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A Survey of the Latest Advances in Oligopoly Games

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Abstract—One of the most important problems of game theory—the game of firms in an oligopoly market—is considered. The survey covers classical and modern formulations for the game-theoretic problem of choosing optimal player's strategies and the recent methodological achievements in oligopoly games with applications, including publications over the past five years.

Keywords: oligopoly, aggregative game, reflexion, conjectural variation

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1. INTRODUCTION

Game theory had not yet become a separate science by 1838 when a game model was used to analyze an oligopoly [1]. In the middle of the 20th century, the formation of game theory was based on oligopoly as one of the first objects of study [2, 3]. There could be several reasons as follows. First, oligopoly is a really observable game, unlike many ephemeral objects with psychological overtones, such as the Prisoners' Dilemma, where payoffs are not measurable but evaluated subjectively. Second, oligopoly is productive for the development of game theory because it represents a market with large circulating capitals; as a consequence, the interests of many individuals are affected through equity, and the choice of an optimal strategy in such a game has practical significance for the entire society. Third, oligopoly is one of the few game situations predetermined by the objective laws of economics, therefore requiring mathematical modeling and analysis.

Presently, it seems topical to generalize modern experience in solving game-theoretic problems of oligopoly: the last surveys of oligopoly models date back to 2009 in the domestic literature [4] and to 2020 in the foreign literature [5]. However, foreign surveys do not mention the achievements of Russian researchers, who have significantly contributed to the development of oligopoly theory in recent years. To be as up-to-date as possible, this survey will cover the publications of the last five years, except for the fundamental concepts. Furthermore, it will highlight the most outstanding results from the author's point of view, which significantly advance the theory and applications of oligopoly games.

As a real-world object, an oligopoly is a market in which a relatively small number of firms sell an identical product. They compete by prices or supply volumes (outputs, quantities) for the choice of an audience of buyers that is incommensurably greater than the set of sellers. (Price competition is equivalent to quantity competition due to their relationship through a decreasing demand function.) Oligopolistic competition occurs when the number of sellers is so small that a change in the quantity (price) of each firm results in a noticeable shift along the total demand curve, affecting the results (profits) of all firms. As a theoretical model, oligopoly is a game in which players, i.e., cost-differentiated selling firms, seek to maximize their payoffs (utilities) that nonmonotonically depend on the quantity due to the synchronous decrease of the common price (aggregate) for all players under nondecreasing costs.

In the coordinate system of game models, the oligopoly game belongs, first, to noncooperative games and, second, to aggregative games in which the payoff of each player depends on the actions of all players. Third, the corpus of oligopoly studies is divided into two roughly equal parts, one formulating the game as nonhierarchical (i.e., the players are assumed to be equal) and the other as hierarchical (i.e., there is a leader among the players). Fourth, oligopoly refers to games with complete information. Due to asymmetric awareness, this aspect needs detailed discussion.

As a rule, in an oligopoly game, the utility functions of players are considered common knowledge. However, each player is a priori unaware of his action supposed by his environment in response to the latter's actions. Of course, it is always possible to compute the best response from the utility function. Meanwhile, if the player believes that everyone else also responds optimally he will compute the best response to the best response and so on. Consequently, the players' response functions always contain a component characterizing the response of the environment; this component is a priori unknown to the player and leads to asymmetric awareness. Formally, this asymmetry is expressed in an infinite sequence of conjectural variations, i.e., the player's supposed changes in the actions of other players in response to his unit action change.

The classical ways to solve the asymmetric awareness problem of players are based on some hypothesis about the behavior of the environment. Modern studies quite often proceed from the following hypotheses:

1) Cournot's hypothesis [1], which neglects the influence of the actions of the environment on the choice of a given player;

2) Stackelberg's hypothesis [6], which states that one player (leader) is aware of that the environment (followers) ignores his actions in accordance with Cournot's hypothesis.[1]

Based on these hypotheses, conjectural variations are computed unambiguously and assumed to be given during analysis. The researchers thereby postulate that the players' factual actions coincide with the conjectured ones. In this case, the vector of players' actions, which is a Nash solution of the game, is regarded as a real resulting equilibrium. Of course, it requires verification, and this problem is an inevitable consequence of the classical approach.

Another way to solve the asymmetric awareness problem is to analyze the mental types of players or reflexive dynamics. The theory of reflexive games is based on the ideas of D.A. Novikov and A.G. Chkhartishvili [7], the pioneers of this approach, as well as on the theoretical foundations of reflexion defined by V.A. Lefebvre [8, 9]. Briefly, the main and most fruitful idea of reflexive analysis can be expressed as follows: players make various assumptions about the actions of their environment, i.e., they play with not real but imaginary (phantom) opponents. Therefore, by describing the set of all possible beliefs of all players, one can calculate the complete set of equilibria (called informational equilibria, i.e., equilibria under some combination of the mental types of players). Therefore, verifying the assumptions (consequently, assessing the real mental types of players) comes to choosing the equilibrium closest to the real state of the market. Hence, the analysis of informational equilibria is the basis for developing informational control algorithms. Informational control is the global goal of creating a variety of game-theoretic models of oligopoly.

Finally, note the following aspect concerning the classification attributes of the oligopoly game. Relating this game to different classes, static and dynamic models are rather modeling methods. Dynamic models describe the competition process of oligopolists as their multi-period actions based on the best responses while static models characterize the result of this process based on the

[1] In the English literature, Stackelberg game is synonymous with hierarchical game. However, in a game with multiple leaders, the hierarchy becomes ambiguous.

same best responses; in both cases, the best response implicitly contains conjectural variations. Therefore, in what follows, a basic model of the oligopoly game is introduced. For this model, different ways to consider market realities as a game (different modifications) are analyzed, and dynamic and static solution methods are discussed.

2. THE BASIC MODEL

The game-theoretic concept of an undifferentiated oligopoly with quantity competition rests on the following basic assumptions.

