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Suboptimal Robust Stabilization of an Unknown Autoregressive Object with Uncertainty and Offset External Perturbation

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Abstract—In this paper, the problem of suboptimal stabilization of an object with discrete time, output and control uncertainties, and bounded external perturbation is considered. The autoregressive nominal model coefficients, uncertainty amplification coefficients, norm and external disturbance offset are assumed to be unknown. The quality indicator is the worst-case asymptotic upper bound of the output modulus of the object. The solution of the problem in conditions of non-identifiability of all unknown parameters is based on the method of recurrent target inequalities and optimal online estimation, in which the quality index of the control problem serves as an identification criterion. A non-linear replacement of the unknown parameter perturbations that reduces the optimal online estimation problem to a fractional linear programming problem is proposed. The performance of adaptive suboptimal control is illustrated by numerical simulation results.

Keywords: robust control, adaptive control, optimal control, bounded perturbation, model verification

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1. INTRODUCTION

In the 1980s, it was shown in the famous [1, 2] articles that the adaptive control algorithms known, by that time, in the deterministic formulation do not guarantee stability of adaptive systems in the presence of any small external perturbations or non-modulated dynamics. During the next two decades, various modifications of standard estimation algorithms were proposed to ensure robust stability of adaptive systems (robust implies stability of the system in the presence of operator perturbations, which are called uncertainty in the theory of robust control). The robust stability was proved for sufficiently small operator perturbations using the apparatus of Lyapunov functions and independently of the robust control theory in the H_∞ formulation, which was developed in the same years at the same time [3]. The works of this field became known as robust adaptive control [4–6]. At the same time, the problem of finding the nominal (i.e. unperturbed) model and levels of uncertainties and external perturbations from measurement data was recognized in the mid-1990s as the main problem for practical applications of H_∞ theory [7]. Therefore, assessing the quality of a given or resulting identification model of the controlled object has been called a central problem in systems identification theory [8]. To this date, this problem remains open and relevant even in the tasks of offline identification, i.e. identification by a given set of measurements, rather than by the course of control [9].

In the early 1990s, fundamental results were obtained in the theory of robust control in the ℓ_1-formulation, in which the basic signal space is the space of ℓ_∞-bounded real sequences and the control quality indicators are formulated in terms of ℓ_∞ norms of the control system output [10, 11]. In the ℓ_1 theory of robust control, in contrast to H_∞ theory, explicit representations for asymptotic quality indices for systems with structured uncertainty, external bounded perturbations, and given command signals were obtained [12–15]. This made it possible to propose a general method for the synthesis of an adaptive optimal robust control with a given accuracy, potentially implementing maximum feedback capabilities and providing, under conditions of unidentifiable unknown parameters, the same asymptotic control quality as for objects with known parameters [16, 17]. The importance and relevance of investigating the maximum possibilities of feedback are noted in a recent review [18]. The mentioned method is based on the method of recurrent target inequalities [19], multiple estimation of unknown parameters, including perturbation parameters, and the use of the control problem quality index as an ideal identification criterion. For general systems, this method is not directly implementable due to the high complexity of the task of calculating the current optimal evaluations under non-convex quality indices and non-convex constraints. That is why the task of searching the classes of objects or less ambitious problem formulations, for which this method is implementable (taking into account the growing power of modern computers), seems relevant. Examples of such less ambitious formulations are, for example, the problem of model validation and optimal perturbation quantification [20, 21] and the problem of synthesizing an optimal robust controller for an object with a known transfer function of the nominal object and unknown norms of the external perturbation and operator perturbations on the output and control [22]. In [23], there are examples of adaptive optimal control problems, in which the optimal values of quality indicators are linear or fractional-linear functions of the estimated unknown parameters. In such problems, the optimal estimation is reduced to linear programming and is implemented online at least for objects of low order.

In this paper, the adaptive optimal robust stabilization problem is considered for a relatively simple object with an autoregressive nominal model, unknown transfer function coefficients, unknown output and control uncertainty gain coefficients, and unknown norm and offset of an external bounded perturbation. The problem is to minimize the worst-case upper bound of the steady-state output modulus of the object in the class of uncertainties and perturbations under consideration. The purpose of this paper is to implement the above general method for synthesizing an adaptive optimal control for the above-described object. The main results of this work are as follows.

1. A special replacement of the unknown parameters of uncertainties and external perturbations is proposed, through which the control quality index in the considered problem becomes a fractional-linear function of the unknown parameters, and the sets of estimates of the unknown parameters consistent with the measurements are described by linear inequalities. The proposed replacement of the unknown parameters makes it possible to solve the optimal estimation problem in online mode.

2. The stability of the closed adaptive system in the optimal region of admissible uncertainty norms, which is universal for all a priori admissible nominal objects, is proved.

3. Under the additional assumption of "unintentionality" of the total perturbation, the optimality of adaptive control with a given accuracy is guaranteed, i.e. the implementation of the maximum feedback possibilities.

4. Numerical simulation results for an object with five unknown transfer function coefficients of the nominal model and four unknown perturbation parameters described above illustrate the performance and optimality of the adaptive control.

5. The problem of quality assessment of the model obtained as a result of identification is solved in the theory of systems identification in the online mode by calculating the optimal assessments with the best value of the quality index consistent with the measurement data and guaranteed in the steady-state mode. The above numerical simulation results clearly illustrate the unfairness

of the traditional criticism of identification by means of set-membership approach as a too crude method. This criticism is based on the traditional and seemingly obligatory assumption that a priori perturbation bounds, the same for all admissible objects, must be known.

6. The use of the growing computing power of modern computers makes it possible, at least for low-order objects, to compute online polyhedral estimates of unknown parameters consistent with measurement data and a priori information about the controlled system. This makes it possible to solve the problem of online verification of the model and/or a priori assumptions and their consistency with achieving the desired control quality. Traditional methods of synthesis of adaptive control based on gradient algorithms and modifications of the least squares method do not consider the problem of model verification and a priori assumptions.

Notation glossary:
$|\varphi|$—Eucledian norm of the vector $\varphi \in \mathbb{R}^n$;
ℓ_e—space of real sequences $x = (\cdots, x_{-1}, x_0, x_1, \cdots)$,
$x_s^t = (x_s, x_{s+1}, \ldots, x_t)$ for $x \in \ell_e$;
$|x_s^t| = \max_{s \leqslant k \leqslant t} |x_k|$;
ℓ_∞—normalized space of bounded real sequences $x = (x_0, x_1, x_2, \ldots)$ with the norm $\|x\| = \sup_t |x_t|$;
$\|x\|_{ss} = \limsup_{t \to +\infty} |x_t|$;
ℓ_1— normalized space of absolutely summarizable sequences with the norm $\|x\|_1 = \sum_{k=0}^{+\infty} |x_k|$;
$\|G\| = \sum_{k=0}^{+\infty} |g_k| = \|g\|_1$—induced norm of stable linear stationary system $G : \ell_\infty \to \ell_\infty$ with transfer function $G(\lambda) = \sum_{k=0}^{+\infty} g_k \lambda^k$.

2. PROBLEM FORMULATION

The control object with discrete time is described by the model

$$a(q^{-1})y_t = b_1 u_{t-1} + v_t, \quad t = 1, 2, 3, \ldots, \tag{2.1}$$

where $y_t \in \mathbb{R}$ is the measured object output at time t, $u_t \in \mathbb{R}$ is the control, $v_t \in \mathbb{R}$ is the total perturbation, q^{-1} is the ($q^{-1}y_t = y_{t-1}$) backward shift operator on the linear space ℓ_e,

$$a(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_n q^{-n}.$$

Initial values $y_{1-n}^0 = (y_{1-n}, \ldots, y_0)$ are arbitrary, $y_k = 0$ when $k < 1 - n$ and $u_k = 0$ when $k < 0$.

