ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2023, Vol. 84, No. 6, pp. 692–708.
Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, 2023.
Russian Text © The Author(s), 2023, published in Avtomatika i Telemekhanika, 2023, No. 6, pp. 79–99.

STOCHASTIC SYSTEMS

Iterative Learning Control of Stochastic Multi-Agent Systems with Variable Reference Trajectory and Topology

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Received November 22, 2022
Revised February 7, 2023
Accepted March 30, 2023

Abstract—In modern smart manufacturing, robots are often connected via a network, and their task can change according to a predetermined program. Iterative learning control (ILC) is widely used for robots executing high-precision operations. Under network conditions, the efficiency of ILC algorithms may decrease if the program is restructured. In particular, the learning error may temporarily increase to an unacceptable value when changing the reference trajectory. This paper considers a networked system with the following features: the reference trajectory and parameters change between passes according to a known program, agents are subjected to random disturbances, and measurements are carried out with noise. In addition, the network topology changes due to the disconnection of some agents from the network and the connection of new agents to the network according to a given program. A distributed ILC design method is proposed based on vector Lyapunov functions for repetitive processes in combination with Kalman filtering. This method ensures the convergence of the learning error and reduces its increase caused by changes in the reference trajectory and network topology. The effectiveness of the proposed method is confirmed by an example.

Keywords: iterative learning control, multi-agent system, variable topology, random disturbances, repetitive processes, stability, stabilization, vector Lyapunov function, linear matrix inequalities

DOI: 10.25728/arcRAS.2023.53.81.001

1. INTRODUCTION

Smart manufacturing (SM) is a new paradigm of modern industry, often referred to as the Fourth Industrial Revolution (Industry 4.0). SM systems integrate the physical components of production with cyberspace. These systems have emerged through developing and using information technologies such as the Industrial Internet-of-Things (IIoT), artificial intelligence (AI), and cloud computing, combined with the significantly increased performance of modern computers. The creation of intelligent industrial processes has been a powerful driver in the development of machine learning (ML) and networked multi-agent systems. ML and networked structures provide flexible adaptation to today's markets, which are characterized by short lead times, tight tolerances on product parameters, cost constraints, frequent changes in demand, and permanent improvements in technology; see [1, 2] and references therein.

For dynamical systems in engineering, the concept of ML was introduced by Ya.Z. Tsypkin [3] back in the early 1960s as the process of developing in a certain system one or another response to external signals through multiple impacts on the system and external corrections. Iterative learning control (ILC) perfectly fits into this concept: being intended for repetitive processes, it is corrected on each repetition (also termed trial or pass) based on information from previous repetitions. This approach allows consistently improving quality indicators, e.g., the tracking accuracy of a reference trajectory. Since the pioneering study [4], ILC has become an actively evolving area of research with numerous applications, primarily in robotics. The surveys [5, 6] can be recommended as starting points for the literature.

In the standard statement of the ILC problem, the reference trajectory does not change between passes and the system always returns to the same initial state after each pass. These assumptions restrict the capabilities of ILC in many SM applications. In modern SM systems, reference trajectories and control objectives can change by certain rules, so control should be appropriately reconfigured. Changing the reference trajectory generates a transient error that often reduces the accuracy below an acceptable level for several passes. Therefore, it is necessary to develop new ILC algorithms to compensate the transient error. The paper [2] considered a control reconfiguration scenario under a preset change in the reference trajectory between passes. A stochastic version of the same scenario was studied in [7, 8]. In the cited works, the transient error was compensated by ILC algorithms based on appropriate switching rules.

Since the publication of [9], distributed (networked) ILC laws have been considered by several authors; see [10–15] and references therein. Good surveys can be found in [16] and [15, 17, 18]. An analysis of the problem's state-of-the-art shows the following: the vast majority of studies on distributed ILC for linear systems involved the Arimoto algorithm or its discrete counterpart and the supervector method [19] as the mathematical apparatus. For nonlinear systems, the same Arimoto algorithm was combined with a priori estimation techniques, which gave conservative results. As a rule, the degree of conservatism of the results cited above cannot be estimated: the purely illustrative examples do not reflect possible applications.

The work [20] was one of the first where the effectiveness of the proposed networked ILC algorithms was confirmed by both simulations and experiments with a group of quadrotors. The results of [20] were further developed in [16]. In [21], the effectiveness of the proposed network ILC algorithms was also confirmed by simulations and experiments with a group of mobile robots. Note that discrete-time models with discrete versions of Arimoto algorithms were used there.

An ILC design for multi-agent systems based on 2D discrete models (linear repetitive processes) was proposed in [17]. The computational complexity of the design procedure was reduced by establishing a 2D counterpart of the Fax–Murray theorem, well known in the theory of networked systems. The approach of [17] was extended in [18] to stochastic multi-agent systems. The accuracy and the rate of convergence of the tracking error were significantly improved compared to known algorithms; the theoretical results were clearly confirmed by examples. The paper [22] considered an ILC design problem for uncertain multi-agent systems in the deterministic statement with a variable network topology. A special ILC switching law was developed to reduce the transient error due to a topology change. In [23], an ILC design problem was proposed for stochastic multi-agent systems with a variable reference trajectory and a fixed network topology.

This paper differs from the known works: for the first time in the literature, a distributed ILC law is designed for a networked multi-agent system in the stochastic statement where the reference trajectory and the network topology change on a given finite interval along the passes according to a known program.

2. PROBLEM STATEMENT

Consider a networked system of N linear dynamical subsystems (agents) that operate in a repeated mode (i.e., execute the same operation over and over again). The network topology may change over time. The dynamics of agent i on pass (iteration or trial) k are described by the discrete state-space model

$$x_i(k,p+1) = A_{\sigma_i(k)}x_i(k,p) + B_{\sigma_i(k)}u_i(k,p) + D_{\sigma_i(k)}\omega_i(k,p), \tag{2.1}$$

$$y_i(k,p) = Cx_i(k,p), \tag{2.2}$$

$$y_{\nu i}(k,p) = y_i(k,p) + G_{\sigma_i(k)}\nu_i(k,p), \quad i \in \mathcal{I}, \quad k \geqslant 0, \quad 0 \leqslant p \leqslant T-1, \tag{2.3}$$

with the following notations: p is the discrete time on pass k, T is the same pass length for all k, and $\mathcal{I} = \{1, 2, \ldots, N\}$ is the set of all agents; at a time instant p on pass k, $x_i(k,p) \in \mathbb{R}^{n_x}$ is the state vector, $u_i(k,p) \in \mathbb{R}^1$ is the scalar control action, and $\omega_i(k,p) \in \mathbb{R}^{n_\omega}$ is the vector of external disturbances affecting agent $i \in \mathcal{I} = \{1, 2, \ldots, N\}$ (plant); in addition, $y_i(k,p) \in \mathbb{R}^1$ is the unobserved scalar output (pass profile), $y_{\nu i}(k,p) \in \mathbb{R}^1$ is the observed (measured) output, and $\nu_i(k,p) \in \mathbb{R}^1$ is the measurement noise. The initial conditions $x_i(k,0)$ and $u_i(0,p)$ are identical for all agents.

By assumption, the disturbances $\omega_i(k,p)$ and the measurement noises $\nu_i(k,p)$ are independent Gaussian white noises with zero mean and the covariances

$$S_{i\omega} = \mathrm{E}\left[\omega_i(k,p)\,\omega_i^\top(k,p)\right],$$
$$S_{i\nu} = \mathrm{E}\left[\nu_i(k,p)^2\right],$$

where E denotes the expectation operator. Suppose also that $\omega_i(k,p)$ is independent of the initial state vector.

The mode-switching signal $\sigma_i(k)$ for agent i is a piecewise constant function that maps $\mathbb{Z}_+$ into $\{1, \ldots, m\}$, where m is the number of possible modes. The discontinuity points of this function will be called *mode switch instants*. Each mode has a particular reference trajectory (output) $y^{ref}_{\sigma_i(k)}(p)$ and particular matrices $A_{\sigma_i(k)} \in \{A_1, \ldots, A_m\}$, $B_{\sigma_i(k)} \in \{B_1, \ldots, B_m\}$, $D_{\sigma_i(k)} \in \{D_1, \ldots, D_m\}$, and $G_{\sigma_i(k)} \in \{G_1, \ldots, G_m\}$ of compatible dimensions. The triples $\left(A_{\sigma_i(k)}, B_{\sigma_i(k)}, C\right)$ are completely controllable and observable, and $CB_{\sigma_i(k)} \neq 0$.