1. Competition: the game involves a relatively small number of players so that the actions (i.e., quantities) of each player affect the utility of all other players (i.e., the environment). The players offer an identical product demanded by an infinite number of buyers, and the demand function is monotonically decreasing.

2. Rationality: individual utilities depend on the actions of the players as concave functions (i.e., nondecreasing cost functions), and each player chooses actions by maximizing his utility function based on the available information about the actions of his environment.

3. Awareness: at the time of choosing his actions, each player knows the utility functions of his environment (i.e., the demand and cost functions), the number of players, and that the environment has the same (equal) awareness level.

4. Action times:

a) In a static game, the players choose their actions simultaneously, once, and independently.

b) In a dynamic game, each player chooses his action repeatedly and depending on the environment's actions manifested earlier.

Under assumptions 1–3, the action choice model of player i can be written as

$$\max_{Q_i \geqslant 0} \Pi_i(Q, Q_i) = P(Q)Q_i - C_i(Q_i), \quad i \in N = \{1, \ldots, n\}, \tag{1}$$

$$Q = \sum_{i \in N} Q_i, \tag{2}$$

with the following notations: Q_i and Π_i are the action and utility function of player i, respectively; Q is the aggregate action; N is the set of players; n is the total number of players; $P(Q)$ is the inverse demand function, $P'_Q < 0$; finally, $C_i(Q_i)$ is the cost function of player i, $C'_{Q_i} \geqslant 0$.

Formally, an oligopoly game Γ is a tuple of the set of players, the set of their actions, and the set of their utility functions:

$$\Gamma = \langle N, \{Q_i, i \in N\}, \{\Pi_i, i \in N\} \rangle. \tag{3}$$

The necessary condition for the existence of a Nash equilibrium solution of the game (3) is the concave utility function of the players [10]. However, a less stringent condition imposed on the demand functions was proved in [11]: the marginal revenue of each player decreases with increasing the action of any player from the environment, i.e.,

$$P'_Q + P''_{QQ} Q < 0. \tag{4}$$

In this case, the Nash equilibrium is determined by solving the following system of reaction equations with a given vector of conjectural variations:

$$\frac{\partial \Pi_i(Q_i^*, \rho_{ij})}{\partial Q_i} = 0, \quad i, j \in N, \tag{5}$$

where $\rho_{ij} = Q'_{jQ_i}$ denotes the conjectural variation[2] of player i, i.e., the supposed change in the quantity of player j in response to the unit quantity increase of player i, and Q_i^* is the equilibrium value. In oligopoly theory, it is customary to consider the optimal (also termed consistent in the foreign literature) conjectural variation computed from Eq. (5) of player j. In other words, this variation corresponds to the best response of player j.

Thus, the computation of equilibrium in the game directly depends on the ability to find conjectural variations. In turn, this ability is predetermined by the peculiarities of the functions $P(Q)$ and $C_i(Q_i)$ that lead to different modifications of the basic model. Whenever the player's serial number does not matter, it will be omitted below: the player's action will be denoted by $q = Q_i\ \forall i \in N$ and his utility by $\pi = \Pi_i \forall i \in N$.

3. MODIFICATIONS OF THE BASIC MODEL

The monotonically decreasing inverse demand function is modeled using the functions given in Table 1. Nondecreasing cost functions are described by the models presented in Table 2. Condition (4) holds for the linear, combined, and exponential demand functions; for the power demand function, the existence of equilibrium can also be ensured by choosing nondecreasing cost functions. The power cost function can be either convex or concave for different degrees; a concave cost function corresponds to the positive scale effect whereas a convex cost function to the negative scale effect. A combination of logarithmic and quadratic functions can serve to model the convex-concave function.

Obviously, in the vast majority of publications, researchers consider the linear models of demand and costs: in this case, it is easy to calculate conjectural variations from the best response functions (reaction functions) in explicit form. However, the accuracy of representing nonlinear processes by linear models becomes relevant here. It was analyzed in several classical studies [12, 13], and the

Table 1. The types of inverse demand function[3]

Type	Formula and parameters	Publications
Linear	$P(Q) = a - bQ$, $a > 0$, $b > 0$, $a >> b$	[14–40]
Linear with several variables	$p_i(Q) = a - \sum_{i\in N} b_i Q_i$, $a > 0$, $b_i > 0$, $a >> b_i$	[33, 34, 36, 52]
Power	$P(Q) = AQ^{\alpha}$, $A > 0$, $\alpha < 0$, $\lvert\alpha\rvert < 1$	[27, 41–47]
Combined	$P(Q) = A - Q^{\alpha}$, $A > 0$, $\alpha > 0$	[48]
Exponential	$P(Q) = Ae^{\alpha Q}$, $A > 0$, $\alpha < 0$	[49]

Table 2. The types of agent's cost function

Type	Formula and parameters	Publications
Linear	$C(q) = B_0 + B_1 q$, $B_0 \geqslant 0$, $B_1 > 0$	[15–19, 23, 29–32, 34, 36–39, 41, 42, 44–46]
Power	$C(q) = B_0 + B_1 q^{\beta}$, $B_0 \geqslant 0$, $B_1 > 0$, $\beta \in (0, 2)$	[24, 25, 43]
Quadratic	$C(q) = B_0 + B_1 q + \frac{B_2}{2} q^2$, $B_0 \geqslant 0$, $B_1, B_2 > 0$	[14, 21, 22, 26–28, 34, 35, 40, 50–52]
Logarithmic	$C(q) = \ln(B_0 + B_1 q)$, $B_0 \geqslant 1$, $B_1 > 0$	[27]

[2] Some studies consider a conjectural price variation as a supposed change in price in response to the unit quantity increase of player i. When analyzing this approach, the variation ρ will be referred to as the quantity variation.