A priori information about the control object includes the following assumptions.

AP1. The column vector of the coefficients of the *nominal model* (i.e. the model without total perturbation v) belongs to the known bounded polyhedron Ξ,

$$\xi := (a_1, \ldots, a_n, b_1)^{\mathrm{T}} \in \Xi = \{ \, \hat{\xi} \mid P\hat{\xi} \geqslant p \, \} \subset \mathbb{R}^{n+1}, \quad P \in \mathbb{R}^{l \times (n+m)}, \quad p \in \mathbb{R}^l,$$

$b_1 \neq 0$ for each $\xi \in \Xi$.

AP2. The total perturbation v is

$$v_t = c^w + \delta^w w_t + \delta^y \Delta^1(y)_t + \delta^u \Delta^2(u)_t, \tag{2.2}$$

$$\|w\|_\infty \leqslant 1, \quad |\Delta^1(y)_t| \leqslant p_t^y = |y_{t-\mu}^{t-1}|, \quad |\Delta^2(u)_t| \leqslant p_t^u = |u_{t-\mu}^{t-1}|, \tag{2.3}$$

where c^w is the offset and δ^w is the norm (upper bound) of the external bounded perturbation $c^w + \delta^w w$, $w \in \ell_\infty$ is the unknown normalized external perturbation, $\delta^y \geqslant 0$ and $\delta^u \geqslant 0$ are the uncertainty amplification coefficients for the output and control, respectively, operators $\Delta^1 : \ell_e \to \ell_e$ and $\Delta^2 : \ell_e \to \ell_e$ are linear non-stationary or non-linear strictly causal operators with bounded memory μ (operator $\Delta : \ell_e \to \ell_e$ is called strictly causal if values z_t of the sequence $z = \Delta(x)$ only depend on $x_{-\infty}^{t-1}$ for all t [10]).

The uncertainty memory μ is chosen by the designer based on a priori information about the controlled system and can be chosen as large as desired, but not infinite, without compromising the quality of the adaptive control synthesized below (see Note 1 at the end of Section 3 and Note 3 in Section 6).

AP3. Vector $\theta = (\xi^{\mathrm{T}}, c^w, \delta^w, \delta^y, \delta^u)^{\mathrm{T}}$ of parameters of the (2.1) object is unknown, $|c^w| \leqslant C^w$ with the known upper boundary C^w.

The substantive formulation of the problem considered in the paper consists in constructing the inverse relation $u_t = U_t(y_{1-n}^t, u_0^{t-1}, \Xi)$, guaranteeing the lowest possible upper boundary for the asymptotic quality index

$$J_\mu(\theta) = \sup_v \limsup_{t\to+\infty} |y_t|, \tag{2.4}$$

where sup is taken on the set of perturbations v satisfying the assumption **AP2**. The inverse relation is subjected to the requirement of its computational feasibility in online mode, which is difficult to formalize in exact terms.

The main difficulty of the formulated optimal problem consists in the unidentifiability of the unknown vector θ in the deterministic formulation under consideration (see Section 4).

3. ROBUST QUALITY OF THE OPTIMAL SYSTEM WITH KNOWN PARAMETERS

For an object with a known vector of coefficients ξ and with a known displacement c^w, the regulator

$$u_t = \frac{1}{b_1}\Big[(a(q^{-1}) - 1)y_{t+1} - c^w\Big] = \frac{1}{b_1}\big[a_1 y_t + a_2 y_{t-1} + \ldots + a_n y_{t-n+1} - c^w\big] \tag{3.1}$$

guarantees, for all t, the equality

$$y_{t+1} = v_{t+1} - c^w = \delta^w w_{t+1} + \delta^y \Delta^1(y)_{t+1} + \delta^u \Delta^2(u)_{t+1}\,. \tag{3.2}$$

Because of the unpredictability and arbitrariness of the value $v_{t+1} - c^w$ at time t of the calculated control u_t, it follows that the regulator (3.1) is *optimal* for the quality index (2.4). Let us introduce notations for the transfer function from y to u of the regulator (3.1):

$$G^\xi(\lambda) = \frac{a(\lambda) - 1}{b_1 \lambda} = \frac{1}{b_1}\sum_{k=0}^{n-1} a_{k+1}\,\lambda^k, \quad \|G^\xi\| = \frac{1}{|b_1|}\sum_{k=1}^{n} |a_k|\,.$$

The closed system (2.1), (3.1) is called *robustly stable* in the uncertainty class (2.3) if the value of the quality index (2.4) is finite. The robust quality of the optimal system (2.1), (3.1) is described by the following theorem.

Theorem 1. *For the closed system* (2.1), (3.1) *the following statements are true:*

1. *The system is robustly stable at $\mu = +\infty$ if and only if*

$$\delta^y + \delta^u \|G^\xi\| < 1\,. \tag{3.3}$$

2. *If the robust stability condition* (3.3) *is satisfied, then*

$$J_\mu(\theta) \nearrow J(\theta) = \frac{\delta^w + \delta^u |c^w/b_1|}{1 - \delta^y - \delta^u \|G^\xi\|} \quad (\mu \to +\infty), \tag{3.4}$$

where the $\nearrow$ sign indicates monotonic convergence from below at $\mu \to +\infty$.

The proof of Theorem 1 is given in the Appendix.

AP4. The unknown parameter vector θ satisfies the inequality

$$\delta^y + \delta^u \|G^\xi\| \leqslant \bar{\delta} < 1 \tag{3.5}$$

with the known number $\bar{\delta}$.

The assumption of the known upper bound $\bar{\delta}$ is not restrictive. In essence, it consists in the a priori choice by the constructor of the value $\bar{\delta}$ as close to one as desired without affecting the control quality and excludes the models near the boundary of the region of robustly stabilizable objects that are unacceptable for practical applications from consideration. Replacing the open set of parameters θ characterized by the necessary condition of robust stabilizability (3.3) with a closed set defined by the inequality (3.5) as close to it as desired allows to formulate rigorous results about the control quality.

Remark 1. The basic results of the ℓ_1 theory of robust control referred to systems with structured uncertainty with infinite memory ($\mu = +\infty$) and only zero initial data [10] and therefore could not be applied to the problems of adaptive control. The second statement of Theorem 1 makes it possible to use the exponent $J(\theta)$ and the uncertainty model with bounded memory of the form (2.3) not only for the formulation and solution of adaptive optimal control problems without loss of control quality, but also for online verification of the object model, including quantification of uncertainties and external perturbation (i.e. estimation of uncertainty amplification coefficients and upper bound of external perturbation).

4. OPTIMAL ESTIMATION UNDER CONDITIONS OF NON-IDENTIFIABILITY

This section explains the non-identifiability of all unknown parameters and substantiates the necessity of using the quality index J as an identification criterion at the content level.

The following simple statement allows us to use the method of recurrent target inequalities to estimate the unknown parameter vector θ.

