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Two-Stage Dynamic Programming in the Routing Problem with Decomposition

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Abstract—This paper considers an optimal movement routing problem with constraints. One such constraint is due to decomposing the original problem into a preliminary subproblem and a final subproblem; the tasks related to the preliminary problem must be executed before the tasks of the final subproblem begin. In particular, this condition may arise in the tool control problem for thermal cutting machines with computer numerical control (CNC): if there are long parts among workpieces, the cutting process near a narrow material boundary should start with these workpieces since such parts are subject to thermal deformations, which may potentially cause rejects. The problem statement under consideration involves two zones for part processing. The aggregate routing process in the original problem includes a starting point, a route (a permutation of indices), and a particular track consistent with the route and the starting point. Each of the subproblems has specific precedence conditions, and the travel cost functions forming the additive criterion may depend on the list of pending tasks. A special two-stage procedure is introduced to apply dynamic programming as a solution method. The structure of the optimal solution is established and an algorithm based on this structure is developed. The algorithm is implemented on a personal computer and a computational experiment is carried out.

Keywords: dynamic programming, route, megalopolis, precedence conditions

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1. INTRODUCTION

In applied problems, it is often necessary to choose the sequence of certain tasks under various constraints. As a result, substantial differences arise even when formulating these problems compared to the natural prototype, the traveling salesman problem (TSP); [1–7] and other publications. Several circumstances, including the presence of constraints, required a special theory for solving applications-oriented routing problems. For example, we refer to the monographs [8–10].

As it turned out [8, § 4.9], the precedence conditions, natural for engineering applications, can serve for reducing computational complexity; this fact was established theoretically. In addition, the main constraint in this paper (also, see [11, 12]) works "positively" as well. This can be observed, e.g., from the results of [11, Sec. 10].

We mean the problem statement in which the entire set of tasks for sequential execution is divided into two (disjunctive) subsets: the execution of tasks belonging to the second subset can be started only after completing all the tasks from the first subset. (Note that this problem may have specific precedence conditions and the travel cost functions forming the additive criterion may depend on the list of pending tasks.) Such a situation occurs in the shaped thermal cutting of parts on CNC machines. The matter concerns the possibility of effective heat rejection, which is motivated by thermal deformation considerations. (More details can be found in [10, § 1.3], including the rules of part stiffness and sheet stiffness.) These considerations particularly lead to the idea of zone cutting ([10, § 1.3.3]); in connection with the implementation of this idea, we note the constructs [12, Sec. 12] on a multistage modification of the dynamic programming-based procedure.

Omitting the details (see [10, Ch. 1]), we discuss one typical case of two zones. Consider the pre-cutting of long parts (see [11, Sec. 10]); these are parts with one of the dimensions exceeding at least 10 times another [10, p. 46]. Such workpieces are subjected to maximum thermal deformation. Therefore, if the workpieces are near a narrow material boundary, the cutting process should be started with them: in this case, there will be quite "a lot" of solid metal near the tool cut-in and switch-out points. The natural implementation of this principle is to form a zone that includes long (and perhaps some other) parts. This zone should be associated with the preliminary problem. The remaining parts form the second (final) zone. Of course, this solution is the simplest only in terms of zone cutting, but we consider it in more detail, continuing the constructs of [11, 12] and accompanying algorithmic design with theoretical justifications. In particular, the framework presented in [11, 12] will be supplemented by some properties of the preliminary problem.

Note that the decomposition of the original problem into a set of two subproblems can be reduced to imposing new precedence conditions. In this case, a "standard" optimal routing problem (in the sense of [8, 13, 14]) arises, which has a well-known structure of the optimal solution. However, in problems of appreciable dimension, one faces difficulties with computational implementation, quite understandable due to the NP-hardness of TSP. The decomposition approach [11, 12] allows overcoming these difficulties to a large extent without losing optimality; see the results of the computational experiment in [11, 12]. Hence, it is reasonable to study the decomposition approach in detail, especially in view of publications on shaped sheet cutting on CNC machines (in particular, see [10, 15–18]). Here, we focus on the modification with the pre-cutting of long parts under the localized precedence conditions for the preliminary and final subproblems.