[3] Most studies of quantity oligopoly consider the inverse demand function. Therefore, it will be briefly called the demand function. When referring to the function $Q(P)$, the term "direct demand function" will be used.

conclusion was not in favor of the adequacy of linear models. In particular, A. Walters summarized over 30 publications confirming the convex-concave nature of cost functions, and P. Ghemawat examined 97 studies with proof of the learning effect (i.e., the concavity of cost curves).

Also, another modification of the basic model is a differentiated oligopoly with the non-identical products (substitutes) of players. In this case, linear demand functions with different rates of substitution b_i are used for different players. As a result, the demand price of player i becomes a function of several variables, i.e., $p_i(Q_1, \ldots, Q_n)$. However, from an economic point of view, such a model represents not an oligopoly but monopolistic competition. Some studies of such models will be discussed below in the context of solution methods inherent in oligopoly games.

4. REFLEXIVE OLIGOPOLY GAMES

Modern oligopoly game theory has two mainstreams: 1) analysis of the process of reaching an equilibrium based on players' interaction (the dynamic game) and 2) study of an equilibrium (the static game). In recent years, the publications of Russian researchers have been actively developing both of them. Hence, the Russian scientific school occupies a leading position in this topical problem of game theory.

4.1. Dynamic Reflexive Games

The fundamental statement of a reflexive game is dynamic because it describes the iterative process of reaching an equilibrium by players. In the literature, this statement is considered in two lines as follows.

The first line formulates the dynamic game based on **finite-difference reaction equations**. The player's best response to the environment is given by a recurrence function with a fractal parameter[4] that corrects the step in the process of reaching an equilibrium. In this case, the player's reflexion is embedded in the best response function, which is derived from the optimum conditions of the players' utility functions (5). For each agent, the solution of problem (1) can be expressed as the reaction function (or the best response function), denoted by $r_i(Q_{-i})$. Then the reflexive game has generalized dynamics of the form [53]

$$Q_i(t) = Q_i(t-1) + \gamma_i^t [r_i(Q_{-i}(t-1)) - Q_i(t-1)], \quad i \in N, \quad t = 1, 2, \ldots, \tau, \tag{6}$$

where $\gamma_i^t \in [0,1]$ is the reflexion step and $-i$ means the environment of player i.

Substantively, process (6) with $\gamma_i^t = 1$ describes a logical transition between strategies at time instants t and $t+1$ by the best response.

Within this line, studies mainly establish convergence conditions to a game equilibrium for the iterative process (6). Convergence is estimated by the residual (error) ε, namely, its value $\varepsilon_i^t = |Q_i(t) - Q_i(t-1)|$ at adjacent time instants or its value $\varepsilon_i^\tau = |Q_i(\tau) - Q_i^*|$ at the final time instant, where τ is the total number of iterations.

Particularly for oligopoly models with linear demand and cost functions, the convergence of process (6) with an appropriately chosen reflexion step was proved in [15; 17] under the Stackelberg leadership of one or several players. In addition, G.I. Algazin and D.G. Algazina [16] examined the process $\boldsymbol{\varepsilon}^{t+1} = \mathbf{B}^t \boldsymbol{\varepsilon}^t$ and established its convergence property in the matrix norm form $\|\mathbf{B}^t\| < 1$, where the errors transition matrix $\mathbf{B}^t = \left\{ b_{ij} = 1 - \gamma_i^{t-1}, b_{ij} = -\frac{\gamma_j^{t-1}}{2} \forall i \neq j \in N \right\}$ consists of the combination of the reflexion steps of different players. The analogy of this matrix with the matrix equation for calculating conjectural variations, $\mathbf{B}^r \boldsymbol{\rho}^r = \mathbf{I}$, where $\boldsymbol{\rho}^r = \{\rho_{ij} \forall i, j \in N\}$ is the

[4] The parameter for correcting the step of the dynamic process is fractional in the range $[0,1]$ or fractal. This shows an analogy of the finite-difference approach with fractal differential equations; see the discussion below.

matrix of variations at reflexion rank r, $\mathbf{B}^r = \{b_{ij} = -2 - S_i^{r-1}, b_{ij} = -1 \forall i \neq j \in N\}$, and $S_i^r = \sum_{j \in N \setminus i} \rho_{ij}^r$, in the static reflexion considered below [54] confirms the affinity of the dynamic and static processes.

Having considered process (6) in the case of players with identical utility functions and $\gamma^t = 1$ (i.e., iterations with the best response $Q_i(t) = r_i(Q_{-i}(t-1))$), R. Cornes et al. [42] derived the convergence condition $\left|\frac{\pi''_{qq}+\pi''_{Qq}}{\pi''_{qQ}+\pi''_{QQ}}\right|$, where $q = \frac{Q}{n}$.

In the tripoly case, Y. Dzhabarova and B. Zlatanov [21] investigated process (6) with $\gamma_i^t = 1$ and considered reaction functions in the form of contracting mappings (those satisfying the triangle inequality $\sum_{i=1,2,3} \varepsilon_i^t \leqslant \sum_{i=1,2,3} k_i \varepsilon_i^t$, $\sum_{i=1,2,3} k_{ij} < 1$). As a result, they proved the residual constraint $\max\limits_{i=1,2,3} \varepsilon_i^\tau \leqslant \frac{k}{1-k} \sum_{i=1,2,3} |Q_i(\tau) - Q_i(\tau - 1)|$, where $k = \max\limits_{j=1,2,3} \sum_{i=1,2,3} k_{ij}$.

Obviously, the main problem of dynamic modeling with finite-difference reaction equations is the need for explicitly expressing the best response function. This is possible only for linear demand and cost functions. To analyze a game with at least one nonlinear function among those mentioned, researchers [33, 34, 45] adopted an iterative process with the gradient of the utility function (1):

$$Q_i(t) = Q_i(t-1) + \gamma_i^t Q_i(t-1) \Pi'_{iQ_i}(t-1). \tag{6a}$$

This process is not as logical as the best response in (6), but it allows obtaining equilibria in analytical form. Process (6a) is based on the property $\lim_{Q_i \to Q_i^*} \Pi'_{iQ_i} = 0$ when approaching the equilibrium; consequently, $\lim_{Q_i \to Q_i^*} \varepsilon_i^t = 0$.