Statement 1. *If, for some estimation*

$$\hat{\theta} = (\hat{\xi}^{\mathrm{T}}, \hat{c}^w, \hat{\delta}^w, \hat{\delta}^y, \hat{\delta}^u)^{\mathrm{T}}, \quad \hat{\xi} \in \Xi, \quad \hat{\delta}^w \geqslant 0, \quad \hat{\delta}^y \geqslant 0, \quad \hat{\delta}^u \geqslant 0,$$

of an unknown vector θ for all t, the inequalities

$$|\hat{a}(q^{-1})y_t - \hat{b}_1 u_{t-1} - \hat{c}^w| \leqslant \hat{\delta}^w + \hat{\delta}^y p_t^y + \hat{\delta}^u p_t^u, \tag{4.1}$$

are valid, then the control object (2.1) *with parameter vector $\hat{\theta}$ satisfies Eq.* (2.1) *and a priori assumptions* **AP1** *and* **AP2** *for all t.*

The proof of Statement 1 is given in the Appendix.

It follows from Statement 1 that for any control of the object (2.1) the complete information about the vector of unknown parameters θ by the time t has the form of inclusion

$$\theta \in \Theta_t = \left\{ \hat{\theta} \in \Theta_0 \mid |\hat{a}(q^{-1})y_k - \hat{b}_1 u_{k-1} - \hat{c}^w| \leqslant \hat{\delta}^w + \hat{\delta}^y p_k^y + \hat{\delta}^u p_k^u \ \forall k \leqslant t \right\},$$

where

$$\Theta_0 = \left\{ \hat{\theta} = (\hat{\xi}^{\mathrm{T}}, \hat{c}^w, \hat{\delta}^w, \hat{\delta}^y, \hat{\delta}^u)^{\mathrm{T}} \mid \hat{\xi} \in \Xi,\ \hat{\delta}^w \geqslant 0,\ \hat{\delta}^y \geqslant 0,\ \hat{\delta}^u \geqslant 0,\ \hat{\delta}^y + \hat{\delta}^u \|G^{\hat{\xi}}\| \leqslant \bar{\delta} \right\}$$

is the a priori set of acceptable parameters.

The sets Θ_t consist of vectors $\hat{\theta} \in \Theta_0$ that satisfy the Eq. (2.1) and the a priori assumptions **AP1**, **AP2**, **AP4** for the available measurements y^t_{1-n} and u^{t-1}_0. Obviously, any vector $\hat{\theta}$ with a sufficiently large component $\hat{\delta}^w$ lies in Θ_t. In particular, it follows from this that the parameters ξ and c^w of the optimal regulator (3.1) are *unidentifiable* under any control of the object (2.1).

The method of recurrent target inequalities consists in constructing a convergent sequence of estimates θ_t satisfying the *target inequalities* (4.1) with sufficiently large t. However, unlike adaptive stabilization problems, this is not sufficient to solve the problem of ensuring as small an estimate as possible for the quality indicator (2.4). Indeed, if the estimates θ_t converge to some marginal estimate θ_∞ and the target inequalities are fulfilled, then from Theorem 1 and Statement 1 it follows that

$$\limsup_{t\to+\infty} |y_t| \leqslant J(\theta_\infty)\,.$$

To solve the set optimal problem of this inequality, it is not enough but necessary to ensure the fulfillment of the following ineqiality with a given accuracy

$$J(\theta_\infty) \leqslant J(\theta)$$

with the unknown and unidentifiable vector θ. It follows from this that it is necessary to use the quality index $J(\theta)$ of the control problem as an identification criterion, i.e. to use the optimal estimation

$$\theta_t = \underset{\hat{\theta}\in\Theta_t}{\operatorname{argmin}}\ J(\hat{\theta}) = \underset{\hat{\theta}\in\Theta_t}{\operatorname{argmin}}\ \frac{\hat{\delta}^w + \hat{\delta}^u|\hat{c}^w/\hat{b}_1|}{1-\hat{\delta}^y - \hat{\delta}^u\|G^{\hat{\xi}}\|}\,. \tag{4.2}$$

The direct online implementation of formula (4.2) is difficult because, first, the number of target inequalities in the description of the sets Θ_t can increase indefinitely and, second, the quality index J and the robust stabilizability condition (3.5) are non-convex. The first difficulty is overcome by using upper polyhedral approximations of Θ_t sets and introducing a dead zone when updating the estimates. The method of getting rid of non-convexity in the optimal estimation problem is described in the next section.

5. REDUCTION TO A MODEL WITH OUTPUT UNCERTAINTY

In this section, we describe the replacement of the unknown perturbation parameters to convert the quality index to a fractional-linear form and the non-convex robust stabilizability condition (3.5) to a relaxed linear condition by reducing it to a model with only output uncertainty.

Let the object (2.1) be controlled in such a way that for all t the inequalities

$$|u_t| \leqslant C_1 + C_2|y^t_{t-n+1}| \tag{5.1}$$

with some constants C_1, C_2. Then, from Eq. (2.1) and the assumption AP2, follows

$$|a(q^{-1})y_t - b_1u_{t-1} - c^w| \leqslant \delta^w + \delta^y|y^{t-1}_{t-\mu}| + \delta^u|u^{t-1}_{t-\mu}| \leqslant \delta^w + \delta^u C_1 + (\delta^y + \delta^u C_2)|y^{t-1}_{t-\mu-n}|, \tag{5.2}$$

and for every $\bar{\mu} \geqslant \mu + n$ we get

$$|a(q^{-1})y_t - b_1u_{t-1} - c^w| \leqslant \delta^w + \delta^u C_1 + (\delta^y + \delta^u C_2)|y^{t-1}_{t-\bar{\mu}}|\,. \tag{5.3}$$

Let us introduce new unknown parameters ζ, δ^e and δ:

$$\zeta = (\xi, c^w, \delta^e, \delta), \quad \delta^e = \delta^w + \delta^u C_1, \quad \delta = \delta^y + \delta^u C_2\,. \tag{5.4}$$

For these new parameters, inequalities (5.3) take the form of

$$|a(q^{-1})y_t - b_1u_{t-1} - c^w| \leqslant \delta^e + \delta|y_{t-\bar{\mu}}^{t-1}|\,. \tag{5.5}$$

Inequalities (5.5) are equivalent to inequalities (4.1) for the modified parameter vector

$$\theta^m = (\xi^{\mathrm{T}}, c^w, \delta^e, \delta, 0)^{\mathrm{T}}\,.$$

According to Statement 1, the inequalities (5.5) mean that if control satisfies the inequalities (5.1), the output y can be considered the output of object (2.1) with modified vector of parameters θ^m (without control uncertainty), and for this object, according to Theorem 1, we have

$$\limsup_{t\to+\infty} |y_t| \leqslant I(\zeta) := J(\theta^m) = \frac{\delta^e}{1-\delta}\,. \tag{5.6}$$

If the object (2.1) is controlled by the optimal regulator (3.1), then

$$|u_t + c^w/b_1| \leqslant \|G^\xi\||y_{t-n+1}^t|,$$

and, therefore,

$$|u_t| \leqslant |c^w/b_1| + \|G^\xi\||y_{t-n+1}^t\|\,. \tag{5.7}$$

Inequalities (5.7) guarantee inequalities (5.1) and (5.3) with constants $C_1 = |c^w/b_1|$ and $C_2 = \|G^\xi\|$, for which the parameters δ^e and δ from (5.4) are equal to

$$\delta^e = \delta^w + \delta^u|c^w/b_1|, \quad \delta = \delta^y + \delta^u\|G^\xi\|\,. \tag{5.8}$$

Then, from formula (3.4) for $J(\theta)$ and formula (5.6) for $I(\zeta)$ with parameters (5.8), it follows that

$$J(\theta) = I(\zeta), \quad \zeta = (\xi^{\mathrm{T}}, c^w, \delta^w + \delta^u|c^w/b_1|, \delta^y + \delta^u\|G^\xi\|)^{\mathrm{T}}\,. \tag{5.9}$$

Thus, for the imaginary system characterized by inequalities (5.5) without control uncertainty, the quality index becomes a fractional-linear function that depends only on the norm δ^e of the imaginary external perturbation and the gain δ of the imaginary output uncertainty.