The topology-switching signal $\rho(k)$ for the network is a piecewise constant function that maps $\mathbb{Z}_+$ into $\{1, \ldots, c\}$, where c is the number of possible topologies. The discontinuity points of this function will be called *topology switch instants*. Each topology is defined by the set of operating agents $\mathcal{I}_{\rho(k)} = \{i_n\}_{n=1}^{N_{\rho(k)}} \subseteq \mathcal{I}$, where $N_{\rho(k)} \leqslant N$ is the number of operating agents in a topology $\rho(k)$, and by their connections represented as a directed graph $\mathcal{G}_{\rho(k)} = \left(\mathcal{I}_{\rho(k)}, \mathcal{E}_{\rho(k)}\right)$, where $\mathcal{E}_{\rho(k)} \subseteq \mathcal{I}_{\rho(k)} \times \mathcal{I}_{\rho(k)}$ are graph edges. The ability of agent i to receive information from agent j $(i, j \in \mathcal{I}_{\rho(k)})$ is determined by a directed edge from vertex j to vertex i and denoted by an ordered pair $(j,i) \in \mathcal{E}_{\rho(k)}$. The elements of the adjacency matrix

$$\mathcal{S}_\rho(\mathcal{G}_\rho) = \begin{bmatrix} s_{i_1 i_1} & s_{i_1 i_2} & \cdots & s_{i_1 i_{N_\rho}} \\ s_{i_2 i_1} & s_{i_2 i_2} & \cdots & s_{i_2 i_{N_\rho}} \\ \vdots & \vdots & \ddots & \vdots \\ s_{i_{N_\rho} i_1} & s_{i_{N_\rho} i_2} & \cdots & s_{i_{N_\rho} i_{N_\rho}} \end{bmatrix}, \quad \rho = \rho(k),$$

are defined as follows: $s_{ij} > 0$ for $(j,i) \in \mathcal{E}_{\rho(k)}$, and $s_{ij} = 0$ otherwise $(s_{ii} = 0)$. The Laplacian matrix of the graph $\mathcal{G}_{\rho(k)}$ is given by

$$\mathcal{L}_\rho(\mathcal{G}_\rho) = \begin{bmatrix} \sum\limits_{j \in \mathcal{I}_\rho} s_{i_1 j} & -s_{i_1 i_2} & \cdots & -s_{i_1 i_{N_\rho}} \\ -s_{i_2 i_1} & \sum\limits_{j \in \mathcal{I}_\rho} s_{i_2 j} & \cdots & -s_{i_2 i_{N_\rho}} \\ \vdots & \vdots & \ddots & \vdots \\ -s_{i_{N_\rho} i_1} & -s_{i_{N_\rho} i_2} & \cdots & \sum\limits_{j \in \mathcal{I}_\rho} s_{i_{N_\rho} j} \end{bmatrix}, \quad \rho = \rho(k).$$

Assume that the reference trajectory $y^{ref}_{\sigma_i(k)}(p)$ is available only for some non-empty subset of agents that can change depending on the network topology. The ability of agents to obtain information about the reference trajectory is specified by the matrix $\mathcal{R}_{\rho(k)} = \mathrm{diag}[r_{i_n}]_{n=1}^{N_{\rho(k)}}$, where $r_i = 1$ if $y^{ref}_{\sigma_i(k)}(p)$ is available to agent i, and $r_i = 0$ otherwise.

The agents with $r_i = 1$ will be called *global leaders*. The remaining agents can receive information from either global leaders or any other agents. Agent i such that $r_i = 0$ and $\exists j : s_{ij} > 0$ will be called a *follower*, where agent j is one of the agents transmitting information to agent i; these agents will be called *local leaders* for follower i. Each follower cannot transmit information to its local leaders.

Assume that the total number of mode switchings and topology switchings is finite: $\mathbf{N}_{\sigma\rho} < \infty$. Such a scenario arises for gantry robots operating by a preset program in a smart manufacturing system.

The learning error

$$e_i(k,p) = y^{ref}_{\sigma_i(k)}(p) - y_i(k,p)$$

is unavailable for measurement and control design. Therefore, consider

$$\hat{e}_i(k,p) = y^{ref}_{\sigma_i(k)}(p) - \hat{y}_i(k,p), \tag{2.4}$$

where $\hat{y}_i(k,p) = C\hat{x}_i(k,p)$ and $\hat{x}_i(k,p)$ is the state estimate of agent i given by the Kalman filter

$$\hat{x}_i(k,p+1) = A_{\sigma_i(k)}\hat{x}_i(k,p) + B_{\sigma_i(k)}u_i(k,p) + F_{i\sigma_i(k)}(y_{\nu i}(k,p) - C\hat{x}_i(k,p)) \tag{2.5}$$

with the initial condition $\hat{x}_i(k,0) = F_{i\sigma_i(k)}y_{\nu i}(k,0),\ i \in \mathcal{I}_{\rho(k)}$, where $F_{i\sigma_i(k)} = A_{\sigma_i(k)}S_{i\sigma_i(k)}C^\top \times \left[CS_{i\sigma_i(k)}C^\top + G_{\sigma_i(k)}S_{i\nu}G^\top_{\sigma_i(k)}\right]^{-1}$ and $S_{i\sigma_i(k)}$ is the solution of the algebraic Riccati equation

$$\begin{aligned} S_{i\sigma_i(k)} &= A_{\sigma_i(k)}S_{i\sigma_i(k)}A^\top_{\sigma_i(k)} \\ &- A_{\sigma_i(k)}S_{i\sigma_i(k)}C^\top\left[CS_{i\sigma_i(k)}C^\top + G_{\sigma_i(k)}S_{i\nu}G^\top_{\sigma_i(k)}\right]^{-1}CS_{i\sigma_i(k)}A^\top_{\sigma_i(k)} \\ &+ D_{\sigma_i(k)}S_{i\omega}D^\top_{\sigma_i(k)}, \quad i \in \mathcal{I}_{\rho(k)}. \end{aligned}$$

Then the ILC design problem is to find a distributed control law (protocol) $u_i(k,p)$ to reach a consensus in the following sense:

$$\mathrm{E}\left[|\hat{e}_i(k,p)|^2\right] \leqslant \kappa\varrho^k + \delta, \quad \kappa > 0, \quad 0 < \varrho < 1, \quad \delta > 0, \tag{2.6}$$

$$\lim_{k\to\infty} \mathrm{E}\left[|u_i(k,p)|^2\right] = \mathrm{E}\left[|u_i(\infty,p)|^2\right] < \infty, \quad i \in \mathcal{I}, \quad 0 \leqslant p \leqslant T-1. \tag{2.7}$$

The limit value $u_i(\infty,p)$ is often called the learned control.

3. BUILDING A 2D MODEL IN INCREMENTAL VARIABLES

3.1. A Fixed Operating Mode of Agents and a Fixed Network Topology

The analysis begins with the case where the operating mode of agents and the network topology are fixed. In other words, consider an interval along the passes on which the signals $\sigma_i(k)$ and $\rho(k)$ have no discontinuity points and their values are equal for all i and k. When solving the problem in such cases, the simplified notations $\sigma = \sigma_i(k)$ and $\rho = \rho(k)$ will be adopted. Following [18, 22], let the ILC law be

$$u_i(k+1,p-1) = u_i(k,p-1) + \Delta u_i(k+1,p-1) \tag{3.1}$$

with a correction (update) law $\Delta u_i\,(k+1,p-1)$ of the form

$$\begin{aligned}\Delta u_i\,(k+1,p-1) &= K_{1\sigma\rho}\,(\hat{x}_i\,(k+1,p-1)-\hat{x}_i\,(k,p-1))\\ &+ K_{2\sigma\rho}\left(\sum_{j\in N_{\rho i}} s_{ij}\,(\hat{y}_j(k,p)-\hat{y}_i(k,p)) + r_i\left(y_\sigma^{ref}(p)-\hat{y}_i(k,p)\right)\right),\end{aligned} \tag{3.2}$$

where $K_{1\sigma\rho}$ and $K_{2\sigma\rho}$ are the protocol matrices in mode σ and topology ρ, $N_{\rho i}=\{j\in\mathcal{I}_\rho \mid (j,i)\in\mathcal{E}_\rho\}$ denotes the set of neighbors available for agent i in topology ρ, and s_{ij} and r_i are the elements of the matrices $\mathcal{S}_\rho$ and $\mathcal{R}_\rho$, respectively.