2. GENERAL CONCEPTS AND NOTATIONS. PROBLEM STATEMENT

In the sequel, we use the following abbreviations: DP (dynamic programming), FS (feasible solution), TSP (Traveling Salesman Problem), RP (routing process), and OP (ordered pair). The problems investigated in this paper, traditionally regarded as intractable, require a thorough formalization for developing optimal procedures. This is all the more essential under constraints that arise in engineering applications and considerably complicate the problem statement compared to TSP-like problems, more traditional in discrete optimization. Therefore, we summarize general definitions, including some results of set theory. They are crucial for the correct formulation of the problem.

We use the conventional set-theoretic symbolism (quantifiers, connectives, etc.) and the notations $\varnothing$ (empty set) and $\triangleq$ (equality by definition). A family is a set whose all elements are sets. As usual [19, Ch. II, § 2], the expression $\{ a \in A \mid \ldots\}$ means the set of all elements $a \in A$ with the property $\ldots$; this convention is widely used below.

Any two objects x and y are assigned an unordered pair $\{x; y\} : \{x; y\}$, i.e., a set containing x and y only. An object z is assigned the singleton $\{z\} \triangleq \{z; z\}$ containing z. A set is an object; therefore, following [19, Ch. II, § 3], any two objects u and v are assigned their ordered pair (OP) $(u, v) \triangleq \{\{u\}; \{u; v\}\}$ with the first u and second v elements. If h is an OP, we denote by $\mathrm{pr}_1(h)$ and $\mathrm{pr}_2(h)$ the first and second elements of h, respectively, $h = (\mathrm{pr}_1(h), \mathrm{pr}_2(h))$. Any three objects x, y, and z are assigned their (ordered) triplet $(x, y, z) \triangleq ((x, y), z)$ [20, Ch. 1, § 3] with the first x, second y, and third z elements. Thus, strictly speaking [20, Ch. 1, § 3], an ordered triplet is defined as a special-form OP (a convention accepted in set theory); sometimes, this will be utilized below.

We denote by $\mathcal{P}(H)$ and $\mathcal{P}'(H)$ the families of all and all non-empty subsets of an arbitrary set H, respectively, and by $\mathrm{Fin}(H)$ the family of all non-empty finite subsets of H; $\mathrm{Fin}(H) \subset \mathcal{P}'(H)$. For a finite set H, we have $\mathrm{Fin}(H) = \mathcal{P}'(H)$. If A, B, and C are three sets, then [20, Ch. 1, § 3] $A \times B \times C \triangleq (A \times B) \times C$; therefore, for $x \in A \times B$ and $y \in C$, we have $(x, y) \in A \times B \times C$. For any non-empty sets S and T, we denote by T^S the set of all mappings (functions) from S into T (see [19, Ch. II, § 6]); the expressions $h \in T^S$ and $h : S \to T$ are identical. The value of a mapping at a certain point of the definitional domain (here, the set S) is denoted in a traditional way: for $g \in T^S$ and $s \in S$, we have $g(s) \in T$. For non-empty sets S, T, and $C \in \mathcal{P}'(S)$ and a mapping $h \in T^S$,

$$h^1(C) \triangleq \{h(x) : x \in C\} \in \mathcal{P}'(T)$$

is the image of C under the action of h. We follow conventional notations for multivariate functions; note that $\psi(h, l) = \psi(\mathrm{pr}_1(h), \mathrm{pr}_2(h), l) \in Q$ is defined for non-empty sets S, T, P, and Q, a mapping $\psi \in Q^{S \times T \times P}$, and points $h \in S \times T$ and $l \in P$. As Q, we often use $\mathbb{R}_+ \triangleq \{\xi \in \mathbb{R} \mid 0 \leqslant \xi\}$, where $\mathbb{R}$ stands for the real line. If H is a non-empty set, then $\mathcal{R}_+[H] \triangleq (\mathbb{R}_+)^H$ is the set of all nonnegative real-valued functions on H.