6. ADAPTIVE CONTROL

Let us proceed to the description of the adaptive suboptimal control algorithm based on the use of new unknown parameters δ^e and δ. After applying the control u_t and measuring the output y_{t+1} at time $t+1$, we will update the vector estimates

$$\zeta_t = (\xi_t^{\mathrm{T}}, c_t^w, \delta_t^e, \delta_t)^{\mathrm{T}}$$

of the unknown vector ζ from (5.9) and polyhedral estimates Z_t composed of a priori constraints and several linear inequalities generated by the new target inequalities (5.5). Initial estimates Z_0 and ζ_0 have the form of

$$Z_0 = \left\{ \hat{\zeta} = (\hat{\xi}^{\mathrm{T}}, \hat{c}^w, \hat{\delta}^e, \hat{\delta})^{\mathrm{T}} \mid \hat{\xi} \in \Xi,\ \hat{\delta}^e \geqslant 0,\ 0 \leqslant \hat{\delta} \leqslant \bar{\delta} \right\}, \quad \zeta_0 = (\xi_0^{\mathrm{T}}, 0, 0, 0)^{\mathrm{T}},$$

where ξ_0 is any vector from the a priori polyhedron Ξ, $\bar{\delta}$ is the upper bound of the parameter δ from the assumption **AP4**.

Let us choose any number $\bar{\mu} \geqslant \mu + n$ of memorized outputs $y^t_{t-\bar{\mu}+1}$ and the parameter $\varepsilon > 0$ of the dead zone guaranteeing a finite number of updates of the estimates. The control u_t at time t is determined by the *adaptive regulator*

$$u_t = \frac{1}{b^t_1}\left(a^t_1 y_t + a^t_2 y_{t-1} + \ldots + a^t_n y_{t-n+1} - c^w_t\right). \tag{6.1}$$

The algorithm for updating the vector estimates ζ_t and the polyhedral estimates Z_t is as follows. After measuring the output y_{t+1} at the moment $t+1$, let us assume that

$$\varphi_t := (-y_t, -y_{t-1}, \ldots, -y_{t-n+1}, u_t)^T, \quad \eta_{t+1} := \operatorname{sgn}(y_{t+1} - \varphi^T_t \xi_t - c^w_t),$$
$$p_{t+1} = |y^t_{t-\bar{\mu}+1}|, \quad \psi_{t+1} := (\eta_{t+1}\varphi^T_t, \eta_{t+1}, 1, p_{t+1})^T, \quad \nu_{t+1} := \eta_{t+1} y_{t+1}.$$

In these notations, the adaptive regulator Eq. (6.1) is equivalent to the equation $\varphi^T_t \xi_t + c^w_t = 0$, so that $\eta_{t+1} = \operatorname{sgn}(y_{t+1})$ and the target inequality (5.5) at time $t+1$ for the current estimate ζ_t is equivalent to the inequality

$$\psi^T_{t+1}\zeta_t \geqslant \nu_{t+1}. \tag{6.2}$$

Let us assume that

$$\zeta_{t+1} := \zeta_t, \quad Z_{t+1} := Z_t, \quad \text{if} \;\; \psi^T_{t+1}\zeta_t \geqslant \nu_{t+1} - \varepsilon|\psi_{t+1}|. \tag{6.3}$$

Otherwise, let us assume that

$$Z_{t+1} := Z_t \cap \Omega_{t+1}, \quad \Omega_{t+1} := \left\{ \hat{\zeta} \mid \psi^T_{t+1}\hat{\zeta} \geqslant \nu_{t+1} \right\}, \tag{6.4}$$

$$\zeta_{t+1} := \underset{\hat{\zeta} \in Z_{t+1}}{\operatorname{argmin}} \; I(\hat{\zeta}), \tag{6.5}$$

where the quality index I is defined in (5.6).

The algorithm for updating the estimates has a simple geometric interpretation. According to formula (6.3), the estimates of Z_t and ζ_t are updated only when the distance from the vector ζ_t to the half-space Ω_{t+1} is greater than the parameter of the dead zone ε (see the proof of Theorem 2). According to formula (6.4) the update of Z_t consists in adding the linear inequality $\psi^T_{t+1}\hat{\zeta} \geqslant \nu_{t+1}$, which is the one of the two linear inequalities that make up the target inequality (5.5) that is violated for the estimation of ζ_t. Calculating the optimal estimate ζ_{t+1} according to (6.5) is a fractional-linear programming problem reduced to a linear programming problem by introducing an auxiliary variable [25].

Theorem 2. *Let the object* (2.1) *with the unknown parameter vector* $\theta = (\xi^{\mathrm{T}}, c^w, \delta^w, \delta^y, \delta^u)^{\mathrm{T}}$ *satisfy the assumptions* **AP1**–**AP4** *and be controlled by the adaptive regulator* (6.1) *with the evaluation algorithm* (6.3)–(6.5) *and with the dead zone parameter satisfying the inequalities*

$$0 < \varepsilon < (1 - \bar{\delta})/(\sqrt{n+1} + G_u), \quad G_u = \sup_{\xi \in \Xi} \|G^{\xi}\|. \tag{6.6}$$

Then the following statements are true:

1) *If the parameters* δ^y *and* δ^u *satisfy the inequality*

$$\delta^y + \delta^u G_u \leqslant \bar{\delta} < 1, \tag{6.7}$$

then multiple estimations Z_t and vector estimations ζ_t converge in a finite time, and

$$\limsup_{t\to+\infty} |y_t| \leqslant I(\zeta_\infty^\varepsilon) < I(\zeta_\infty) + K_{\zeta_\infty}\varepsilon, \tag{6.8}$$

$$I(\zeta_\infty) \leqslant \bar{I} = \frac{\delta^w + \delta^u \max_t |c_t^w/b_1^t|}{1 - \delta^y - \delta^u \max_t \|G^{\xi_t}\|} \leqslant \frac{\delta^w + \delta^u \max_t |c_t^w/b_1^t|}{1 - \delta^y - \delta^u G_u}, \tag{6.9}$$

where $\zeta_\infty = (\xi_\infty^{\mathrm{T}}, c_\infty^w, \delta_\infty^e, \delta_\infty)$ is the limit value of ζ_t,

$$\zeta_\infty^\varepsilon = \left(\xi_\infty^{\mathrm{T}},\, c_\infty^w, \delta_\infty^e + \varepsilon(\sqrt{2} + |c_\infty^w/b_1^\infty|),\, \delta_\infty + \varepsilon(\sqrt{n+1} + \|G^{\xi_\infty}\|)\right)^{\mathrm{T}}, \tag{6.10}$$

$$K_{\zeta_\infty} = \frac{\sqrt{2} + |c_\infty^w/b_1^\infty| + \delta_\infty^e(\sqrt{n+1} + \|G^{\xi_\infty}\|)}{(1 - \delta_\infty - \varepsilon(\sqrt{n+1} + \|G^{\xi_\infty}\|))^2}. \tag{6.11}$$

2) *If, for all t, the inequalities*

$$|u_t| \leqslant \bar{u}_t = |c^w/b_1| + \|G^\xi\| |y_{t-\bar{\mu}+1}^t|, \tag{6.12}$$

are true, then the multiple estimations Z_t and the vector estimations ζ_t converge in a finite time, and

$$\limsup_{t\to+\infty} |y_t| \leqslant I(\zeta_\infty^\varepsilon) < I(\zeta_\infty) + K_{\zeta_\infty}\varepsilon \leqslant J(\theta) + K_{\zeta_\infty}\varepsilon, \tag{6.13}$$

where $\zeta_\infty = (\xi_\infty^{\mathrm{T}}, c_\infty^w, \delta_\infty^e, \delta_\infty)$ is the limit value of ζ_t; ζ_∞^ε and K_{ζ_∞} are equal to (6.10) *and* (6.11), *respectively.*

The proof of Theorem 2 is given in the Appendix.