With the increment vector

$$\hat{\eta}_i\,(k+1,p+1)=\hat{x}_i\,(k+1,p)-\hat{x}_i\,(k,p)$$

of the state estimate, the estimation error $\tilde{x}_i\,(k,p)=x_i\,(k,p)-\hat{x}_i\,(k,p)$, and the increment of the estimation error

$$\tilde{\eta}_i\,(k+1,p+1)=\tilde{x}_i\,(k+1,p)-\tilde{x}_i\,(k,p)\,,$$

system (2.1)–(2.3) can be written in terms of the increments and the learning error estimate (2.4) as

$$\begin{aligned}\hat{\eta}_i\,(k+1,p+1) &= A_\sigma\hat{\eta}_i\,(k+1,p)+F_{i\sigma}C\tilde{\eta}_i\,(k+1,p)\\ &+ B_\sigma\Delta u_i\,(k+1,p-1)+F_{i\sigma}G_\sigma\Delta\nu_i\,(k+1,p-1)\,,\end{aligned} \tag{3.3}$$

$$\begin{aligned}\tilde{\eta}_i\,(k+1,p+1) &= (A_\sigma-F_{i\sigma}C)\,\tilde{\eta}_i\,(k+1,p)\\ &+ D_\sigma\Delta\omega_i\,(k+1,p-1)-F_{i\sigma}G_\sigma\Delta\nu_i\,(k+1,p-1)\,,\end{aligned} \tag{3.4}$$

$$\begin{aligned}\hat{e}_i\,(k+1,p) &= -CA_\sigma\hat{\eta}_i\,(k+1,p)-CF_{i\sigma}C\tilde{\eta}_i\,(k+1,p)+\hat{e}_i\,(k,p)\\ &- CB_\sigma\Delta u_i\,(k+1,p-1)-CF_{i\sigma}G_\sigma\Delta\nu_i\,(k+1,p-1)\,,\end{aligned} \tag{3.5}$$

where $\Delta\nu_i(k+1,p-1)=\nu_i(k+1,p-1)-\nu_i(k,p-1)$ and $\Delta\omega_i(k+1,p-1)=\omega_i(k+1,p-1)-\omega_i\,(k,p-1)$. The second equation in (3.3)–(3.5) does not depend on the others. Hence, the well-known separation principle holds here: the filter and the controller can be designed independently. Therefore, $\tilde{\eta}_i$ can be treated as a bounded external variable, and the ILC design procedure will involve the system

$$\hat{\eta}_i\,(k+1,p+1)=A_\sigma\hat{\eta}_i\,(k+1,p)+B_\sigma\Delta u_i\,(k+1,p-1)+F_{i\sigma}G_\sigma\Delta\nu_i\,(k+1,p-1)\,, \tag{3.6}$$

$$\hat{e}_i(k+1,p)=-CA_\sigma\hat{\eta}_i(k+1,p)+\hat{e}_i(k,p)-CB_\sigma\Delta u_i(k+1,p-1)-CF_{i\sigma}G_\sigma\Delta\nu_i(k+1,p-1). \tag{3.7}$$

Introducing the extended vectors

$$\begin{aligned}\hat{x}\,(k,p) &= \left[\hat{x}_{i_1}^\top\,(k,p)\ \ \ldots\ \ \hat{x}_{i_{N_\rho}}^\top\,(k,p)\right]^\top,\\ \hat{\eta}\,(k,p) &= \left[\hat{\eta}_{i_1}^\top\,(k,p)\ \ \ldots\ \ \hat{\eta}_{i_{N_\rho}}^\top\,(k,p)\right]^\top,\\ \hat{e}\,(k,p) &= \left[\hat{e}_{i_1}\,(k,p)\ \ \ldots\ \ \hat{e}_{i_{N_\rho}}\,(k,p)\right]^\top,\\ \Delta\nu\,(k,p) &= \left[\Delta\nu_{i_1}\,(k,p)\ \ \ldots\ \ \Delta\nu_{i_{N_\rho}}\,(k,p)\right]^\top\end{aligned}$$

and using (3.2) allow writing the extended system (3.6)–(3.7) as

$$\begin{aligned} \hat{\eta}(k+1,p+1) &= \left(\bar{A}_{11\sigma\rho} + \bar{B}_{1\sigma\rho}\bar{K}_{1\sigma\rho}\bar{\mathcal{H}}_{1\rho}\right)\hat{\eta}(k+1,p) \\ &+ \left(\bar{A}_{12\sigma\rho} + \bar{B}_{1\sigma\rho}\bar{K}_{2\sigma\rho}\bar{\mathcal{H}}_{2\rho}\right)\hat{e}(k,p) + \bar{F}_{1\sigma\rho}\Delta\nu(k+1,p-1), \end{aligned} \tag{3.8}$$

$$\begin{aligned} \hat{e}(k+1,p) &= \left(\bar{A}_{21\sigma\rho} + \bar{B}_{2\sigma\rho}\bar{K}_{1\sigma\rho}\bar{\mathcal{H}}_{1\rho}\right)\hat{\eta}(k+1,p) \\ &+ \left(\bar{A}_{22\sigma\rho} + \bar{B}_{2\sigma\rho}\bar{K}_{2\sigma\rho}\bar{\mathcal{H}}_{2\rho}\right)\hat{e}(k,p) + \bar{F}_{2\sigma\rho}\Delta\nu(k+1,p-1), \end{aligned} \tag{3.9}$$

where

$$\begin{gathered} \bar{A}_{11\sigma\rho} = I_{N_\rho} \otimes A_\sigma, \quad \bar{A}_{12\sigma\rho} = 0, \quad \bar{A}_{21\sigma\rho} = I_{N_\rho} \otimes (-CA_\sigma), \quad \bar{A}_{22\sigma\rho} = I_{N_\rho}, \\ \bar{B}_{1\sigma\rho} = I_{N_\rho} \otimes B_\sigma, \quad \bar{B}_{2\sigma\rho} = I_{N_\rho} \otimes (-CB_\sigma), \\ \bar{K}_{1\sigma\rho} = I_{N_\rho} \otimes K_{1\sigma\rho}, \quad \bar{K}_{2\sigma\rho} = I_{N_\rho} \otimes K_{2\sigma\rho}, \\ \bar{\mathcal{H}}_{1\rho} = I_{N_\rho} \otimes \mathcal{H}_1, \quad \mathcal{H}_1 = I_{n_x}, \quad \bar{\mathcal{H}}_{2\rho} = (\mathcal{L}_\rho + \mathcal{R}_\rho) \otimes \mathcal{H}_2, \quad \mathcal{H}_2 = 1, \\ \bar{F}_{1\sigma\rho} = \mathrm{diag}[F_{i_n\sigma}G_\sigma]_{n=1}^{N_\rho}, \quad \bar{F}_{2\sigma\rho} = \mathrm{diag}[-CF_{i_n\sigma}G_\sigma]_{n=1}^{N_\rho}, \end{gathered}$$

and $\otimes$ stands for the Kronecker product.

The incremental system (3.8)–(3.9) has the standard repetitive process form. Further convergence analysis will employ the stability theory for switched stochastic repetitive processes [24].

3.2. Operating Mode Switching and Topology Switching

Consider the case of operating mode switching. The operating mode of agents may change under a fixed topology. For brevity, the topology-switching signal will be denoted by ρ (i.e., the same on all passes under consideration). Let $(k+1)$ be a switch instant of the global leader i. The controlled dynamics of the global leader are described by

$$\begin{aligned} \hat{x}_i(k+1,p) &= A_{\sigma_i(k+1)}\hat{x}_i(k+1,p-1) + F_{i\sigma_i(k+1)}C\tilde{x}_i(k+1,p-1) \\ &+ B_{\sigma_i(k+1)}u_i(k+1,p-1) + F_{i\sigma_i(k+1)}G_{\sigma_i(k+1)}\nu_i(k+1,\ p-1), \end{aligned} \tag{3.10}$$

$$\begin{aligned} \tilde{x}_i(k+1,p) &= \left(A_{\sigma_i(k+1)} - F_{i\sigma_i(k+1)}C\right)\tilde{x}_i(k+1,p-1) \\ &+ D_{\sigma_i(k+1)}\omega_i(k+1,p-1) - F_{i\sigma_i(k+1)}G_{\sigma_i(k+1)}\nu_i(k+1,\ p-1), \end{aligned} \tag{3.11}$$

$$\begin{aligned} \hat{e}_i(k+1,p) &= -C\left(A_{\sigma_i(k+1)}\hat{x}_i(k+1,p-1) - A_{\sigma_i(k)}\hat{x}_i(k,p-1)\right) \\ &- C\left(F_{i\sigma_i(k+1)}C\tilde{x}_i(k+1,p-1) - F_{i\sigma_i(k)}C\tilde{x}_i(k,p-1)\right) + \hat{e}_i(k,p) \\ &- C\left(B_{\sigma_i(k+1)} - B_{\sigma_i(k)}\right)u_i(k,p-1) - CB_{\sigma_i(k+1)}\Delta u_i(k+1,p-1) \\ &- C\left(F_{i\sigma_i(k+1)}G_{\sigma_i(k+1)}\nu_i(k+1,p-1) - F_{i\sigma_i(k)}G_{\sigma_i(k)}\nu_i(k,p-1)\right) \\ &+ \left(y^{ref}_{\sigma_i(k+1)}(p) - y^{ref}_{\sigma_i(k)}(p)\right). \end{aligned} \tag{3.12}$$

The variable $\tilde{x}_i(k+1,p)$ does not depend on the others and is unavailable for measurement; see the previous section. Therefore, it will be excluded from (3.10) and (3.12) for control design.