In the sequel, $\mathbb{N} \triangleq \{1; 2; \ldots\} \in \mathcal{P}'(\mathbb{R}_+)$ and $\mathbb{N}_0 \triangleq \{0\} \cup \mathbb{N} = \{0; 1; 2; \ldots\} \in \mathcal{P}'(\mathbb{R}_+)$; for $K \in \mathcal{P}'(\mathbb{N})$ and $m \in \mathbb{N}$, we have $K \oplus m \triangleq \{k + m : k \in K\} \in \mathcal{P}'(\mathbb{N})$. For $p \in \mathbb{N}_0$ and $q \in \mathbb{N}_0$,

$$\overline{p, q} \triangleq \{k \in \mathbb{N}_0 \mid (p \leqslant k) \& (k \leqslant q)\} \in \mathcal{P}(\mathbb{N}_0).$$

(The case $\overline{p, q} = \varnothing$ is not ruled out.) Note that $\overline{1, 0} = \varnothing$ and $\overline{1, m} = \{k \in \mathbb{N} \mid k \leqslant m\}$ for $m \in \mathbb{N}$. For a non-empty finite set K, $|K| \in \mathbb{N}$ is the cardinality of K and $\overline{1, |K|} = \{j \in \mathbb{N} \mid j \leqslant |K|\}$ is a non-empty discrete interval of $\mathbb{N}$. If $m \in \mathbb{N}$, then $(\mathrm{bi})[\overline{1, m}]$ is the (non-empty) set of all permutations [21, Ch. 5] of the discrete interval $\overline{1, m}$; for $\alpha \in (\mathrm{bi})[\overline{1, m}]$, $\alpha^{-1} \in (\mathrm{bi})[\overline{1, m}]$ is the permutation inverse to α:

$$\alpha(\alpha^{-1}(k)) = \alpha^{-1}(\alpha(k)) = k \ \ \forall k \in \overline{1, m}.$$

Any mappings defined on finite subsets $\mathbb{N}_0$ are called tuples, according to the notion of an image of non-empty subsets of the definitional domain. In particular, this applies to permutations. We often adopt the index representation of mappings, particularly tuples (an indexed family; see [22, I. 1]). The symbols $\diamond$ and $\square$ indicate tuple gluing operations.

Following tradition in set theory [19], a set consisting of OPs is called a relation.

2.1. Problem Statement

In the sequel, X is a non-empty set, $X^0 \in \mathrm{Fin}(X)$ is the set of possible starting points of the processes under consideration, and $\mathbf{n} \in N$, $\mathbf{n} \geqslant 4$; the sets

$$M_1 \in \mathrm{Fin}(X), \ldots, M_{\mathbf{n}} \in \mathrm{Fin}(X), \tag{2.1}$$

called megalopolises, and the relations

$$\mathbb{M}_1 \in \mathcal{P}'(M_1 \times M_1), \ldots, \mathbb{M}_{\mathbf{n}} \in \mathcal{P}'(M_{\mathbf{n}} \times M_{\mathbf{n}}), \tag{2.2}$$

are fixed. For each $j \in \overline{1, N}$, the elements of the relation $\mathbb{M}_j$ are OPs, including the arrival point at the megalopolis M_j and the departure point from M_j; $\mathbb{M}_j$ is the set of all OPs of this form. Concerning X^0 and the megalopolises (2.1), we assume that

$$\left(M_j \cap X^0 = \varnothing \ \ \forall j \in \overline{1, \mathbf{n}}\right) \& \left(M_p \cap M_q = \varnothing \ \ \forall p \in \overline{1, \mathbf{n}} \ \ \forall q \in \overline{1, \mathbf{n}} \setminus \{p\}\right). \tag{2.3}$$

Conditions (2.3) are typical for the routing problems discussed. As $\mathcal{M} \triangleq \{M_i : i \in \overline{1, \mathbf{n}}\}$ we obtain the family of megalopolises of the original problem to be visited from some point of X^0. Let $\mathcal{M}$ be implemented as the sum of two non-empty subfamilies. To introduce them, we fix a number $N \in \overline{2, \mathbf{n} - 2}$ and assume that

$$\mathcal{M}_1 \triangleq \{M_i : i \in \overline{1, N}\}, \quad \mathcal{M}_2 \triangleq \mathcal{M} \setminus \mathcal{M}_1 = \{M_i : i \in \overline{N+1, \mathbf{n}}\}. \tag{2.4}$$

(The latter property in (2.4) can be easily verified in view of (2.3).) Each of the families (2.4) contains at least two megalopolises. Defining the families (2.4), we consider the problem of visiting megalopolises from $\mathcal{M}$ as the set of two interconnected subproblems: the problems of visiting megalopolises from $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. In this regard, for $j \in \overline{1, \mathbf{n} - N}$, we suppose that

$$M^{(j)} \triangleq M_{N+j} \in \mathrm{Fin}(X) \ \ \forall j \in \overline{1, \mathbf{n} - N}$$