In particular, Y. Peng, Y. Xiao, et al. [33, 34, 45] varied the form of process (6a) with a time-independent step γ_i in the duopoly game. Having considered the process $Q_i(t) = Q_i(t-1) + \gamma_i \Pi'_{iQ_i}(t-1)$, the authors obtained *two* explicit-form equilibria in the case of linear demand function [33] (including one equilibrium with a zero component); moreover, they proved an existence condition for one equilibrium in the case of power demand function [45] (violation of this condition makes the process chaotic). Differential oligopoly analysis for the process $Q_i(t) = Q_i(t-1) + \gamma_i Q_i(t-1) \Pi'_{iQ_i}(t-1)$ yielded *four* equilibria [34], including three with zero components. In all cases, numerical experiments showed the ranges of the step γ_i where the process is stable. Note that bifurcations appear beyond these ranges, leading to an infinite set of equilibria.

The second line considers the dynamic game based on **differential equations**. In this case, the utility function (1) integrates the players' payoffs over the interval T (the game duration) with a discount rate δ, i.e.,

$$J_i = \int_0^T e^{-\delta t} \Pi_i(Q, Q_i, u_i, t) dt. \tag{7}$$

In addition, the process of changing the utility functions $\Pi_i(t)$ is given by differential equations of either actions [39] or equilibrium price [52] dynamics:

$$\dot{Q}_i = f(P(t), Q_i(t), u_i(t)), \tag{8}$$

$$\dot{P} = f(P(t), u_i(t)). \tag{9}$$

The control parameters u_i introduced in model (7) is an important advantage of the differential approach: the optimal control of processes (8) or (9) can be found.

Based on an equation of the type (8), N.I. Aizenberg et al. [14] simulated the process of equilibrium stabilization in an oligopoly with a linear demand function and quadratic cost functions. According to the results of numerical experiments, the equilibrium was reached starting from $t = 5$

under two control scenarios. First, the deviations of the players' marginal costs from the price, i.e., the value $u_i = P(t) - C'_{iQ_i}(t)$, were considered as the control parameters: the process was described by the linear differential equation $\dot{Q}_i = Q_i(t)(P(t) - C'_{iQ_i}(t))$. Second, the cited authors studied control by the deviation of marginal costs from marginal revenue, i.e., $u_i = (P(t)Q_i(t))'_{Q_i} - C'_{iQ_i}(t)$, for the process formalized by the equation $\dot{Q}_i = Q_i(t)((P(t)Q_i(t))'_{Q_i} - C'_{iQ_i}(t))$.

Also based on model (8), G.A. Ougolnitsky and A.B. Usov [39] considered the marginal costs $u_i = B_{1i}$ of the player's linear cost functions as control parameters and used the linear differential Eqs. (8) in the form $\dot{Q}_i = a_i u_i(t) - m_i Q_i(t)$, where a_i and m_i are constants. As a result, the piecewise-constant optimal control of process (8) was found analytically; explicit formulas were obtained for Nash equilibria and Stackelberg equilibria were simulated within a numerical experiment under different numbers of control switchings.

On the other hand, M. Raoufinia et al. [52] analyzed players' advertising actions as control parameters for the dynamics described by the linear differential Eq. (9) of the form $\dot{P} = a - \sum_{i \in N} b_i Q_i(t) + \sum_{i \in N} u_i(t) - P(t)$ with the quadratic cost functions $C(Q_i) = B_0 + B_1 Q_i + \frac{B_2}{2} Q_i^2 + \frac{B_3}{2} u_i^2$. Due to the chosen form of differential equations, the researchers found that the optimal control is constant and there exists a unique Nash equilibrium.

A non-trivial variety of the dynamic game was formulated using fractal differential equations. Based on such equations, A. Al-Khedhairi [18] combined process (6) with the gradient of the utility function (1) and process (8) as follows:

$$\frac{d^{\varphi} Q_i}{dt^{\varphi}} = \gamma_i Q_i(t) \Pi'_{iQ_i}(t), \tag{10}$$

where φ is the order of the fractal derivative, $\varphi \in (0, 1]$. As a result, *four* equilibria (as in the finite-difference model [34]) were established even for the simplest duopoly game with linear demand and cost functions, and numerical simulations demonstrated bifurcations.

Although the dynamic model (7)–(9) is not reflexive in the full sense, in essence, processes (8) and (9) are a continuous analog of process (6). Nevertheless, it is necessary to consider important differences between these approaches. Direct comparison of the finite-difference and differential approaches to modeling the dynamics of the oligopoly game indicates the following: within the former approach, the process of reaching equilibrium is based on the endogenous reaction function, i.e., the one derived from the utility functions of players. According to the second approach, the dynamics characteristics are set exogenously, i.e., they cannot be derived from the basic model (1) but require additional conditions of the type (8) and (9). Therefore, the main difficulty of the differential approach consists in selecting appropriate (reality-adequate) coefficients for the differential equations of dynamics to match the reflexive process. Moreover, the scope of the finite-difference approach is restricted to the case of linear demand and cost functions whereas the application of the differential approach is limited to the linear differential equations of the dynamic process.