Remark 2. The first statement of Theorem 2 guarantees the stability of adaptive control in the narrowed set (6.7) of parameters (δ^y, δ^u) as compared to the set of parameters (3.5) corresponding to robustly stabilized objects with known nominal model parameters. The upper estimate $\bar{I}$ in (6.9) is highly overestimated compared to the upper estimates $I(\zeta_t^\varepsilon)$ computed during control and converging to measurements in finite time. Despite this, it is better than the a priori universal estimate for the entire class of admissible coefficient vectors Ξ that can be obtained for projective or least squares estimation algorithms and that is valid in the much larger two-dimensional domain of acceptable parameters (δ^y, δ^u).

Remark 3. The second statement of Theorem 2 is based on the condition (6.12). This condition is not verifiable by measurement data, since the parameters c^w, b_1 and ξ are unknown. The adaptive regulator (6.1) guarantees the validity of these inequalities for the current estimates c_t^w, b_1^t and ξ_t. However, due to the fact that, in the chain of inequalities (5.2) and (5.3), each of the inequalities is substantially coarsened (including the admissible choice of any $\bar{\mu} \geqslant \mu + n$), and given that the current optimal estimates ζ_t minimize the quality index I, inequalities (6.12) are not actually violated, as shown by numerous numerical experiments. The formal proof of this is complicated by the fact that, although the replacement of variables (5.4) allows one to move to a "good" fractional-linear quality index $I(\zeta)$ under linear target inequalities (5.5), the original non-linearity of the quality index $J(\theta)$ and the non-linearity of the robust stabilizability condition (3.3) are "hidden" when variables (5.4) are replaced by constants C_1 and C_2 in the additional condition (5.7).

Remark 4. The most important exclusive advantage of the considered method of synthesis of adaptive robust control is the verifiability of the a priori assumptions about the controlled object. An indicator of the acceptability of the a priori assumptions is the non-decreasing sequence of the smallest values $I(\zeta_t)$ consistent with the a priori assumptions and measurement data. At no point in time t is it known whether the current value of $I(\zeta_t)$ is the limiting value of $I(\zeta_\infty)$. But if the estimates of ζ_t do not change over a long period of time, this means that the current estimate satisfies the target inequalities and hence guarantees this best upper bound $I(\zeta_t)$ for $|y_t|$ after the decay of the transients. The invariance of the ζ_t estimate over a long period of time also guarantees that the a priori assumptions are consistent with the measured data at the current guaranteed asymptotic control quality $I(\zeta_t)$. Traditional methods for synthesizing adaptive control in both deterministic and stochastic formulations leave the problems of model verification and a priori assumptions out of consideration.

Remark 5. The formula (6.11) for the constant K_∞ is given for the sole purpose of explaining that the accuracy of the solution of the optimal problem is of the order of ε. It is easier to control this accuracy by direct calculation of the difference $I(\zeta_t^\varepsilon) - I(\zeta_t)$ and correct it, if necessary, by appropriate change of the dead zone parameter ε. Note, however, that as ε decreases, the number of possible updates of the estimates and, consequently, the number of inequalities in the description of Z_t estimates may increase exponentially with respect to the number of estimated parameters, since the volumes of the balls excluded from Z_t are proportional to $\varepsilon^{\dim \zeta}$. The question about the coarseness of the exponential estimator $\varepsilon^{\dim \zeta}$ remains open, given that the addition of new linear inequalities cuts off much larger sets of falsified parameters by measurements.

7. NUMERICAL MODELING

In this section, the results of numerical modeling of the adaptive suboptimal control described above are presented. The effectiveness of this control is illustrated by a comparison with adaptive control based on least squares estimation (LSE). The LSE algorithm underlies the stochastic theory of adaptive optimal control for systems with random external perturbations [26, 27]. However, attempts to generalize this theory to systems with uncertainty in general have been unsuccessful even for stochastic uncertainties.

In the case of the object (2.1), the least squares estimates $(\xi_t^{\mathrm{T}}, c_t^w)$ minimize the mean square of the unrelatedness (i.e., the centered total perturbation) of the model:

$$(\xi_t, c_t^w) = \operatorname*{argmin}_{\hat\xi \in \Xi, \hat c^w} \frac{1}{t} \sum_{k=1}^{t} \left(\hat a(q^{-1}) y_{t+1} - \hat b(q^{-1}) u_t - \hat c^w \right)^2, \tag{7.1}$$

without considering fundamentally different effects of the external perturbation w and uncertainties Δ^1 and Δ^2 on the system dynamics, and so cannot form a basis for estimation algorithms oriented at minimizing the upper bound of the object's output. The least squares estimates are calculated using the recurrence formulas

$$\xi_{t+1} = \xi_t + K_t(y_{t+1} - \xi_t^{\mathrm{T}} \varphi_t), \quad K_t = \frac{P_t \varphi_t}{1 + \varphi_t^{\mathrm{T}} P_t \varphi}, \quad P_{t+1} = (I - K_t \varphi_t^{\mathrm{T}}) P_t$$

with the addition of a projection of the estimates ξ_t onto the a priori set Ξ. When modeling adaptive control with initial matrices $P_0 = cI$, where I is a unit matrix and $c > 0$, relatively better average results for LSE were observed when $c \in [1, 5]$. The results for $c = 1$ are given below.

Numerical modeling is illustrated by the example of the object (2.1) with unstable poles, i.e. by the roots $a(\lambda)$, 0.9 , 0.8, $0.7 \pm 0.6\,i$ and the coefficient $b_1 = 2$. These parameters correspond (with an accuracy of 10^{-4}) to the vector of coefficients

$$\xi = (-4.0082,\ 6.4542,\ -5.0654,\ 1.634,\ 2)^\top \in \mathbb{R}^5 .$$

A priori restrictions: $0.1 \leqslant b_1 \leqslant 10$, $|a_k| \leqslant 20$ for all k, $|c^w| \leqslant 100$. The dead zone parameter ε in (6.3) is equal to 0.001.

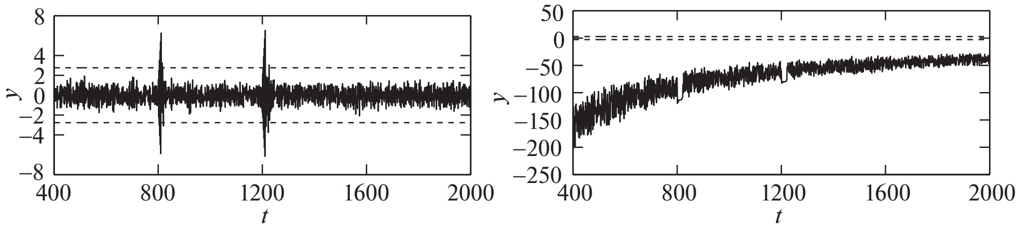

Fig. 1. Typical outputs of y_t under the LSE algorithm and random perturbations.