(see (2.2)). From (2.4) it follows that $\mathcal{M}_2 = \{M^{(j)} : j \in \overline{1, \mathbf{n} - N}\}$. Within this approach, let us accept an appropriate convention: the precedence conditions, possibly present in the $\mathcal{M}$-problem, are localized in the $\mathcal{M}_1$- and $\mathcal{M}_2$-problems. Taking these considerations into account, we introduce the sets

$$(\mathbf{K}_1 \in \mathcal{P}(\overline{1, N} \times \overline{1, N}))\&(\mathbf{K}_2 \in \mathcal{P}(\overline{1, \mathbf{n} - N} \times \overline{1, \mathbf{n} - N}));$$

their elements (OPs) are called address pairs. In each such pair, the first element is called the sender and the second one the recipient. From this point onwards, we assume that

$$\begin{gathered}(\forall \mathbf{K}^0 \in \mathcal{P}'(\mathbf{K}_1) \ \ \exists z^0 \in \mathbf{K}^0 : \mathrm{pr}_1(z^0) \neq \mathrm{pr}_2(z) \ \ \forall z \in \mathbf{K}^0) \\ \&(\forall \tilde{\mathbf{K}}^0 \in \mathcal{P}'(\mathbf{K}_2) \ \ \exists z^0 \in \tilde{\mathbf{K}}^0 : \mathrm{pr}_1(z^0) \neq \mathrm{pr}_2(z) \ \ \forall z \in \tilde{\mathbf{K}}^0).\end{gathered} \tag{2.5}$$

Conditions (2.5) usually hold in practical problems; see the discussion in [8, part 2]. For example, in the case of sheet cutting, the following traditional requirement is reduced to this condition: if the part to be cut has inner contours, they must be cut before the outer (enclosing) contour; see [8, Remark 2.2.1].

Remark 1. Introducing $\tilde{\mathbf{K}}_2 \triangleq \{(\mathrm{pr}_1(z) + N, \mathrm{pr}_2(z) + N) : z \in \mathbf{K}_2\}$, as $\mathbf{K}_1$ and $\tilde{\mathbf{K}}_2$ we obtain two subsets of $\overline{1, \mathbf{n}} \times \overline{1, \mathbf{n}}$. It is possible to consider the case in which $\mathbf{K}_1 \subset \tilde{\mathbf{K}}$ and $\tilde{\mathbf{K}}_2 \subset \tilde{\mathbf{K}}$, where $\tilde{\mathbf{K}} \subset \overline{1, \mathbf{n}} \times \overline{1, \mathbf{n}}$, and $\tilde{\mathbf{K}} \in \mathcal{P}(\overline{1, \mathbf{n}} \times \overline{1, \mathbf{n}})$ can be treated as the aggregate set of address pairs and used to define the aggregate precedence conditions. If an OP $\hat{z} \in \tilde{\mathbf{K}}$ is such that $\hat{k} \triangleq \mathrm{pr}_1(\hat{z}) \in \overline{1, N}$ and $\hat{l} \triangleq \mathrm{pr}_2(\hat{z}) \in \overline{N+1, \mathbf{n}}$, under the decomposition into the $\mathcal{M}_1$- and $\mathcal{M}_2$- problems, $M_{\hat{k}}$ will automatically be visited before $M_{\hat{l}}$. (In other words, no special consideration of $\hat{z}$ as an address pair is required.) If $\tilde{z} \in \tilde{\mathbf{K}}$ has the property $\tilde{k} \triangleq \mathrm{pr}_1(\tilde{z}) \in \overline{N+1, \mathbf{n}}$ and $\tilde{l} \triangleq \mathrm{pr}_2(\tilde{z}) \in \overline{1, N}$, the problem with the precedence conditions defined through $\tilde{\mathbf{K}}$ will be infeasible when solving the $\mathcal{M}_1$- and $\mathcal{M}_2$-problems sequentially. Therefore, within the problem statement with the decomposition into the $\mathcal{M}_1$- and $\mathcal{M}_2$-problems, a natural case is $\tilde{\mathbf{K}} = \mathbf{K}_1 \cup \tilde{\mathbf{K}}_2$: no additional consideration of the cross precedence conditions is required if the aggregate problem is feasible under this decomposition.