In contrast to Section 3.1, the perturbation $\left(y^{ref}_{\sigma_i(k+1)}(p) - y^{ref}_{\sigma_i(k)}(p)\right)$ appears in the last equation of (3.12) at the switch instant. This perturbation generates a transient that can significantly increase the learning error, which is an undesirable effect. Hence, at this instant, it is reasonable to construct a control law minimizing the perturbation effect and then return to the original control law ensuring convergence. Such a control law can be obtained by minimizing the deviation of the agent's output from the available reference trajectory. A similar situation occurs when switching modes for followers and when changing the topology. The control design procedure in all these cases will be described below.

4. THE CONVERGENCE THEOREM

Convergence conditions are based on the results of [24]. In contrast to the cited study, the problem under consideration has finitely many switchings and there is no need to estimate the average dwell time. In accordance with [24], these conditions will be obtained by using a vector Lyapunov function

$$V_{\sigma\rho}(\xi,\epsilon) = \begin{bmatrix} V_{1\sigma\rho}(\xi) \\ V_{2\sigma\rho}(\epsilon) \end{bmatrix}, \tag{4.1}$$

where $V_{1\sigma\rho}(\xi) > 0$ for $\xi \neq 0$, $V_{2\sigma\rho}(\epsilon) > 0$ for $\epsilon \neq 0$, and $V_{1\sigma\rho}(0) = 0$ and $V_{2\sigma\rho}(0) = 0$. The discrete counterpart of the divergence of (4.1) along the trajectories of system (3.8)–(3.9) is defined as

$$\begin{aligned} \mathcal{D}V_{\sigma\rho}(\xi,\epsilon) &= \mathrm{E}\left[V_{1\sigma\rho}(\hat{\eta}(k+1,p+1))\,|\hat{\eta}(k+1,p) = \xi, \hat{e}(k,p) = \epsilon\right] \\ &- V_{1\sigma\rho}(\xi) + \mathrm{E}\left[V_{2\sigma\rho}(\hat{e}(k+1,p))\,|\hat{\eta}(k+1,p) = \xi, \hat{e}(k,p) = \epsilon\right] - V_{2\sigma\rho}(\epsilon). \end{aligned}$$

Theorem 1. *Assume that there exist a vector Lyapunov function of the form* (4.1) *and positive scalars c_1, c_2, c_3, and γ such that*

$$\begin{gathered} c_1\|\xi\|^2 \leqslant V_{1\sigma\rho}(\xi) \leqslant c_2\|\xi\|^2, \\ c_1|\epsilon|^2 \leqslant V_{2\sigma\rho}(\epsilon) \leqslant c_2|\epsilon|^2, \\ \mathcal{D}V_{\sigma\rho}(\xi,\epsilon) \leqslant \gamma - c_3\left(\|\xi\|^2 + |\epsilon|^2\right) \end{gathered}$$

along the trajectories of system (3.8)–(3.9) *for all pairs $\sigma\rho$. Then the ILC law* (3.1) *with the update law* (3.2) *ensures the convergence conditions* (2.6).

Proof. Calculating $\mathcal{D}V_{\sigma\rho}(\xi,\epsilon)$ along the trajectories of system (3.8)–(3.9) and following the proof of [24, Theorem 1] give the estimate

$$\mathrm{E}\left[|\hat{e}(k,p-1)|^2\right] \leqslant \mu^{\mathbf{N}_{\sigma\rho}+1}\left[\lambda^k \sum_{q=0}^{p-1} \lambda^{p-1-q}|\hat{e}(0,q)|^2\right] + \frac{\gamma}{c_1(1-\lambda)^2}, \quad 0 < \lambda < 1, \tag{4.2}$$

where $\mu = c_2/c_1 \geqslant 1$, for all k and p. This inequality implies (2.6) with $\varrho = \lambda$, $\kappa = \frac{\mu^{\mathbf{N}_{\sigma\rho}+1}|\bar{e}|^2}{1-\lambda}$, $|\bar{e}|^2 = \max\limits_q |e(0,q)|^2$, and $\delta = \frac{\gamma}{c_1(1-\lambda)^2}$. The parameter λ (hence, ϱ) depends on c_2 and c_3 and determines the rate of convergence for the learning error; the parameter μ depends on c_1 and c_2 and determines the initial estimate. The proof of Theorem 1 is complete.

In view of (4.2), from (2.4) it follows that

$$\mathrm{E}\left[|C\hat{x}_i(k,p)|^2\right] = \mathrm{E}\left[|\hat{y}_i(k,p)|^2\right] \leqslant 2|y_\sigma^{ref}(p)|^2 + 2\mathrm{E}\left[|\hat{e}_i(k,p)|^2\right] < \infty \tag{4.3}$$

for any σ. This upper bound will serve for proving condition (2.7).

5. CONTROL DESIGN

5.1. Control under a Fixed Operating Mode of Agents and a Fixed Network Topology

Let the entries of the vector Lyapunov function be the quadratic forms

$$V_{1\sigma\rho}(\xi) = \xi^\top \bar{P}_{1\sigma\rho}\xi,$$
$$V_{2\sigma\rho}(\epsilon) = \epsilon^\top \bar{P}_{2\sigma\rho}\epsilon,$$

where $\bar{P}_{1\sigma\rho} = I_{N_\rho} \otimes P_{1\sigma\rho}$ and $\bar{P}_{2\sigma\rho} = I_{N_\rho} \otimes P_{2\sigma\rho}$, that satisfy the inequality

$$\mathcal{D}V_{\sigma\rho}(\xi,\epsilon) \leqslant \gamma - \left(\xi^\top \bar{Q}_{1\rho}\xi^\top + \epsilon^\top \bar{Q}_{2\rho}\epsilon + \Delta u^\top \bar{R}_\rho \Delta u\right), \tag{5.1}$$

where $\bar{Q}_{1\rho} \succ 0$, $\bar{Q}_{2\rho} \succ 0$, $\bar{R}_\rho \succ 0$, $\bar{Q}_{1\rho} = I_{N_\rho} \otimes Q_{1\rho}$, $\bar{Q}_{2\rho} = I_{N_\rho} \otimes Q_{2\rho}$, $\bar{R}_\rho = I_{N_\rho} \otimes R_\rho$, and $\Delta u = \bar{K}_{1\sigma\rho}\bar{\mathcal{H}}_{1\rho}\xi + \bar{K}_{2\sigma\rho}\bar{\mathcal{H}}_{2\rho}\epsilon$.

Then calculating the discrete counterpart of the divergence of the vector Lyapunov function yields

$$\mathcal{D}V_{\sigma\rho}(\xi,\epsilon) = \begin{bmatrix} \xi \\ \epsilon \end{bmatrix}^\top \Big((\bar{A}_{\sigma\rho} + \bar{B}_{\sigma\rho}\bar{K}_{\sigma\rho}\bar{\mathcal{H}}_\rho)^\top \bar{P}_{\sigma\rho}(\bar{A}_{\sigma\rho} + \bar{B}_{\sigma\rho}\bar{K}_{\sigma\rho}\bar{\mathcal{H}}_\rho) - \bar{P}_{\sigma\rho}$$
$$+ \bar{Q}_\rho + (\bar{K}_{\sigma\rho}\bar{\mathcal{H}}_\rho)^\top \bar{\mathcal{R}}_\rho \bar{K}_{\sigma\rho}\bar{\mathcal{H}}_\rho\Big) \begin{bmatrix} \xi \\ \epsilon \end{bmatrix} + 2\left(\mathrm{tr}\left[\bar{P}_{1\sigma\rho}S_{1\sigma}\right] + \mathrm{tr}\left[\bar{P}_{2\sigma\rho}S_{2\sigma}\right]\right),$$