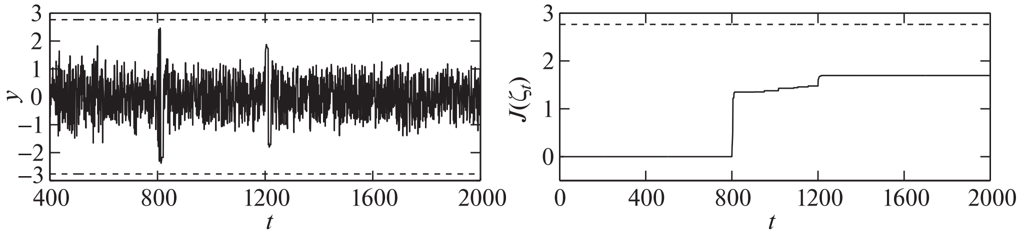

Fig. 2. Left—output of y_t under optimal estimation and random perturbations; right—plot of $I(\zeta_t)$.

The modeling results are given below for random perturbations and deterministic perturbations in the form of

$$v_t = c^w + \cos(5t) + \delta^y \sin(3\sqrt{t})\,|y_{t-\mu}^{t-1}| + \delta^u \sin(\ln(0.3t + \pi/2))\,|u_{t-\mu}^{t-1}| \tag{7.2}$$

with parameters $c^w = 1,\ \delta^y = 0.2,\ \delta^u = 0.05,\ \mu = 20$. "Exotic" perturbations of the form (7.2) with non-stationary frequencies were chosen to illustrate possible "bad" dynamics of a closed-loop adaptive system with LSE algorithm.

In all numerical experiments, the initial data y_{1-n}^0 was chosen randomly (with uniform distribution) from the interval $[-\delta^w, \delta^w]$ and "locally bad" perturbations v_{t+1} maximizing $|y_{t+1}|$ at time intervals [800, 810] and [1200, 1210] were modeled to illustrate the quality of steady-state estimates. The optimal interval $[-J(\theta, J(\theta)]$ is denoted in all of the following plots of the y_t output by dashed lines with ordinates $\pm J(\theta)$.

In experiments with independent random perturbations $w_t,\ \Delta^1(y)_t,\ \Delta^2(u)_t$, uniformly distributed at their respective intervals, the adaptive control based on LSE on most implementations of initial data y_{1-n}^0 demonstrates the required inequality $|y_t| \leqslant I(\zeta) = J(\theta) \approx 2.76$ at relatively small offsets of the mean output value from zero in steady-state mode. However, quite often there are bursts with outliers beyond the optimal interval due to the maximizing $|y_{t+1}|$ perturbation v_{t+1} at the time intervals mentioned above. This effect is shown in the left plot of Fig. 1. It is also quite common for the output of y_t to fall outside the optimal interval. It is known that the external perturbation offset presents a great difficulty for LSE due to insufficient excitation of the corresponding estimate of the component (equal to 1) of the regression vector. In the numerical experiment shown in the right plot of Fig. 1, the steady-state output of y_t exceeds the optimal upper bound by more than one order of magnitude. For comparison, the plot of the output of the adaptive control with the optimal estimation algorithm (6.3)–(6.5) under the same implementations of all random variables is presented in the left plot of Fig. 2, illustrating the optimality of the adaptive control. The right plot of Fig. 2 illustrates the exceptional advantage of the optimal algorithm in the form of a plot of the current optimal values of $I(\zeta_t)$ consistent with the current measurement data (y_{1-n}^t, u_1^{t-1}) and satisfying the inequalities $I(\zeta_t) \leqslant J(\theta)$. We emphasize that the actual guaranteed control quality is markedly better than the optimal value of $J(\theta)$ when the

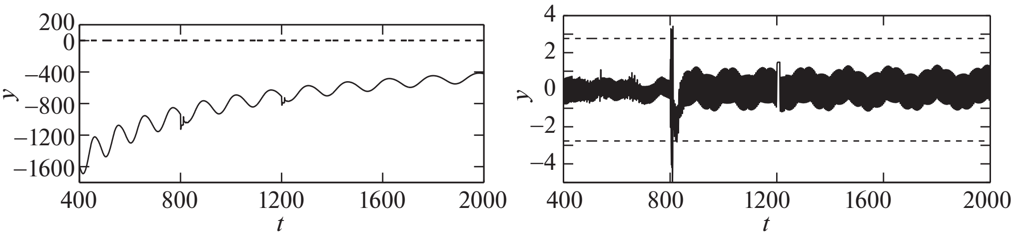

Fig. 3. Plots of the output of y_t under LSE (left) and optimal (right) estimation algorithms and deterministic perturbations (7.2).

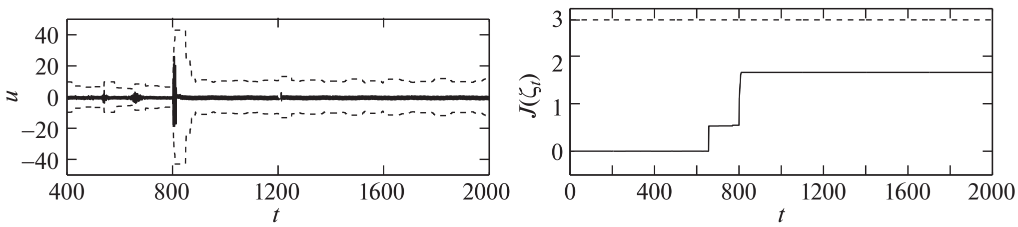

Fig. 4. Left—plots of u_t and $\pm\bar{u}_t$; right—plot of $I(\zeta_t)$.

simulated perturbations are not actually worst-case. These plots illustrate the unfairness of the traditional criticism of identification by means of multiple-valuation (set-membership approach) as a too crude method.

Figure 3 presents plots of the output of y_t using LSE (left) and optimal (right) estimation algorithms with deterministic perturbations (7.2) and the same initial data y_{1-n}^0. The left plot illustrates that the output y_t exceeds the optimal upper bound $J(\theta) \approx 2.76$ by two orders of magnitude. The right plot of Fig. 3 shows an infrequently observed small burst of the adaptive optimal control output beyond the optimal interval $[-J(\theta, J(\theta)]$.

The left plot in Fig. 4 illustrates the fulfillment of the inequalities (6.12) (due to large bursts in the initial period, only the steady-state mode is presented). The plot u_t (solid line) at all t is located in the tube $[-\bar{u}_t, \bar{u}_t]$ bounded by the dashed lines. Note that numerous experiments have failed to find an example of perturbations in which the inequalities (6.12) are violated. The right plot illustrates the fulfillment of the inequalities $I(\zeta_t) \leqslant J(\theta)$ guaranteeing the optimality of the adaptive control.

It is known that transients in stable linear stationary systems with and without limited external perturbations can be accompanied by significant bursts under unfavorable initial data [28, 29]. It should be noted that due to the "integral" nature of the identification criterion (7.1), LSE, as a rule, generates significantly smaller bursts at the initial time interval. The optimal algorithm (6.3)–(6.5) attributes large bursts to the presence of uncertainty with δ_t estimates close to the upper bound of 0.9 for quite a long time. This is illustrated by the right-hand plots in Figs. 2 and 4, where, for quite a long time, $I(\zeta_t) = 0$ (which is equivalent to estimates of $\delta_t^w = 0$). Numerical experiments have shown that using the LSE algorithm instead of the optimal estimation (6.3)–(6.5) for initial time segments from $2 \dim \zeta$ to $10 \dim \zeta$ in most cases improves the transients in the adaptive optimal system.