Let $\mathbb{P}_1 \triangleq (\mathrm{bi})[\overline{1, N}]$ and $\mathbb{P}_2 \triangleq (\mathrm{bi})[\overline{1, \mathbf{n} - N}]$; then, according to [8, formulas (2.1.5) and (2.2.53)], [10, formula (4.4.6)], and (2.5),

$$\mathcal{A}_1 \triangleq \Big\{\alpha \in \mathbb{P}_1 \mid \alpha^{-1}(\mathrm{pr}_1(z)) < \alpha^{-1}(\mathrm{pr}_2(z)) \ \ \forall z \in \mathbf{K}_1\Big\} \in \mathcal{P}'(\mathbb{P}_1), \tag{2.6}$$

$$\mathcal{A}_2 \triangleq \Big\{\alpha \in \mathbb{P}_2 \mid \alpha^{-1}(\mathrm{pr}_1(z)) < \alpha^{-1}(\mathrm{pr}_2(z)) \ \ \forall z \in \mathbf{K}_2\Big\} \in \mathcal{P}'(\mathbb{P}_2). \tag{2.7}$$

Thus, under conditions (2.5), we have non-empty sets of precedence-feasible routes (index permutations) in the $\mathcal{M}_1$- and $\mathcal{M}_2$-problems. Now, let $\mathbb{P} \triangleq (\text{bi})[\overline{1, \mathbf{n}}]$ (the set of all routes of the aggregate problem). For $\alpha \in \mathbb{P}_1$ and $\beta \in \mathbb{P}_2$, a (glued) route $\alpha \diamondsuit \beta \in \mathbb{P}$ is given by the rule

$$\Big((\alpha \diamondsuit \beta)(k) \triangleq \alpha(k) \ \ \forall k \in \overline{1, N}\Big) \,\&\, \Big((\alpha \diamondsuit \beta)(l) \triangleq \beta(l-N)+N \ \ \forall l \in \overline{N+1, \mathbf{n}}\Big). \tag{2.8}$$

Here, the matter concerns gluing special-form permutations (shift gluing). Below we will introduce another gluing operation, not for permutations but for fragments of tracks. For this reason, a different notation will be used there. In (2.8), in particular, it is possible to use routes from the sets (2.6) and (2.7). Considering this aspect, we obtain

$$\mathbf{P} \triangleq \{\alpha \diamond \beta : \alpha \in \mathcal{A}_1, \beta \in \mathcal{A}_2\} = \{\mathrm{pr}_1(z) \diamond \mathrm{pr}_2(z) : z \in \mathcal{A}_1 \times \mathcal{A}_2\} \in \mathcal{P}'(\mathbb{P}). \tag{2.9}$$

The routes from $\mathbf{P}$ (2.9) are treated as feasible in the $\mathcal{M}$-problem (based on the sequential solution of the $\mathcal{M}_1$- and $\mathcal{M}_2$-problems). Following [11, formulas (2.11)–(2.13)], we naturally arrive at the tracks described by [11, formula (2.14)]. To this end, let $\mathfrak{Z} \triangleq (X \times X)^{\overline{0,\mathbf{n}}}$. According to [11, formulas (2.11)–(2.13)], for $x \in X^0$ and $\gamma \in \mathbf{P}$, the tracks of the bundle

$$\mathcal{Z}_\gamma[x] \triangleq \Big\{(z_t)_{t \in \overline{0,\mathbf{n}}} \in \mathfrak{Z} \mid (z_0 = (x,x)) \& (z_\tau \in \mathbb{M}_{\gamma(\tau)} \ \ \forall \tau \in \overline{1,\mathbf{n}})\Big\} \in \mathrm{Fin}(\mathfrak{Z}) \tag{2.10}$$

start from the point x (in the notations of [11, 12], from (x,x); this difference is nonessential) and are consistent with γ. In addition, for $x \in X^0$,

$$\tilde{\mathbf{D}}[x] \triangleq \Big\{(\gamma, (z_t)_{t \in \overline{0,\mathbf{n}}}) \in \mathbf{P} \times \mathfrak{Z} \mid (z_t)_{t \in \overline{0,\mathbf{n}}} \in \mathcal{Z}_\gamma[x]\Big\} \in \mathrm{Fin}(\mathbf{P} \times \mathfrak{Z}) \tag{2.11}$$