4.2. Static Reflexive Games

In contrast to the dynamic game, the static setting assumes that the players act simultaneously and independently. As a consequence, they directly come to an equilibrium "in one step," bypassing the iterative process. However, the players also rest on some assumptions about the actions of the environment, so static oligopoly games are currently developing predominantly in the stream of reflexive games: the dynamics of the action process are transferred to the thought process of the players. This process is formalized through iterations of assumptions leading to the following characteristics of players: 1) follower, making no assumptions about the strategies of his environment (his conjectural variation is zero); 2) Stackelberg leader (of the first level), supposing that he is surrounded by followers; 3) Stackelberg leader of the second level (or higher levels), supposing that he is surrounded by Stackelberg leaders of the first level (or lower levels). Therefore, *reflexion* [7] will refer to the operation of calculating the variation $\rho^r_{\eta_r j}$ from system (5) by the η_rth player at the rth step of the thought process under the assumption that all other players are at the $(r-1)$th step of the thought process with their conjectural variations $\rho^{r-1}_{ij} \forall i \in N \backslash \eta_r$, $j \in N \backslash i$. Accordingly, the serial number r of the player performing this operation is *the reflexion rank*. Consequently, the vector of players' ranks $\{r_i, i \in N\}$ is introduced into the structure of the game (3): it transforms into a game with imaginary (*phantom*) environment players written as $\Gamma = \langle N, \{Q_i, i \in N\}, \{\Pi_i, i \in N\}, \{r_i, i \in N\}\rangle$. Therefore, the solution of a static reflexive game is not a real but *informational equilibrium*, i.e., a vector of actions of a real player and phantom players existing in his mind in which the player maximizes the utility based on his awareness of the environment (*as if* the environment would choose the actions imagined by this player). The solutions of all possible games with phantom players form a set of informational equilibria and are used for subsequent comparison with the parameters of real markets in order to estimate the reflexion ranks of real players.

In the studies [22, 24, 25, 27, 38, 40, 43] of static reflexive games, system (5) is usually written as follows:

$$P(Q) + (1 + S^r_i) Q_i P'_Q - C_{iQ_i} = 0, \quad i \in N, \quad S^r_i = \sum_{j \in N \backslash i} \rho^r_{ij}, \tag{11}$$

where S^r_i is the sum of the conjectural variations of player i at reflexion rank r.

The principal vector of modern investigations of static games aims at a comparative analysis of equilibria in the game of Stackelberg followers and leaders for certain types of demand and cost functions.

Starting from the second aspect, the analysis of different demand and cost functions has yielded the following results by now. Models with a linear demand function and a convex, particularly polynomial (quadratic), cost function [22, 27, 40] are productive as they lead to a concave unimodal utility function of a player, guaranteeing the uniqueness of equilibrium; for these models, the monotonicity of equilibrium in the players' conjectures was proved [22]. Models with a linear demand function and a convex-concave (power) cost function [24, 25, 43] are more difficult to analyze because multiple equilibria are possible in the case of a concave cost function. Such situations have been studied to date in sufficient detail: necessary and sufficient conditions of Nash equilibrium in the presence of first- and second-level leaders were established [43]; approximate formulas were derived for calculating the conjectural variations of an arbitrary number of players at an arbitrary reflexion rank [24]; ways to approach the approximate calculation of equilibria in explicit form were outlined [25]. Models with the nonlinear demand and cost functions appear to be most complicated; in this case, only necessary conditions of Nash equilibrium for several first-level leaders were defined [43].

Concerning the aspect of leadership analysis, note the studies of deepening reflexion and increasing the number of reflexing players. They lead to the emergence of many Stackelberg leaders in the game as well as leaders of higher levels.

Having considered the interaction between the set of followers and the set of first-level Stackelberg leaders, L. Julien [27] solved within each of the sets two simultaneous nonhierarchical Cournot games embedded in the hierarchical Stackelberg game. Having restricted the players' cost trends to the negative scale effect (i.e., convex cost functions), L. Julien established the range of possible values for the sum of conjectural variations at the first rank: $S^1_i \in (-1, 0]$; he proved that if there exists an active Stackelberg equilibrium (i.e., with $Q_i > 0 \ \forall i \in N$), then it is unique. An interesting analysis of the game with two followers (the subscript F) and two leaders (the subscript L) under particular demand and cost functions led to the following outcomes: for the functions $P(Q) = 1 - Q$,

$C_F(q) = 0.5q^2$, and $C_L(q) = \ln(1 + 0.5q)$ (the positive scale effect for the leaders), an equilibrium exists and is unique, as for the functions $P(Q) = (Q + 1)^{\alpha}, \alpha < -2$ and $C_F(q) = C_L(q) = 0$; for the functions $P(Q) = 1 - Q$, $C_L(q) = 0$, and $C_F(q) = 1 + B_1 q - 0.5q^2$ when costs decrease for $q > B_1$, there is no equilibrium; for the functions $P(Q) = 1 - Q$, $C_F(q) = 0.5q^2$, and $C_L(q) = \frac{1}{16}\ln(\frac{1}{8} + 0.5q)$ (the positive scale effect for the leaders, but their marginal costs decrease faster than the price goes down), two equilibria occur.

Within reflexive analysis, the problem of calculating conjectural variations becomes the key one: they have to be determined not for one or two kinds of assumptions, as in the classical approach, but for an infinite variety of mental types. Here, difficulties arise in the reality-closest case of nonlinear (convex or concave) cost functions, where it is necessary to solve a system of nonlinear equations in order to find these variations. Some progress has been recently achieved in developing approximate methods for calculating conjectural variations [24]. However, this problem has not been completely solved because it requires an explicit representation of the players' best response functions, which is fundamentally impossible in the case of nonlinear cost functions.

In the studies of deeper reflexion leading to multilevel leadership, M.I. Geraskin [25] analyzed a model with a linear demand function and convex-concave cost functions; he derived the following recurrence formula for calculating the sum of conjectural variations at an arbitrary reflexion rank:

$$S_i^r = \left(\frac{1}{\sum\limits_{j \in N \setminus i} \frac{1}{u_j - S_j^{(r-1)} + 1}} - 1 \right)^{-1}, \tag{12}$$

$$u_i = -1 + \frac{P'_{Q_i} + (1 + S_i^{r-1}) Q_i P''_{QQ_i} - C''_{iQ_iQ_i}{}^{5}}{|P'_Q|}.$$

According to the analysis of (12), the value S_i^r is negative and bounded modulo by 1 under the negative scale effect, i.e., when the player supposes a decrease in the actions of the environment in response to an increase in his actions. (This result coincides with the conclusions of L. Julien [27].) If the scale effect is positive, this value can be positive and unbounded: it may be optimal for the environment to increase its actions in response to the growth of the player's action.