The time to model one experiment on a laptop computer with a 4xIntelCore i5-7200U processor CPU@2.50GHz was less than 0.2 s for LSE estimation and around 2 s for optimal estimation. The number of estimation updates on the interval [1, 2000] was in the range of 50–75 and increased slightly on the interval $[1, 10^5]$. These figures illustrate the performance of the online optimal estimation algorithm on the example of a system with ten unknowns and nine adjustable parameters, of which 3 parameters (norms of external bounded perturbation and output and control uncertainties) are not estimated within traditional methods of adaptive control synthesis.

8. CONCLUSION

According to [30], "the main control problem for a given process can be formulated as follows: having some a priori information about the process and a finite set of measurements, it is required to construct a feedback regulator that provides a given control quality." A more ambitious variant of this problem is the requirement for asymptotic control optimality due to specification of information about the controlled process. Many practical control problems are formulated in terms of tolerances on deviations of control system outputs from the set values. Such problems correspond to the basic signal space of ℓ_∞ bounded sequences and its corresponding theory of robust control in the ℓ_1 formulation. The present work is devoted to the solution for an object with autoregressive nominal model of the problem of application of ℓ_1 theory in the standard for applications case when a priori information about the parameters of nominal model and levels of disturbances and uncertainties is insufficient for the a priori synthesis of the controller and the missing information has to be extracted from the current measurement data y_{1-n}^t, u_0^{t-1}. Under clearly stated a priori assumptions, a solution for the optimal stabilization problem under conditions of strong a priori uncertainty and non-identifiability of the unknown parameters, based on the multiple estimation of the unknown parameters and using the quality index of the control problem as an identification criterion, is proposed.

APPENDIX

Proof of Theorem 1. The robust stability condition (3.3) follows from Theorem 7 [24] applied to system (2.1), (3.1). To prove the second statement of the Theorem, it is sufficient to apply Theorems 5 and 6 [24] (see also [15]). To do this, we have to present the system (2.1), (3.1) in the standard M–Δ form given in Fig. 5 and having a block form.

$$\begin{pmatrix} y \\ z \end{pmatrix} = M \begin{pmatrix} r \\ w \\ \xi \end{pmatrix} = \begin{pmatrix} M_{yr} & M_{yw} & M_{y\xi} \\ M_{zr} & M_{zw} & M_{z\xi} \end{pmatrix} \begin{pmatrix} r \\ w \\ \xi \end{pmatrix}, \quad \xi = \Delta z. \tag{A.1}$$

For the system (2.1), (3.1) the signal $r = c^w \mathbf{1}$, $\mathbf{1} = (1, 1, \ldots)$, and M–Δ form looks like

$$\begin{pmatrix} y \\ z^1 \\ z^2 \end{pmatrix} = M \begin{pmatrix} \mathbf{1} \\ w \\ \xi^1 \\ \xi^2 \end{pmatrix} = \begin{pmatrix} 0 & \delta^w & \delta^y & \delta^u \\ 0 & \delta^w & \delta^y & \delta^u \\ -\dfrac{c^w q}{b_1} & \delta^w G^\xi(q^{-1}) & \delta^y G^\xi(q^{-1}) & \delta^u G^\xi(q^{-1}) \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ w \\ \xi^1 \\ \xi^2 \end{pmatrix}, \tag{A.2}$$

where

$$z_t = \begin{pmatrix} y_t \\ u_t \end{pmatrix}, \quad \xi = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = \begin{pmatrix} \Delta^1 & 0 \\ 0 & \Delta^2 \end{pmatrix} z = \begin{pmatrix} \Delta^1(y) \\ \Delta^2(u) \end{pmatrix}.$$

The first and second lines of the matrix M in (A.2) correspond to Eq. (3.2). The third row of M corresponds to the representation of the optimal regulator (3.1) in the form of

$$u_t = -c^w/b_1 + G^\xi(q^{-1}) y_t = -c^w/b_1 + \delta^w G^\xi(q^{-1}) w_t + \delta^y G^\xi(q^{-1}) \xi_t^1 + \delta^u G^\xi(q^{-1}) \xi_t^2 .$$

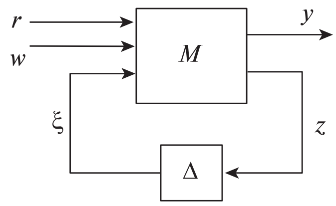

Fig. 5. M–Δ form of the system (2.1), (3.1).

The formula for $J(\theta)$ in (3.4) corresponds to the quality index (2.4), in which sup is taken on the perturbation set v with uncertainties Δ^1 and Δ^1 with finite memory (see [12]), and is derived by Theorem 5 [24] as follows. Let us assume that $\|z\|_{ss} = (\|z^1\|_{ss}, \cdots, \|z^p\|_{ss})^{\mathrm{T}}$ for the vector sequence $z \in \ell_e^p$, and

$$[M]_1 := \begin{pmatrix} \|M_{11}\|_1 & \cdots & \|M_{1p}\|_1 \\ \vdots & \vdots & \vdots \\ \|M_{q1}\|_1 & \cdots & \|M_{qp}\|_1 \end{pmatrix}$$

for a stable $q \times p$ response matrix M of impulses $M_{ij} \in \ell_1$. For the matrix M from (A.1) we will assume that

$$M_{ss}(r) := \begin{pmatrix} [M_{yr}r]_{ss} + [M_{yw}]_1 & [M_{y\xi}]_1 \\ [M_{zr}r]_{ss} + [M_{zw}]_1 & [M_{z\xi}]_1 \end{pmatrix}.$$

According to Theorem 5 from [24]

$$J(\theta) = [M_{yr}r]_{ss} + [M_{yw}]_1 + [M_{y\xi}]_1(I - [M_{z\xi}]_1)^{-1}([M_{zr}r]_{ss} + [M_{zw}]_1).$$

Then, for the system (A.2), we have

$$J(\theta) = \delta^w + (\delta^y \;\; \delta^u)\left(I - \begin{pmatrix} \delta^y & \delta^u \\ \delta^y\|G^\xi\| & \delta^u\|G^\xi\| \end{pmatrix}\right)^{-1} \begin{pmatrix} \delta^w \\ |c^w|\|\frac{-q}{b_1}r\|_{ss} + \delta^w\|G^\xi\| \end{pmatrix}$$

$$= \delta^w + (\delta^y \;\; \delta^u) \begin{pmatrix} 1-\delta^y & -\delta^u \\ -\delta^y\|G^\xi\| & 1-\delta^u\|G^\xi\| \end{pmatrix}^{-1} \begin{pmatrix} \delta^w \\ |c^w/b_1| + \delta^w\|G^\xi\| \end{pmatrix}$$

$$= \delta^w + \frac{1}{1-\delta^y-\delta^u\|G^\xi\|}(\delta^y \;\; \delta^u) \begin{pmatrix} 1-\delta^u\|G^\xi\| & \delta^u \\ \delta^y\|G^\xi\| & 1-\delta^y \end{pmatrix} \begin{pmatrix} \delta^w \\ |c^w/b_1| + \delta^w\|G^\xi\| \end{pmatrix}$$

$$= \delta^w + \frac{1}{1-\delta^y-\delta^u\|G^\xi\|}\;(\delta^y \;\; \delta^u) \begin{pmatrix} \delta^w + \delta^u|c^w/b_1| \\ (1-\delta^y)|c^w/b_1| + \delta^w\|G^\xi\| \end{pmatrix}$$

$$= \delta^w + \frac{\delta^y\delta^w + \delta^u|c^w|\|G^\xi\| + \delta^u\delta^w\|G^\xi\|}{1-\delta^y-\delta^u\|G^\xi\|} = \frac{\delta^w + \delta^u|c^w/b_1|)}{1-\delta^y-\delta^u\|G^\xi\|}.$$

Finally, the monotonic convergence of $J_\mu(\theta)$ to $J(\theta)$ in (3.4) is guaranteed by Theorem 6 from [24].