is the set of all feasible solutions (FSs) in the $\mathcal{M}$-problem with the starting point x, i.e., feasible in the $(\mathcal{M}, x)$-problem. Finally,

$$\mathbf{D} \triangleq \Big\{(\gamma, (z_t)_{t \in \overline{0,\mathbf{n}}}, x) \in \mathbf{P} \times \mathfrak{Z} \times X^0 \mid (\gamma, (z_t)_{t \in \overline{0,\mathbf{n}}}) \in \tilde{\mathbf{D}}[x]\Big\} \in \mathrm{Fin}(\mathbf{P} \times \mathfrak{Z} \times X^0) \tag{2.12}$$

is the set of all FSs in the $\mathcal{M}$-problem, called routing processes (RPs). For substantive details, see [11, 12]. Following [11, formula (2.17)], for $j \in \overline{1,\mathbf{n}}$, we introduce the sets

$$\Big(\mathfrak{M}_j \triangleq \{\mathrm{pr}_1(z) : z \in \mathbb{M}_j\} \in \mathrm{Fin}(M_j)\Big) \,\&\, \Big(\mathbf{M}_j \triangleq \{\mathrm{pr}_2(z) : z \in \mathbb{M}_j\} \in \mathrm{Fin}(M_j)\Big)$$

with the property $\mathbb{M}_j \subset \mathfrak{M}_j \times \mathbf{M}_j \subset M_j \times M_j$; see (2.2). For $j \in \overline{1, \mathbf{n} - N}$, it is obvious that

$$\begin{gathered} M^{(j)} = M_{N+j} \in \mathrm{Fin}(X), \quad \mathbb{M}^{(j)} \triangleq \mathbb{M}_{N+j} \in \mathcal{P}'(M^{(j)} \times M^{(j)}), \\ \mathfrak{M}^{(j)} \triangleq \mathfrak{M}_{N+j} \in \mathrm{Fin}(M^{(j)}), \quad \mathbf{M}^{(j)} \triangleq \mathbf{M}_{N+j} \in \mathrm{Fin}(M^{(j)}). \end{gathered} \tag{2.13}$$

Here, the sets used in the final problem are simply renumbered. In view of (2.13), we also suppose [11, formula (2.18)] that $\bar{\mathbf{M}}$ is the union of all sets $\mathbf{M}^{(i)}$, $i \in \overline{1, \mathbf{n} - N}$. Finally, let $\mathfrak{N} \triangleq \mathcal{P}'(\overline{1,\mathbf{n}})$; the sets from $\mathfrak{N}$ are called lists. Following [11, formula (2.19)], we have

$$\left(\mathbb{X} \triangleq \bigcup_{i=1}^{\mathbf{n}} \mathfrak{M}_i \in \mathrm{Fin}(X)\right) \,\&\, \left(\mathbf{X} \triangleq \left(\bigcup_{i=1}^{\mathbf{n}} \mathbf{M}_i\right) \cup X^0 \in \mathrm{Fin}(X)\right). \tag{2.14}$$

Also, we fix the real-valued functions

$$\mathbf{c} \in \mathcal{R}_+[\mathbf{X} \times \mathbb{X} \times \mathfrak{N}], \ c_1 \in \mathcal{R}_+[\mathbb{M}_1 \times \mathfrak{N}], \ \ldots, c_{\mathbf{n}} \in \mathcal{R}_+[\mathbb{M}_{\mathbf{n}} \times \mathfrak{N}], \quad f \in \mathcal{R}_+[\bar{\mathbf{M}}]. \tag{2.15}$$

Note in this regard that for $\gamma \in \mathbf{P}$ and $\tau \in \overline{1, \mathbf{n}}$, $\gamma^1(\overline{\tau, \mathbf{n}}) \in \mathfrak{N}$ is an image of the discrete interval $\overline{\tau, \mathbf{n}}$ under the action of γ. The values of $\mathbf{c}$ serve to estimate external movements; the values of $c_1, \ldots, c_{\mathbf{n}}$, to estimate internal work associated with visiting megalopolises; the values of f, to estimate the terminal state. One of the arguments of the functions $\mathbf{c}, c_1, \ldots, c_N$ is the list of pending tasks. This dependence may arise in the sheet cutting problem when considering possible thermal deformations of the parts during cutting by introducing penalties. Speaking about the general problem statement, we emphasize that the dependence on the list of pending tasks may also arise for other reasons. (For example, in the sequential disassembly of nuclear facilities in case of accidents, only those objects radiate that are not dismantled at the moment.) Now, let us proceed to constructing the additive criterion. Following [11, formula (2.21)], for $x \in X^0$, $\gamma \in \mathbf{P}$, and $(z_t)_{t \in \overline{0, \mathbf{n}}} \in \mathcal{Z}_\gamma[x]$, we have