An unorthodox interpretation[6] of the conjectural *price variation* P'_{Q_i} as a supposed price change in response to a unit quantity increase of player i was proposed by V.A. Bulavsky and V.V. Kalashnikov [50, 51]. Since the authors considered the negative scale effect when this variation is negative, they defined the variation by absolute value $v_i = -P'_{Q_i}$. In this case, Eqs. (11) become simpler:

$$P(Q) + Q_i P'_{Q_i} - C'_{iQ_i} = 0, \quad i \in N. \tag{13}$$

For the direct demand function $Q(P) = G(P) + D$, where $G'_p \leqslant 0$ and $D = \text{const} > 0$ (with a jump discontinuity [51]), and the convex cost functions of players, i.e., $C'_{iQ_i} > 0$ and $C''_{iQ_iQ_i} > 0$ (quadratic cost functions [51]), the researchers proved the existence of a unique equilibrium in the case of conjectural variations independent of the players' actions as well as established the following

[5] In [25], it has the form $u_i = -2 - \frac{C''_{iQ_iQ_i}}{b}$ because it was obtained for a linear demand function: in this case, $P'_Q = P'_{Q_i} = -b$ and $P''_{QQ_i} = 0$. In addition, formula (12) is presented for the conjectural variations that do not depend on players' actions (i.e., for $\rho'_{ijQ_i} = 0$), whereas the paper [25] described a more general case where $\rho'_{ijQ_i} \neq 0$.

[6] This approach is conceptually close to the inclusive best response in aggregative games [5], which represents the optimal response of a player to the total action of all players (including himself), i.e., $\widetilde{r}_i(Q)$, in contrast to the best response to the environment $r_i(Q_{-i})$.

condition for price variations:

$$\upsilon_i = \frac{1}{\sum\limits_{j \in N \setminus i} \frac{1}{\Theta'_{jQ_j}} - G'_p}, \quad i \in N. \tag{14}$$

Here, $\Theta_i = -Q_i P'_{Q_i} + C'_{iQ_i}$ is the right-hand side of the transformed Eq. (13), i.e., $P(Q) = \Theta_i$. Hence, $\Theta'_{jQ_j} = P'_{Q_j} = -\upsilon_j$ in (14), and the latter expression, unlike (12), is not a formula for computing price variations but a system of equations from which they can be found. This system is not analytically solvable, and the problem of computing conjectural variations remains open.

Obviously, there is a relationship between the quantity variation ρ_{ij} in (11) and the price variation υ_i in (13). Since $Q_j = Q - \sum_{k \neq j} Q_k$, it follows that $\rho_{ij} = Q'_{jQ_i} = Q'_{Q_i} - \sum_{k \neq i,j} \rho_{ik} - 1$. This means that $-\upsilon_i = P'_{Q_i} = P'_Q Q'_{Q_i} = P'_Q \left(\rho_{ij} + \sum_{k \neq i,j} \rho_{ik} + 1\right) = P'_Q(S_i + 1)$; therefore, $\upsilon_i = -P'_Q(S_i + 1)$.

The advantage of applying price variations in an oligopoly game is that the analysis involves the aggregate reflexion of all players in the environment, which simplifies the verification procedure of these variations: the price is common knowledge in the real game, which cannot be claimed about the actions of the players. On the other hand, this advantage generates the problem of differentiated reflexion analysis since the price variation levels out the differences in the conjectures of player i about the strategies of players j and k.

Some papers in the foreign literature compared the Stackelberg and Cournot leadership models (SvsC). In one of the recent publications, J. Zouhar and M. Zouharova [38] compared these equilibria in terms of social welfare and total profits and indicated the ranges of the marginal cost relationships of the players in which one equilibrium is preferable to the other. However, the comparison of these equilibria is of no practical importance since the players cannot choose one of these models: the game situation is set a priori.

On the other hand, the SvsC scheme can be productive. Having analyzed the difference between the Cournot and Stackelberg models for oligopoly at an arbitrary number n, E. Cumbul [20] considered an interesting variant of informational reflexion concerning the unknown random parameter θ of the linear demand function $P(Q) = a + \theta - bQ$. Each player takes the parameter θ into account as a signal s_i in his utility function, and reflexion is formalized through its choice. The signal is defined as the sum of θ and white noise, and the variances of the distributions of θ and white noise are introduced exogenously. As a result, E. Cumbul showed that the total action of players is lower and the equilibrium price and total profit are higher in Stackelberg equilibrium than in Cournot equilibrium. This result is opposite to the classical solutions of the corresponding games with complete awareness [55].

An uncommon comparison of the noncooperative game within the Cournot and Stackelberg leadership models with the cooperative solution based on the Shapley value under different characteristic functions was conducted by A.V. Korolev and G.A. Ougolnitsky [56] for the case of three players with linear demand and cost functions. As a result, the authors ordered the game situations as follows: the largest total payoff is provided by cooperation, the smaller one by the noncooperative Cournot model, and the smallest total payoff by hierarchy with Stackelberg leadership. Note that this ordering corresponds to the classical analysis [55].

Generally speaking, the stumbling block in the analysis of static reflexive games is the same problem as in dynamic games: striving for more realistic convex-concave cost models is limited by the possibility of calculating the players' best responses explicitly from the system of nonlinear equations.