where

$$\bar{A}_{\sigma\rho} = \begin{bmatrix} \bar{A}_{11\sigma\rho} & \bar{A}_{12\sigma\rho} \\ \bar{A}_{21\sigma\rho} & \bar{A}_{22\sigma\rho} \end{bmatrix}, \quad \bar{B}_{\sigma\rho} = \begin{bmatrix} \bar{B}_{1\sigma\rho} \\ \bar{B}_{2\sigma\rho} \end{bmatrix}, \quad \bar{K}_{\sigma\rho} = \begin{bmatrix} \bar{K}_{1\sigma\rho} & \bar{K}_{2\sigma\rho} \end{bmatrix},$$
$$\bar{Q}_\rho = \mathrm{diag}\begin{bmatrix} \bar{Q}_{1\rho} & \bar{Q}_{2\rho} \end{bmatrix}, \quad \bar{\mathcal{H}}_\rho = \mathrm{diag}\begin{bmatrix} \bar{\mathcal{H}}_{1\rho} & \bar{\mathcal{H}}_{2\rho} \end{bmatrix}, \quad \bar{P}_{\sigma\rho} = \mathrm{diag}\begin{bmatrix} \bar{P}_{1\sigma\rho} & \bar{P}_{2\sigma\rho} \end{bmatrix},$$
$$S_{1\sigma} = \mathrm{diag}[S_{1i_n\sigma}]_{n=1}^{N_\rho}, \quad S_{2\sigma} = \mathrm{diag}[S_{2i_n\sigma}]_{n=1}^{N_\rho},$$
$$S_{1i\sigma} = F_{i\sigma}G_\sigma S_{i\nu}G_\sigma^\top F_{i\sigma}^\top, \quad S_{2i\sigma} = CF_{i\sigma}G_\sigma S_{i\nu}G_\sigma^\top F_{i\sigma}^\top C^\top.$$

The conditions of Theorem 1 will hold with $\gamma = 2\left(\mathrm{tr}\left[\bar{P}_{1\sigma\rho}S_{1\sigma}\right] + \mathrm{tr}\left[\bar{P}_{2\sigma\rho}S_{2\sigma}\right]\right)$ if the inequality

$$\begin{bmatrix} \xi \\ \epsilon \end{bmatrix}^\top \Big((\bar{A}_{\sigma\rho} + \bar{B}_{\sigma\rho}\bar{K}_{\sigma\rho}\bar{\mathcal{H}}_\rho)^\top \bar{P}_{\sigma\rho}(\bar{A}_{\sigma\rho} + \bar{B}_{\sigma\rho}\bar{K}_{\sigma\rho}\bar{\mathcal{H}}_\rho) - \bar{P}_{\sigma\rho} + \bar{Q}_\rho + (\bar{K}_{\sigma\rho}\bar{\mathcal{H}}_\rho)^\top \bar{\mathcal{R}}_\rho \bar{K}_{\sigma\rho}\bar{\mathcal{H}}_\rho\Big) \begin{bmatrix} \xi \\ \epsilon \end{bmatrix} \leqslant 0$$

is solvable in a positive definite matrix $\bar{P}_{\sigma\rho}$. This inequality is tantamount to the matrix inequality

$$(\bar{A}_{\sigma\rho} + \bar{B}_{\sigma\rho}\bar{K}_{\sigma\rho}\bar{\mathcal{H}}_\rho)^\top \bar{P}_{\sigma\rho}(\bar{A}_{\sigma\rho} + \bar{B}_{\sigma\rho}\bar{K}_{\sigma\rho}\bar{\mathcal{H}}_\rho) - \bar{P}_{\sigma\rho} + \bar{Q}_\rho + (\bar{K}_{\sigma\rho}\bar{\mathcal{H}}_\rho)^\top \bar{R}_\rho \bar{K}_{\sigma\rho}\bar{\mathcal{H}}_\rho \preccurlyeq 0. \tag{5.2}$$

Consider the variables

$$\bar{X}_{\sigma\rho} = \mathrm{diag}\begin{bmatrix} \bar{X}_{1\sigma\rho} & \bar{X}_{2\sigma\rho} \end{bmatrix} = \bar{P}_{\sigma\rho}^{-1}, \quad \bar{X}_{1\sigma\rho} = I_{N_\rho} \otimes X_{1\sigma\rho}, \quad \bar{X}_{2\sigma\rho} = I_{N_\rho} \otimes X_{2\sigma\rho},$$
$$\bar{Z}_{\sigma\rho} = \mathrm{diag}\begin{bmatrix} \bar{Z}_{1\sigma\rho} & \bar{Z}_{2\sigma\rho} \end{bmatrix}, \quad \bar{Z}_{1\sigma\rho} = I_{N_\rho} \otimes Z_{1\sigma\rho}, \quad \bar{Z}_{2\sigma\rho} = I_{N_\rho} \otimes Z_{2\sigma\rho},$$
$$\bar{Y}_{\sigma\rho} = \begin{bmatrix} \bar{Y}_{1\sigma\rho} & \bar{Y}_{2\sigma\rho} \end{bmatrix} = \bar{K}_{\sigma\rho}\bar{Z}_{\sigma\rho}, \quad \bar{Y}_{1\sigma\rho} = I_{N_\rho} \otimes Y_{1\sigma\rho}, \quad \bar{Y}_{2\sigma\rho} = I_{N_\rho} \otimes Y_{2\sigma\rho},$$

where $\bar{Z}_{\sigma\rho}$ is the solution of the equation

$$\bar{Z}_{\sigma\rho}\bar{\mathcal{H}}_\rho = \bar{\mathcal{H}}_\rho \bar{X}_{\sigma\rho}.$$

Multiplying (5.2) on both sides by $\bar{P}_{\sigma\rho}^{-1}$ and applying Schur's complement lemma give the following system of matrix equations and inequalities:

$$\begin{bmatrix} \bar{X}_{\sigma\rho} & (\bar{A}_{\sigma\rho}\bar{X}_{\sigma\rho} + \bar{B}_{\sigma\rho}\bar{Y}_{\sigma\rho}\bar{\mathcal{H}}_\rho)^\top & \bar{X}_{\sigma\rho} & (\bar{Y}_{\sigma\rho}\bar{\mathcal{H}}_\rho)^\top \\ \bar{A}_{\sigma\rho}\bar{X}_{\sigma\rho} + \bar{B}_{\sigma\rho}\bar{Y}_{\sigma\rho}\bar{\mathcal{H}}_\rho & \bar{X}_{\sigma\rho} & 0 & 0 \\ \bar{X}_{\sigma\rho} & 0 & \bar{Q}_\rho^{-1} & 0 \\ \bar{Y}_{\sigma\rho}\bar{\mathcal{H}}_\rho & 0 & 0 & \bar{R}_\rho^{-1} \end{bmatrix} \succcurlyeq 0,$$

$$\bar{Z}_{\sigma\rho}\bar{\mathcal{H}}_\rho = \bar{\mathcal{H}}_\rho\bar{X}_{\sigma\rho},\ \bar{X}_{\sigma\rho} \succ 0. \quad (5.3)$$

Thus, the ILC law (3.1) with the update law (3.2) and the protocol matrices

$$K_{1\sigma\rho} = Y_{1\sigma\rho}Z_{1\sigma\rho}^{-1}, \quad K_{2\sigma\rho} = Y_{2\sigma\rho}Z_{2\sigma\rho}^{-1},$$

where $Z_{1\sigma\rho}$, $Z_{2\sigma\rho}$, $Y_{1\sigma\rho}$, and $Y_{2\sigma\rho}$ are found by solving (5.3), ensures the convergence conditions (2.6) in the case under consideration. Here, the matrices $\bar{Q}_\rho$ and $\bar{R}_\rho$ play the same role as weight matrices in linear quadratic control design. By varying these matrices, it is possible to tune the control signal and achieve desired performance characteristics.

5.2. Control under Operating Mode Switching

As has been noted, mode switching causes a perturbation generating a transient. This transient increases the achieved value of the learning error for several passes and slows down convergence. Therefore, at the switch instant, the update law will be constructed by minimizing the error. In the case of a global leader, the update law will be found by minimizing the objective functional

$$J_{ml} = \mathrm{E}\left[|\hat{e}_i\,(k+1,p)\,|^2|(*)\right]$$

subject to the constraint arising from (3.10) and (3.12):

$$\begin{aligned} \hat{e}_i\,(k+1,p) &= -C\Big(A_{\sigma_i(k+1)}\hat{x}_i\,(k+1,p-1) - A_{\sigma_i(k)}\hat{x}_i\,(k,p-1)\Big) + \hat{e}_i\,(k,p) \\ &- C\Big(B_{\sigma_i(k+1)} - B_{\sigma_i(k)}\Big)u_i\,(k,p-1) - CB_{\sigma_i(k+1)}\Delta u_i\,(k+1,p-1) \\ &- C\Big(F_{i\sigma_i(k+1)}G_{\sigma_i(k+1)}\nu_i\,(k+1,p-1) - F_{i\sigma_i(k)}G_{\sigma_i(k)}\nu_i\,(k,p-1)\Big) \\ &+ \Big(y^{ref}_{\sigma_i(k+1)}\,(p) - y^{ref}_{\sigma_i(k)}\,(p)\Big). \end{aligned} \quad (5.4)$$

Here, $(*)$ means that the expectation is taken under fixed values of the state and control variables in the right-hand side of (5.4). The resulting update law has the form

$$\begin{aligned} \Delta u_i\,(k+1,p-1) &= \Big(CB_{\sigma_i(k+1)}\Big)^{-1}\hat{e}_i\,(k,p) \\ &- \Big(CB_{\sigma_i(k+1)}\Big)^{-1}C\Big(A_{\sigma_i(k+1)}\hat{x}_i\,(k+1,p-1) - A_{\sigma_i(k)}\hat{x}_i\,(k,p-1)\Big) \\ &- \Big(CB_{\sigma_i(k+1)}\Big)^{-1}C\,\Big(B_{\sigma_i(k+1)} - B_{\sigma_i(k)}\Big)\,u_i\,(k,p-1) \\ &+ \Big(CB_{\sigma_i(k+1)}\Big)^{-1}\,\Big(y^{ref}_{\sigma_i(k+1)}\,(p) - y^{ref}_{\sigma_i(k)}\,(p)\Big) \end{aligned} \quad (5.5)$$

for the ILC law (3.1).