$$\mathfrak{C}_\gamma\left[(z_t)_{t \in \overline{0,\mathbf{n}}}\right] \triangleq \sum_{t=1}^{\mathbf{n}} \Big[\mathbf{c}(\mathrm{pr}_2(z_{t-1}), \mathrm{pr}_1(z_t), \gamma^1(\overline{t, \mathbf{n}})) + c_{\gamma(t)}(z_t, \gamma^1(\overline{t, \mathbf{n}}))\Big] + f(\mathrm{pr}_2(z_{\mathbf{n}})). \tag{2.16}$$

Of course, see (2.11), the value of (2.16) is defined for $x \in X^0$ and $\left(\gamma, (z_t)_{t \in \overline{0,\mathbf{n}}}\right) \in \tilde{\mathbf{D}}[x]$. For $x \in X^0$, we introduce the $(\mathcal{M}, x)$-problem

$$\mathfrak{C}_\gamma\left[(z_t)_{t \in \overline{0,\mathbf{n}}}\right] \to \min, \quad \left(\gamma, (z_t)_{t \in \overline{0,\mathbf{n}}}\right) \in \tilde{\mathbf{D}}[x]. \tag{2.17}$$

Its optimum $\tilde{V}[x] \in \mathbb{R}_+$ and the non-empty set $(\mathrm{sol})[x]$ of all its optimal solutions are given by

$$\tilde{V}[x] \triangleq \min_{\left(\gamma, (z_t)_{t \in \overline{0,\mathbf{n}}}\right) \in \tilde{\mathbf{D}}[x]} \mathfrak{C}_\gamma\left[(z_t)_{t \in \overline{0,\mathbf{n}}}\right] \in \mathbb{R}_+, \tag{2.18}$$

$$(\mathrm{sol})[x] \triangleq \left\{\left(\gamma^0, (z_t^0)_{t \in \overline{0,\mathbf{n}}}\right) \in \tilde{\mathbf{D}}[x] \mid \mathfrak{C}_{\gamma^0}\left[(z_t^0)_{t \in \overline{0,\mathbf{n}}}\right] = \tilde{V}[x]\right\} \in \mathrm{Fin}(\tilde{\mathbf{D}}[x]). \tag{2.19}$$

In (2.17), the total travel cost of external movements, internal works, and terminal state implementation is optimized; in the case of sheet cutting, these types of travel cost are associated with non-cutting stroke, cutting in the operating mode, and, e.g., tool "parking," respectively. In the simplest case, the components mentioned can be characterized by the execution times of the corresponding operations. Generally, we obtain a minimization problem for the additive criterion (2.16) in the class of FSs defined each as a route–track OP; (2.19) is the set of all optimal route–track OPs in the mentioned sense. We admit the possibility of varying $x \in X^0$. As a consequence, the $\mathcal{M}$-problem (the original problem) arises:

$$\mathfrak{C}_\gamma\left[(z_t)_{t \in \overline{0,\mathbf{n}}}\right] \to \min, \quad \left(\gamma, (z_t)_{t \in \overline{0,\mathbf{n}}}, x\right) \in \mathbf{D}. \tag{2.20}$$

