one, further iterations make only minor adjustments. A similar situation was noted earlier in fully linear-in-state optimal control problems [12, 17, 18].

10. CONCLUSION

Thus, the problems of probabilistic and quantile optimization of multidimensional controlled jump diffusion were studied. Thanks to Chebyshev-type multidimensional estimates, the considered problems were reduced to one auxiliary deterministic optimal contol problem in terms of the moment characteristics of the random process. To solve this problem, the method of successive global improvements by V.F. Krotov was applied. The effectiveness of the approach was demonstrated on a number of examples.

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