5. APPLICATIONS OF OLIGOPOLY GAME MODELS

5.1. Applications to Real Markets

The game-theoretic problem of oligopoly is historically linked to the problem of analyzing the specifics of markets in which each firm's profits depend on the actions of all firms (i.e., markets with a relatively small number of large firms).

The telecommunications market has become a relevant platform for testing game-theoretic models of oligopoly due to several reasons. First, it has demonstrated the fastest growth among all markets over the past two decades [57]. Second, it represents a typical oligopoly: in most countries, this market has no more than three dominant firms [58]. Third, it attracts the attention of researchers by its social significance, affecting the interests of the whole society. The analysis of the Russian telecommunications market oligopoly [43] showed the adequacy of the linear demand function and convex-concave cost functions of the players and also confirmed the prevalence of the follower's mental type in static reflexion.

Having studied the electricity market, T.-H. Yoo et al. [35] compared three game scenarios in the case of quadratic cost functions, with each subsequent scenario forming a lower equilibrium price than the previous one: 1) the game of power suppliers as followers with an exogenous linear demand function; 2) two successive games resulting in a Stackelberg game: in the first one, power consumers form an endogenous linear demand function, and the second game is played by power suppliers subsequently; 3) two games as in the second scenario but with an incentive introduced into the consumers' utility function; the incentive shifts the demand curve downwards further.

B. Kanieski da Silva et al. [44] analyzed the timber market in the case of the power demand and cost functions independent of the players' actions. The authors compared duopoly equilibria in the cases of cartel, Cournot competition, and Stackelberg leadership game with capacity constraints. The game reflects the specifics of the timber market: the logger's supply is characterized by felling outturn and is constrained by the volume of plantations, whereas the costs are due to the need to restore forest areas.

X. Zhou et al. [47] considered the banking market under the assumption that the volume of demand depends not only on the price (interest rate) but also on the gross domestic product (GDP). They used linear cost functions and a power demand function in the nontrivial form $\log Q = e_0 + e \log P + e_{GDP} \log GDP$, where e_0, e, e_{GDP} are constants and e is the price elasticity of demand. The factual estimates of the conjectural variations were approximated by the linear regressions $\rho_{ij}\frac{Q_i}{Q_j} = \beta_j + \gamma_j\frac{Q_i}{Q}$, where the coefficients β_j and γ_j were determined from the trends of real players' actions for 2007–2016 (assumed equal to the equilibria in the game). Having examined two types of players, namely, leaders (the five largest banks of China) and followers (small banks), the authors empirically showed that the conjectural variations of leaders relative to other leaders are positive, the variances of leaders relative to followers are negative, and the followers have zero variations.

5.2. Applications to Original Problems

The oligopoly model is so fruitful that researchers have often adopted it as a basic known-solution scheme for related problems of economic behavior analysis, usually in microeconomics.

The merger of firms originally competing as followers with the emergence of a Stackelberg leader was studied by W. Ferrarese [23] in the case of linear demand and cost functions. He proved that the merger of firms maximizes the leader's profit regardless of the number of coalition participants; the formation of several leading coalitions is rational if the number of their participants does not exceed some threshold.

On the other hand, R. Fauli-Oller [48] investigated the problem of divisionalization, i.e., the partition of asymmetric-cost duopoly firms into branches competing as followers, in the case of linear cost functions. Having compared the simultaneous choice of the number of divisions by the players in the case of the combined demand function with the sequential choice under the Stackelberg hierarchy in the case of the linear demand function, the author concluded that the equilibrium price is higher when divisionalization runs sequentially.

Some foreign studies deal with a mixed (private-state) oligopoly; for example, it was considered by M.H. Lin and T. Matsumura [28] in the context of optimal privatization as well as by J. Haraguchi and T. Matsumura [26] within the optimal entry problem. The authors optimized the degree of privatization [28], i.e., the share of private firms in the oligopoly, by the social welfare criterion $W = \int_0^Q P(Q)dQ - P(Q)Q + \sum_{i\in N}\Pi_i$ and proved the following: if the state firm is a follower and the private one a leader (in the Stackelberg sense), then in the case of linear cost functions, the zero degree of privatization is optimal for any demand function with $P'_Q < 0 \wedge P''_Q \leqslant 0$; for quadratic cost functions and linear demand functions, the optimal degree of privatization grows with increasing the number of players. Having studied the emergence of a new private firm in a mixed oligopoly, J. Haraguchi and T. Matsumura [26] showed that in the case of a linear demand function and quadratic cost functions, private firm's profit goes down with increasing the number of firms under an exogenously given privatization degree. On the other hand, if the degree of privatization is calculated endogenously from the zero entry profit condition for a new firm ($\Pi_i(n) = F$, where F is the entry barrier investment), then there may exist one to three equilibria satisfying this condition.

A. Mukherjee and C. Zeng [30] also analyzed the new firm's entry problem but in a bilateral oligopoly game. In this game, retailer oligopolists (the subscript d) interact with supplier oligopolists (the subscript[7] u), and the new retailer can bear sunk costs. In the case of linear demand and cost functions, the authors proved that the free entry condition (i.e., the one of the fixed point $\Pi_i(\theta F) = \theta F$, where θ is the share of sunk costs) corresponds to zero profit of the new firm for $\theta = 0$ but provides a positive profit for $\theta > 0$. Having solved the Nash bargaining problem for the seller and supplier with the criterion $\max\limits_{\theta\in[0,1]}\{\Pi_{ui}^{\alpha}(\Pi_{di} + \theta F)^{1-\alpha}\}$, they also found that the entry of a new retailer increases social welfare for $\theta = 1$ if the suppliers' bargaining power α is greater than the firm's quantity elasticity with respect to the number of firms and reduces social welfare for $\theta = 0$.