The update law for a follower is calculated by analogy. However, such agents have no direct access to information about the reference trajectory, and the weighted sum of the output estimates of the corresponding local leaders from the previous pass is used instead. In this regard, (2.4) is replaced by the deviation of the follower's output estimate from those of its local leaders:

$$\hat{\varepsilon}_i(k+1,p) = \sum_{j \in N_{\rho i}} s_{ij}\left(\hat{y}_j(k,p) - \hat{y}_i(k+1,p)\right),$$

where s_{ij} is the element of the matrix $\mathcal{S}_\rho$. Accordingly, the update law will be found by minimizing the objective functional

$$J_{mf} = \mathrm{E}\left[|\hat{\varepsilon}_i(k+1,p)|^2 |(**)\right]$$

subject to the constraint

$$\begin{aligned}
\hat{\varepsilon}_i(k+1,p) &= -\ell_{ii} C\Big(A_{\sigma_i(k+1)}\hat{x}_i(k+1,p-1) - A_{\sigma_i(k)}\hat{x}_i(k,p-1)\Big) \\
&- \ell_{ii} C\left(B_{\sigma_i(k+1)} - B_{\sigma_i(k)}\right) u_i(k,p-1) - \ell_{ii} C B_{\sigma_i(k+1)} \Delta u_i(k+1,p-1) \\
&+ \sum_{j \in N_{\rho i}} s_{ij}\left(\hat{y}_j(k,p) - \hat{y}_i(k,p)\right) - \ell_{ii} C\Big(F_{i\sigma_i(k+1)} G_{\sigma_i(k+1)} \nu_i(k+1,p-1) \\
&\qquad - F_{i\sigma_i(k)} G_{\sigma_i(k)} \nu_i(k,p-1)\Big),
\end{aligned} \tag{5.6}$$

where ℓ_{ii} is the element of the matrix $\mathcal{L}_\rho$ and $(**)$ means that the expectation is taken under fixed values of the state estimate and control in the right-hand side of (5.6). As a result, the update law is given by

$$\begin{aligned}
\Delta u_i(k+1,p-1) &= \ell_{ii}^{-1}\left(CB_{\sigma_i(k+1)}\right)^{-1} \sum_{j \in N_{\rho i}} s_{ij}\left(\hat{y}_j(k,p) - \hat{y}_i(k,p)\right) \\
&- \left(CB_{\sigma_i(k+1)}\right)^{-1} C\left(A_{\sigma_i(k+1)}\hat{x}_i(k+1,p-1) - A_{\sigma_i(k)}\hat{x}_i(k,p-1)\right) \\
&\qquad - \left(CB_{\sigma_i(k+1)}\right)^{-1} C\left(B_{\sigma_i(k+1)} - B_{\sigma_i(k)}\right) u_i(k,p-1).
\end{aligned} \tag{5.7}$$

Recall that local leaders transmit the data received on the previous pass. Consequently, the switching of a follower must be delayed relative to the switching of its local leader in the system so that the information about the reference trajectory corresponds to the new mode. Similar to the signal $\sigma_i(k)$, now called the *local mode-switching signal*, consider the *global signal* $\sigma(k)$, which triggers the mode-switching process for the agents. The switch instants $\sigma_i(k)$ of the global leaders coincide with the switch instants $\sigma(k)$, i.e., $\sigma_i(k) = \sigma(k)\ \forall i : r_i = 1$. For followers, the local signal will be delayed relative to the global signal: $\sigma_i(k) = \sigma_j(k-1)\ \forall i, j : r_i = 0,\ j \in N_{\rho i}$.

Thus, at the mode switch instant, the ILC law (3.1) is used with the update laws (5.5) and (5.7) for the global leaders and followers, respectively. This rule applies to mode switching.

5.3. Control under Network Topology Switching

In the case where the network topology is switched, all agents operate in the same mode on all passes under study. Therefore, for brevity, the mode-switching signal will be denoted by σ.

At a topology switch instant, an agent may perform one of the following actions: connect to the network through operating agents, disconnect from the network, or change local leaders. In the second and third scenarios, the agents are controlled using the ILC law (3.1) with the update law (3.2) with the protocol matrices corresponding to the new topology. In the first scenario, the error of the connected agent using the same law may significantly differ from that achieved by other agents over several passes after the connection.

Let $(k+1)$ be the topology switch instant due to connecting agent i. The same approach as in the previous subsection can be used to find an appropriate ILC update law that will reduce the agent's error caused by the connection. In this case, however, it is easier to find the ILC law (rather than its update law) using the objective functional

$$J_c = \mathrm{E}\left[|\hat{\varepsilon}_i\,(k+1,p)\,|^2|(***)\right]$$

and construct the control law for the connected agent by minimizing this functional subject to the constraint

$$\begin{aligned}\hat{\varepsilon}_i\,(k+1,p) = \sum_{j\in N_{\rho(k+1)i}} s_{ij}\hat{y}_j\,(k,p) - \ell_{ii}CA_{\sigma_i(k+1)}\hat{x}_i\,(k+1,p-1)\\ - \ell_{ii}CB_{\sigma_i(k+1)}u_i\,(k+1,p-1) - \ell_{ii}CF_{i\sigma_i(k+1)}G_{\sigma_i(k+1)}\nu_i\,(k+1,p-1)\,.\end{aligned} \tag{5.8}$$

Here, s_{ij} and ℓ_{ii} are the elements of the matrices $\mathcal{S}_{\rho(k+1)}$ and $\mathcal{L}_{\rho(k+1)}$, respectively; $(***)$ means that the expectation is taken under fixed values of the state estimate and control in the right-hand side of (5.8). The resulting ILC law for the connected subsystem has the form

$$\begin{aligned}u_i\,(k+1,p-1) = \ell_{ii}^{-1}\Big(CB_{\sigma_i(k+1)}\Big)^{-1}\sum_{j\in N_{\rho(k+1)i}} s_{ij}\hat{y}_j\,(k,p)\\ - \Big(CB_{\sigma_i(k+1)}\Big)^{-1}CA_{\sigma_i(k+1)}\hat{x}_i\,(k+1,p-1)\,.\end{aligned} \tag{5.9}$$

5.4. General Control Law

The general control law is based on the following switching rules for the operating mode of agents, the network topology, and the ILC law.

Switching is launched by the signals $\sigma\,(k)$ and $\rho\,(k)$. The signal $\sigma\,(k)$, referred to as the global mode-switching signal, triggers the switching process for the operating mode of agents. The operating mode of the global leaders is switched when launching the switching process by the global signal, i.e., $\sigma_i\,(k) = \sigma\,(k)\ \forall i : r_i = 1$. The operating mode of the other agents is switched with a one-pass delay after switching of their local leaders, i.e., $\sigma_i\,(k) = \sigma_j\,(k-1)\ \forall i,j : r_i = 0,\ j \in N_{\rho(k)i}$. The signal $\rho\,(k)$ switches the network topology.

The ILC signal on pass $(k+1)$ has the form (3.1) with different update laws as follows: the update law (3.2) whose protocol matrices are obtained by solving system (5.3) if $\sigma_i\,(k+1) = \sigma_i\,(k)$ and $i \in \mathcal{I}_{\rho(k)}$; the update law (5.5) if $r_i = 1$, $\sigma_i\,(k+1) \neq \sigma_i\,(k)$, and $i \in \mathcal{I}_{\rho(k)}$; the update law (5.7) if $r_i = 0$, $\sigma_i\,(k+1) \neq \sigma_i\,(k)$, and $i \in \mathcal{I}_{\rho(k)}$, where r_i is the element of the matrix $\mathcal{R}_{\rho(k+1)}$. If $i \notin \mathcal{I}_{\rho(k)}$, then the ILC signal on pass $(k+1)$ has the form (5.9).