Proof of Statement 1. The vector $\hat{\theta}$ satisfies the a priori assumption **AP1** due to the conditions of Statement 1. For all $t > 0$, let us assume $\hat{v}_t = \hat{a}(q^{-1})y_t - \hat{b}_1 u_{t-1}$. Then the control object with the parameter vector $\hat{\theta}$ and the total perturbation $\hat{v}$ satisfies Eq. (2.1), and due to (4.1) the perturbation $\hat{v}$ satisfies the inequalities

$$|\hat{v}_t - \hat{c}^w| \leqslant \hat{\delta}^w + \hat{\delta}^y p_t^y + \hat{\delta}^u p_t^u .$$

The values of $\hat{v}_t$ can be represented in the form of (2.2) by choosing suitable values of w_t, $\Delta^1(y)_t$, $\Delta^2(u)_t$ that satisfy the inequalities (2.3), and thereby ensure that the a priori assumption of **AP2** is true.

Proof of Theorem 2. Let us prove that for each update of the estimates, the distance from ζ_t to the half-space Ω_{t+1} is greater than ε. Since ζ_t only changes when $\psi_{t+1}^T\zeta_t < \nu_{t+1} - \varepsilon|\psi_{t+1}|$ and $\psi_{t+1}^T\hat{\zeta} \geqslant \nu_{t+1}$ for all $\hat{\zeta} \in \Omega_{t+1}$, then

$$\varepsilon|\psi_{t+1}| < |\psi_{t+1}^{\mathrm{T}}(\hat{\zeta} - \zeta_t)| \leqslant |\psi_{t+1}||\hat{\zeta} - \zeta_t|$$

and, therefore, $|\hat{\zeta} - \zeta_t| > \varepsilon$ for all $\hat{\zeta} \in \Omega_{t+1}$. Thus, after adding the inequality $\psi_{t+1}^T\hat{\zeta} \geqslant \nu_{t+1}$ describing the half-space Ω_{t+1} to the description of Z_t, the polyhedron Z_{t+1} and all subsequent ones do not intersect the neigborhood of ε of the vector $\zeta_t \in Z_t$. It follows that the $\varepsilon/2$-neigborhoods of the various estimates of ζ_t do not intersect each other. Since $Z_{t+1} \subset Z_t$ for all t, the number of changes in the estimates of Z_t and ζ_t will be finite if the estimates of ζ_t lie in a bounded set. From the equation of the adaptive regulator (6.1) for all t we have

$$|u_t| \leqslant |c_t^w/b_1^t| + \|G^{\xi_t}\|\|y_{t-n+1}^t\| .$$

Then, for the object (2.1) on the time interval $[0, t]$, the inequalities (5.5) with the parameters

$$\tilde{\delta}_t^e = \delta^w + \delta^u \max_{s\leqslant t} |c_s^w/b_1^s|, \quad \tilde{\delta}_t = \delta^y + \delta^u \max_{s\leqslant t} \|G^{\xi_s}\|$$

are true. Therefore, $\tilde{\zeta}_t = (\xi^{\mathrm{T}}, c^w, \tilde{\delta}_t^e, \tilde{\delta}_t)^{\mathrm{T}} \in Z_t$ for all t. If the assumption (6.7) is satisfied, then $I(\tilde{\zeta}_t) \leqslant \bar{I}$, where $\bar{I}$ is defined in (6.9) (with the right-hand inequality in (6.9) obviously followed from the definition of G_u in (6.6)). From (6.5) for all t, it follows that

$$I(\zeta_t) \leqslant I(\tilde{\zeta}_t)$$

and then $I(\zeta_t) \leqslant \bar{I}$. From the boundedness of $I(\zeta_t)$, there follows the boundedness of the estimates ζ_t and thus the finiteness of the number of updates of the estimates ξ_t and Z_t. Then $\zeta_t = \zeta_\infty = (\xi_\infty^{\mathrm{T}}, c_\infty^w, \delta_\infty^e, \delta_\infty)$ from some point of time t_∞ and

$$\psi_{t+1}^T\zeta_\infty \geqslant \nu_{t+1} - \varepsilon|\psi_{t+1}| \quad \forall t \geqslant t_\infty . \tag{A.3}$$

From (A.3), it follows that, for all $t \geqslant t_\infty$,

$$\begin{aligned} |a_\infty(q^{-1})y_{t+1} - b_\infty(q^{-1})u_t - c_\infty^w| &\leqslant \delta_\infty^e + \delta_\infty p_{t+1} + \varepsilon|\psi_{t+1}| \\ &\leqslant \delta_\infty^e + \delta_\infty p_{t+1} + \varepsilon(\sqrt{n+1}\, p_{t+1} + \sqrt{2} + |u_t|) \\ &\leqslant \delta_\infty^e + \varepsilon(\sqrt{2} + |c_\infty^w/b_1^\infty|) + [\delta_\infty + \varepsilon(\sqrt{n+1} + \|G^{\xi_\infty}\|)]p_{t+1} . \end{aligned}$$

Given Statement 1 of Section 4, it follows from the obtained inequality that the output of y at all $t \geqslant t_*$ satisfies the Eq. (2.1) with the parameter vector ζ_∞^ε of the form (6.10). Then Theorem 1 guarantees the left-hand inequality in (6.8). To prove the right-hand inequality in (6.8), we estimate the difference $I(\zeta_\infty^\varepsilon) - I(\zeta_\infty)$ from above using the inequality

$$\frac{C_1 + \varepsilon_1}{C_2 - \varepsilon_2} - \frac{C_1}{C_2} = \frac{C_2\varepsilon_1 + C_1\varepsilon_2}{C_2(C_2 - \varepsilon_2)} < \frac{\varepsilon_1 + C_1\varepsilon_2}{(C_2 - \varepsilon_2)^2}$$

with the parameters $C_1 = \delta^e_\infty$, $C_2 = 1 - \delta_\infty \leqslant 1$, $\varepsilon_1 = \varepsilon(\sqrt{2} + |c^w_\infty / b^\infty_1|)$, $\varepsilon_2 = \varepsilon(\sqrt{n+1} + \|G^{\xi_\infty}\|)$. Then

$$I(\zeta^\varepsilon_\infty) - I(\zeta_\infty) < \frac{\sqrt{2} + |c^w_\infty / b^\infty_1| + \delta^e_\infty(\sqrt{n+1} + \|G^{\xi_\infty}\|)}{(1 - \delta_\infty - \varepsilon(\sqrt{n+1} + \|G^{\xi_\infty}\|))^2}\,\varepsilon$$

and, therefore, K_{ζ_∞} has the form of (6.11). The first statement of Theorem 2 is proved.

Let us prove the second statement. Now let the inequalities (6.12) be satisfied in the closed adaptive system. Then inequalities (5.3) with constants $C_1 = |c^w/b_1|$ and $C_2 = \|G^\xi\|$ follow from the object Eq. (2.1). This means that for the unknown parameter vector ζ defined in (5.9), the target inequalities (5.5) with the parameters δ^e, δ of the form (5.8) and inclusion $\zeta \in Z_t$ are satisfied for all t. Then, due to the choice of optimal estimates ζ_t according to (6.5), at all t,

$$I(\zeta_t) \leqslant I(\zeta) = J(\theta),$$

where the equality $I(\zeta) = J(\theta)$ is established in (5.9). Hence, as in the first statement of Theorem 2, there follows the convergence of the estimates ξ_t and Z_t in finite time and the inequalities (6.13).

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