The problem of introducing a new technology is adjacent to the entry problem. Y. Zhang [36] analyzed it for a differentiated oligopoly with a linear demand function of several variables and linear cost functions and compared two technologies: one deterministically reduces marginal costs by a value Δ, whereas the other makes a random reduction with a mean Δ and a constant variance. If the first technology is chosen by risk-averse players and the second one by risk-seeking players, according to the author's conclusions, the equilibrium number of risk-seeking players will grow with increasing the rate of substitution b_i in the demand function, and there will be more such players (in the duopoly, two) compared to the risk-averse ones.

In the context of the so-called green economy, the maximization of social welfare is interpreted more widely considering negative externalities, particularly firms' impact on the environment [59–61]. Having formulated the ecological load constraint proportional to the total action of the players, G.A. Ougolnitsky et al. showed that the least ecological damage is achieved under cooperation and the greatest under Stackelberg leadership, both in the static game and in the differential dynamic game. This result is due to the corresponding relationship of action aggregates in the games [55].

[7] In the literature concerning vertical interaction, the retailer and supplier are called the downstream firm and upstream firm, respectively.

An unusual problem of oligopoly in the case of mixed bundling (selling the sets of products) was investigated by J. Zhou [37]. Having assumed a known joint probability function for the utilities U_i of two products that are sold in a set at a discount compared to the sum of their separate prices, he determined the equilibrium relationship of the prices of products and proved that the negative correlation of utilities facilitates bundling. J. Shuai et al. [32] developed this problem under the linear direct demand function $Q_i = U_i + a - P_i$ ($a = 0$ in [37]) and zero costs in the case of a two-product oligopoly of asymmetric players consisting of one quality-dominant firm and a set of symmetric firms ($a = 0$ for them). The authors demonstrated that the dominant firm's striving for bundling grows with increasing the level of dominance a.

Several researchers used the game-theoretic model of oligopoly for macroeconomic analysis. For example, having analyzed the neoclassical economic growth model proposed by R.M. Solow [62], W.-B. Zhang [46] formulated a duopoly model with a power demand function in the case of the Cobb–Douglas production functions $Q_i = \prod_{j=1,2} x_{ij}^{\alpha j}$, $\alpha_j > 0$, $\alpha_1 + \alpha_2 = 1$, that are unresolved with respect to the players' costs. Therefore, in contrast to model (1), the players' actions are the resource costs x_{ij}, and the utility function of player i is considered in the form $\Pi_i = P(Q)Q_i - \sum_{j=1,2} p_{ij}x_{ij}$, where p_{ij} denotes the price of resource j. As is known [55], such a model leads to a power cost function. Based on numerical experiments, the author arrived at a nontrivial conclusion: the Stackelberg duopoly provides higher GDP than perfect competition.

Another macroeconomic problem of tax policy was touched upon by D.R. Collie [19], who modernized model (1) into a dividend return to shareholders from net income by introducing the tax rates $\Pi_i = d(P)\left[(1 - t_1)\left(\frac{P(Q)}{1+t_2} - t_3 - C_i\right)Q_i - t_4\right]$ with the following notations: t_1, is income tax rate, t_2 is ad valorem VAT rate, t_3 is specific VAT rate, t_4 is lump-sum tax rate, and $d(P)$ is the dividend function. In the case of a general-type decreasing demand function and linear cost functions of players, the author derived Cournot equilibrium actions as functions of tax rates. Having analyzed the sensitivity of these functions, he showed that the lump-sum tax leads to higher prices and lower social welfare, whereas the income tax causes lower prices and higher social welfare; in addition, ad valorem VAT is always higher than specific VAT under the same equilibrium price.

Generally speaking, the applications of oligopoly theory demonstrate, on the one hand, the possibilities of this game for interpreting the diversity of economic processes and, on the other, the tendency of researchers to reduce this diversity to a relatively narrow spectrum of game behavior. (As a rule, they are confined to the game of followers.) Thus, Cournot competition, which generalizes monopolistic and perfect forms of competition, is implicitly assumed to be a determinant of general equilibrium.

6. CONCLUSIONS. THE CONCEPTUAL EVOLUTION OF OLIGOPOLY GAMES

The oligopoly game has a fundamental problem of the asymmetric awareness of players, causing the ambiguously determined mental types of players (the mental profile). Reflexive analysis allows forming an exhaustive set of all possible optimal actions of players (informational equilibria), comparing them with the parameters of a real equilibrium, and concluding on the reflexion depth of real players. As a result, the information about the mental profile of players is aggregated, and the players' responses can be predicted. Hence, the players can be controlled.

Informational control aims at implementing the mental profile of players when elaborating public administration decisions to improve market mechanisms. The equilibrium actions of players with exogenous mental types are expressed as unambiguous functions of the market system parameters. Therefore, it is possible to quantify the impact of changes in taxation, budget transfers, and interest rates on the amount of total market supply, and, ultimately, on the equilibrium price.

Furthermore, informational control has a larger aim of purposefully producing a given way of thinking for players in the interests of society. Indeed, under a given equilibrium action of a player as a function of his mental type, it is possible to calculate an appropriate (optimal) mental profile for the target value of the aggregate of equilibrium actions. Next, special information exchange mechanisms can be used to pass from the real mental profile of players to the optimal one. Information exchange mechanisms are formed within the scientific school of D.A. Novikov and A.G. Chkhartishvili: intensive investigations are carried out towards transition from reflexive analysis to reflexive control. The idea of informational control [63] is based on forming a purposeful sequence of opinions in a social group depending on the opinions of the so-called influence agents. As a result, a system of finite-difference equations of opinion dynamics with polarization effects was developed [64].

According to this survey, the nonhierarchical oligopoly game formulation dominating presently predetermines the main vector of studies on improving methods for calculating the equilibria necessary to form optimal laws or control programs by the individual utility criteria of players. A logical evolution of this paradigm is seen in the transition to interpreting oligopoly as a hierarchical game with the Principal having its own interests; they are expected to predetermine the purposeful influence on the action choice mechanisms of players. Therefore, future research will form an oligopoly model with exogenous control optimizing social welfare.

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