It remains to prove the mean-squared boundedness of the ILC law; see condition (2.7). Consider the interval along passes before the first switching. From (2.5) it follows that

$$\begin{aligned}u_i(k,p-1) = (CB_\sigma)^{-1}\Bigg[C\hat{x}_i(k,p) - CA_\sigma^p\hat{x}_i(k,0)\\ - \sum_{q=0}^{p-2} CA_\sigma^{p-1-q}B_\sigma u_i(k,p) - \sum_{q=0}^{p-1} CA_\sigma^{p-1-q}(F_\sigma C\tilde{x}_i(k,q) + F_\sigma G_\sigma \nu_i(k,p))\Bigg]\,.\end{aligned} \tag{5.10}$$

Particularly for $p = 2$,

$$u_i(k,1) = (CB_\sigma)^{-1}\Bigg[C\hat{x}_i(k,2) - CA_\sigma^2\hat{x}_i(k,0) - CA_\sigma B_\sigma u_i(k,0) - \sum_{q=0}^{1} CA_\sigma^{p-1-q}(F_\sigma C\tilde{x}_i(k,q) + F_\sigma G_\sigma \nu_i(k,p))\Bigg].$$

Raising both sides of the last equality to the square, let us estimate the right-hand side using the well-known algebraic inequality $\left(\left(\sum_{i=1}^n a_i\right)^2 \leqslant n\sum_{i=1}^n a_i^2\right)$. Due to (4.3), $\mathrm{E}[|C\hat{x}_i(k,2)|^2] < \infty$, and the values $||\hat{x}_i(k,0)||^2$, $|u_i(k,0)|^2$, $||\tilde{x}_i(k,q)||^2$, and $|\nu_i(k,q)|^2$ have finite means; therefore, applying the expectation operator to the resulting inequality gives $\mathrm{E}[|u_i(k,1)|^2] < \infty$. The sequential variation of p between 3 and T in (5.10) gives $\mathrm{E}[|u_i(k,p)|^2] < \infty$ for all k up to the first switching. According to the control choice approach, (4.3) holds at the switch instants at the beginning of each interval between switchings. Hence, considering the intervals between switchings sequentially shows that $\mathrm{E}[|u_i(k,p)|^2] < \infty$ for all k until the last switching on pass k_f. Repeating the same procedure for $k \geqslant k_f$ and letting $k \to \infty$ finally give (2.7). Thus, the ILC law ensures consensus in the sense of conditions (2.6) and (2.7).

6. AN EXAMPLE

Consider a networked system of identical gantry robots (agents) with a flexible rotating link that moves in the horizontal plane with a constant repetition period. The problem is to design an ILC law for this system as proposed above. The dynamics of each agent on pass k are described by the state-space model

$$\dot{x}_i(k,t) = A_{\sigma_i(k)}^{cont} x_i(k,t) + B^{cont}(u_i(k,t) + \mu_i(k,t)), \tag{6.1}$$

$$y_i(k,t) = Cx_i(k,t) + \rho_i(k,t) \tag{6.2}$$

with the following notations: $x = \begin{bmatrix} \theta & \alpha & \dot{\theta} & \dot{\alpha} \end{bmatrix}^\top$, θ is the servo angle, and α is the flexible link angle; $u = \tau$ is the load gear torque applied to the link; μ and ρ are independent continuous Gaussian white noises of the plant and measurement, respectively, with constant intensities Q_n and R_n. The system matrices have the form

$$A_\sigma^{cont} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & K_s/J_{eq} & -B_{eq}/J_{eq} & 0 \\ 0 & -K_s(J_l(\sigma) + J_{eq})/J_{eq}J_l(\sigma) & B_{eq}/J_{eq} & 0 \end{bmatrix},$$

$$B^{cont} = \begin{bmatrix} 0 & 0 & 1/J_{eq} & -1/J_{eq} \end{bmatrix}^\top, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

where K_s is the stiffness of the flexible link, J_{eq} is the moment of inertia of the servo, B_{eq} is the viscous friction coefficient of the servo, and $J_l(\sigma)$ is the moment of inertia of the flexible link relative to center of mass, $\sigma = \sigma_i(k)$.

The results demonstrated below correspond to the following parameter values:

$$K_s = 1.3\ \mathrm{N} \times \mathrm{m/rad}, \quad J_{eq} = 2.08 \times 10^{-3}\ \mathrm{kg} \times \mathrm{m}^2, \quad B_{eq} = 0.004\ \mathrm{N} \times \mathrm{m/(rad/s)}\ [25],$$
$$Q_n = 5 \times 10^{-5}, \quad \text{and} \quad R_n = 10^{-6}.$$

The agents have two operating modes (pick-and-place operations) with particular reference trajectories of the output:

$$y^{ref}_{\sigma_i(k)}(t) = \begin{cases} \pi\left(\dfrac{t^2}{6} - \dfrac{t^3}{27}\right), & \sigma_i(k) = 1, \\ \dfrac{\pi}{2}\sin\dfrac{\pi t}{6}, & \sigma_i(k) = 2. \end{cases}$$

In addition, the modes differ by the moment of inertia of the flexible link:

$$J_l(\sigma_i(k)) = \begin{cases} 0.0038 \text{ kg} \times \text{m}^2, & \sigma_i(k) = 1, \\ 0.008 \text{ kg} \times \text{m}^2, & \sigma_i(k) = 2. \end{cases}$$

The repetition period is 3 s. The time discretization of the differential dynamics (6.1)–(6.2) gives the state-space model (2.1)–(2.3) for the ILC design with the matrices

$$A_{\sigma_i(k)} = \exp A^{cont}_{\sigma_i(k)} T_s, \quad B_{\sigma_i(k)} = \int_0^{T_s} \exp\left(A^{cont}_{\sigma_i(k)}\tau\right) B^{cont} d\tau,$$

$$D_{\sigma_i(k)} = \left[\int_o^{T_s} \exp\left(A^{cont}_{\sigma_i(k)}\tau\right) B^{cont} Q_n \left(B^{cont}\right)^\top \left(\exp\left(A^{cont}_{\sigma_i(k)}\tau\right)\right)^\top d\tau\right]^{\frac{1}{2}},$$

$$G_{\sigma_i(k)} = \left(\frac{R_n}{T_s}\right)^{\frac{1}{2}},$$

where T_s is a sampling period (0.01 s) and the noises $\omega_i(k,p)$ and $\nu_i(k,p)$ have the unit covariances $S_{i\omega} = I_{n_x}$ and $S_{i\nu} = 1$, respectively.

Consider the networked system with three gantry robots and the following scenario: first, one global leader operates, then the first follower connects to the global leader, and subsequently the second follower connects to the first follower. This scenario corresponds to an SM system with variable production volume: new agents are connected to the network if the volume increases and are disconnected otherwise (when they become superfluous). The variable network topology corresponding to this program is described by

$$\mathcal{I}_{\rho(k)} = \{1\}, \ \mathcal{L}_{\rho(k)} = 0, \text{ and } \mathcal{R}_{\rho(k)} = 1 \text{ for } \rho(k) = 1,$$

$$\mathcal{I}_{\rho(k)} = \{1,2\}, \ \mathcal{L}_{\rho(k)} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}, \text{ and } \mathcal{R}_{\rho(k)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ for } \rho(k) = 2,$$

$$\mathcal{I}_{\rho(k)} = \{1,2,3\}, \ \mathcal{L}_{\rho(k)} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \text{ and } \mathcal{R}_{\rho(k)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ for } \rho(k) = 3.$$

Computing the gain matrices for the Kalman filter yields

$$F_{i\sigma_i(k)} = \begin{bmatrix} 0.7106 & -0.5711 & 16.0723 & -12.8866 \end{bmatrix}^\top \text{ for } \sigma_i(k) = 1,$$

$$F_{i\sigma_i(k)} = \begin{bmatrix} 0.7038 & -0.6084 & 15.5888 & -13.9268 \end{bmatrix}^\top \text{ for } \sigma_i(k) = 2.$$

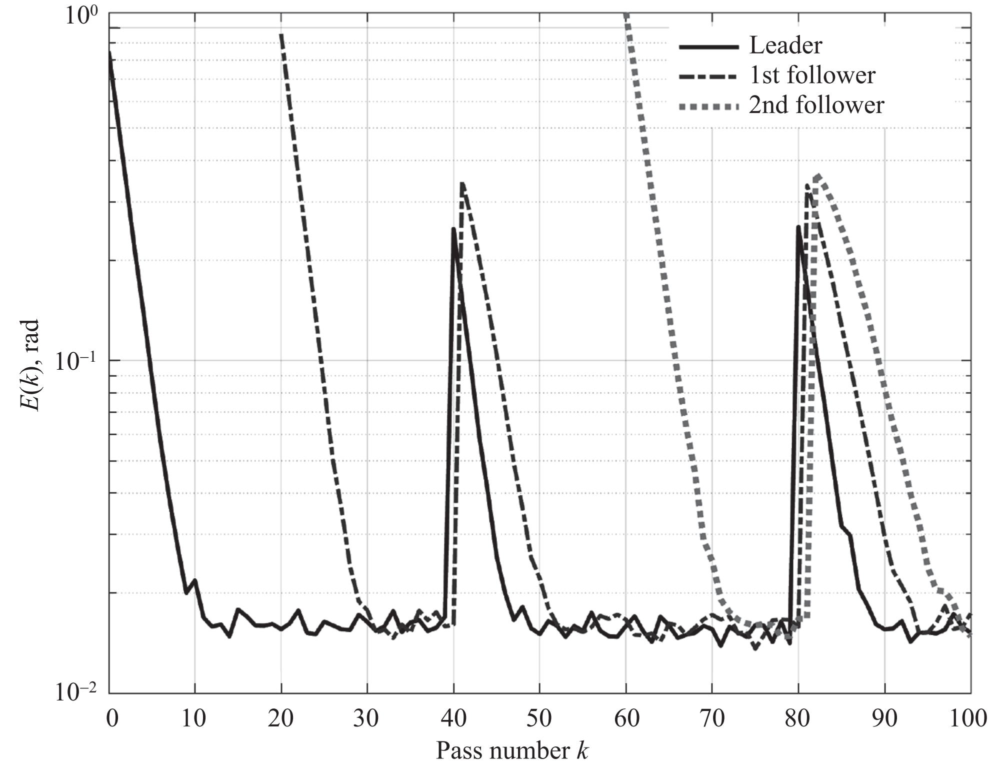

Fig. 1. Root mean square errors for different agents without control switching.

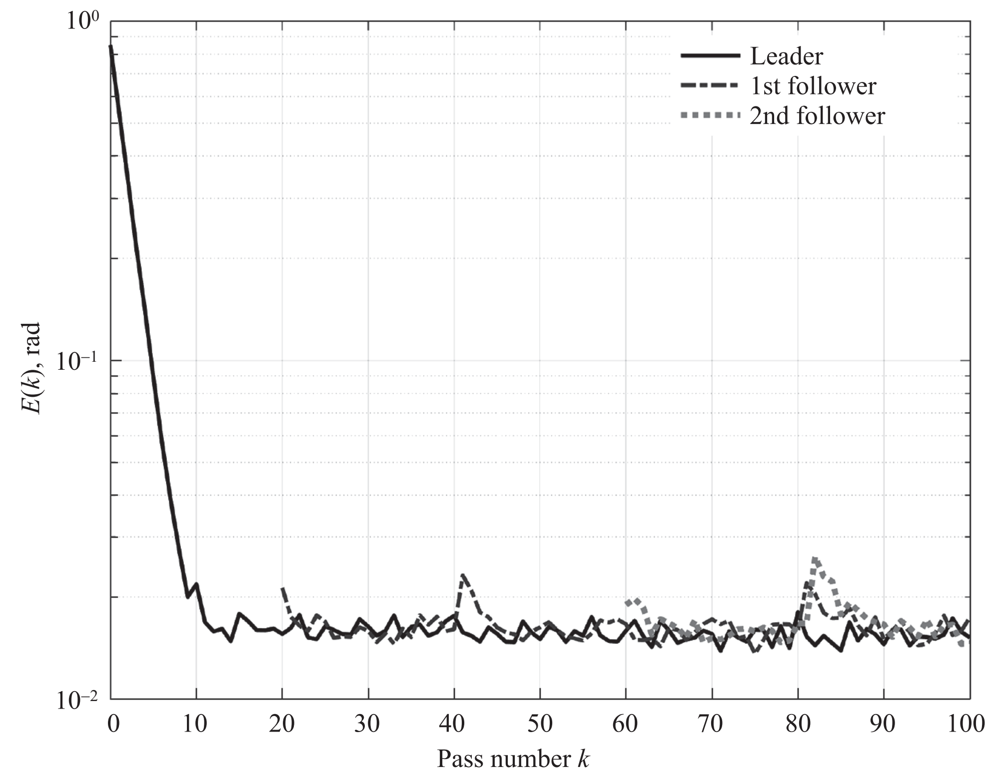

Fig. 2. Root mean square errors for different agents with control switching.

Solving (5.3) with the weight matrices

$$Q_{1\rho(k)} = \begin{cases} 10I_{n_x}, & \rho(k) = 1, \\ 10^{-4}I_{n_x}, & \rho(k) \neq 1, \end{cases}$$
$$Q_{2\rho(k)} = 10^5, \quad R_{\rho(k)} = 10^{-3}$$

gives the following protocol matrices for (3.2):

$K_{1\sigma\rho} = \begin{bmatrix} -25.3429 & -1.2471 & -0.3469 & -0.0181 \end{bmatrix}$ and $K_{2\sigma\rho} = 9.2965$ for $\sigma = 1$ and $\rho = 1$.

$K_{1\sigma\rho} = \begin{bmatrix} -41.8954 & -1.2959 & -0.416 & -0.0044 \end{bmatrix}$ and $K_{2\sigma\rho} = 16.2678$ for $\sigma = 1$ and $\rho = 2$.

$K_{1\sigma\rho} = \begin{bmatrix} -41.9283 & -1.296 & -0.4161 & -0.0044 \end{bmatrix}$ and $K_{2\sigma\rho} = 14.3248$ for $\sigma = 1$ and $\rho = 3$.

$K_{1\sigma\rho} = \begin{bmatrix} -25.3841 & -1.2813 & -0.3428 & -0.0139 \end{bmatrix}$ and $K_{2\sigma\rho} = 9.2971$ for $\sigma = 2$ and $\rho = 1$.

$K_{1\sigma\rho} = \begin{bmatrix} -41.889 & -1.2981 & -0.4159 & -0.0044 \end{bmatrix}$ and $K_{2\sigma\rho} = 16.4272$ for $\sigma = 2$ and $\rho = 2$.

$K_{1\sigma\rho} = \begin{bmatrix} -41.9326 & -1.2982 & -0.4161 & -0.0044 \end{bmatrix}$ and $K_{2\sigma\rho} = 14.2538$ for $\sigma = 2$ and $\rho = 3$,

where $\sigma = \sigma_i(k)$ and $\rho = \rho(k)$. Let the switching signals be

$$\sigma(k) = \begin{cases} 1, & k < 40, \\ 2, & 40 \leqslant k < 80, \\ 1, & k \geqslant 80, \end{cases} \qquad \rho(k) = \begin{cases} 1, & k < 20, \\ 2, & 20 \leqslant k < 60, \\ 3, & k \geqslant 60. \end{cases}$$

The performance of this ILC law can be evaluated using the root mean square error (RMSE) for each trial:

$$E_i(k) = \sqrt{\frac{1}{T}\sum_{p=0}^{T-1} |e_i(k,p)|^2}.$$

Figure 1 shows the RMSE progression for different agents without control switching when changing the operating mode and network topology, i.e., the ILC law (3.1) with the update law (3.2) is applied throughout the system operation. Figure 2 presents the corresponding graphs for different agents with control switching. According to these results, the ILC law designed in this paper allows reducing the transient error at the switch instants.

7. CONCLUSIONS

The ILC algorithm proposed above reduces the transient error at the instants of mode switching and connecting new agents to the network. However, it imposes some restrictions on the network topology. First, mutual information exchange between agents is impossible. Indeed, in order to switch the operating mode of a follower, the mode of its local leaders on the previous pass must correspond to the desired one. For this reason, mutually exchanging their information, the agents will wait for each other's mode to change, and eventually, it will not happen for any of them. In the illustrative example, this situation would arise if the first follower transmitted its output to the second one and the latter to the former. In this case, the first follower would not be able to switch because of waiting for the second one to do it; the second follower, because of waiting for the first one. Also, for this reason, it is impossible to implement a closed information exchange chain. Second, if this algorithm is applied to a system with serially connected agents, the process of switching the entire network system to a new operating mode may take an unacceptably long time due to switching delays of followers.

FUNDING

This work was financially supported by the Russian Science Foundation, project no. 22-21-00612, https://rscf.ru/project/22-21-00612.

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This paper was recommended for publication by A.I. Kibzun, a member of the Editorial Board