Problem (2.20) is assigned the global optimum $\mathbb{V}$ and the non-empty set $\mathbf{SOL}$ of all optimal RPs:

$$\begin{aligned}\mathbb{V} &\triangleq \min_{(\gamma, (z_t)_{t \in \overline{0,\mathbf{n}}}, x) \in \mathbf{D}} \mathfrak{C}_\gamma\left[(z_t)_{t \in \overline{0,\mathbf{n}}}\right] \\ &= \min_{x \in X^0} \min_{\left(\gamma, (z_t)_{t \in \overline{0,\mathbf{n}}}\right) \in \tilde{\mathbf{D}}[x]} \mathfrak{C}_\gamma\left[(z_t)_{t \in \overline{0,\mathbf{n}}}\right] = \min_{x \in X^0} \tilde{V}[x] \in \mathbb{R}_+,\end{aligned} \tag{2.21}$$

$$\mathbf{SOL} \triangleq \left\{(\gamma^0, (z_t^0)_{t \in \overline{0,\mathbf{n}}}, x^0) \in \mathbf{D} \mid \mathfrak{C}_{\gamma^0}[(z_t^0)_{t \in \overline{0,\mathbf{n}}}] = \mathbb{V}\right\} \in \mathrm{Fin}(\mathbf{D}). \tag{2.22}$$

(In (2.21), we take advantage of (2.12) and (2.18).) The elements of the set (2.22) represent optimal RPs. They are triplets, in contrast to the elements of (2.19). With the dependence

$$\tilde{V}[\cdot] \triangleq (\tilde{V}[x])_{x \in X^0} \in \mathcal{R}_+[X^0] \tag{2.23}$$

we associate the starting point minimization problem:

$$\tilde{V}[x] \to \min, \quad x \in X^0. \tag{2.24}$$

In this problem, the criterion is given by (2.23) and the optimum coincides with $\mathbb{V}$ (see (2.21)); the optimum set has the form

$$X^0_{\text{opt}} \triangleq \left\{ x \in X^0 \mid \tilde{V}[x] = \mathbb{V} \right\} \in \mathcal{P}'(X^0). \tag{2.25}$$

In (2.24), the starting point is optimized provided that, in the case of its particular choice, the FS (the route–track OP) is also selected optimally. As a result, the global maximum is reached.

Proposition 1. *If $x \in X^0_{\text{opt}}$ and $(\gamma, (z_t)_{t \in \overline{0,\mathbf{n}}}) \in (\text{sol})[x]$, then $(\gamma, (z_t)_{t \in \overline{0,\mathbf{n}}}, x) \in \mathbf{SOL}$.*

The proof is obvious by combining (2.19), (2.22), and (2.25). Indeed, in (2.21), we sequentially reach the minimum, first in the FS of the elements (2.11) under a fixed starting point and then in the starting point itself (see (2.24) and (2.25)).

3. PRELIMINARY AND FINAL ROUTING PROBLEMS

As already noted, to solve problem (2.20), we actually decompose it into the set of the $\mathcal{M}_1$- and $\mathcal{M}_2$-problems. This decomposition will be optimal in some sense: we find (2.21) and some RP from the set (2.22). Let us discuss this optimal method (see [11, 12]) on a substantive level after formulating the partial problems mentioned. The statement of the $\mathcal{M}_1$-problem depends on the set of parameters defined when solving the $\mathcal{M}_2$-problem. For this reason, we first discuss the $\mathcal{M}_2$-problem (the upper-level problem or the final problem). However, its statement incorporates an object related very simply to the $\mathcal{M}_1$-problem. We begin with this object, i.e., the set of starting points in the $\mathcal{M}_2$-problem. Letting

$$\tilde{\mathbf{K}}_1 \triangleq \{\text{pr}_1(h) : h \in \mathbf{K}_1\}, \tag{3.1}$$

we obtain $\overline{1,N} \setminus \tilde{\mathbf{K}}_1 \in \mathcal{P}'(\overline{1,N})$ by [8, formula (4.9.9), Proposition 4.9.3]. Then

$$X^{00} \triangleq \bigcup_{i \in \overline{1,N} \setminus \tilde{\mathbf{K}}_1} \mathbf{M}_i \in \text{Fin}(\mathbf{X}) \tag{3.2}$$

is the set of all possible starting points of the $\mathcal{M}_2$-problem in which $M^{(1)}, \ldots, M^{(\mathbf{n}-N)}$ form the family of megalopolises to be visited (see (2.14)).

The set X^{00} plays the role of "input variable" for the $\mathcal{M}_2$-problem. Right after obtaining X^{00} (3.2), this problem is well defined in terms of the set of parameters. It is solved by determining the necessary fragments (layers) of the Bellman function (the optimal solution of the $\mathcal{M}_2$-problem is postponed). As a result, we find the value (optimum) function defined on X^{00}. This function then determines the terminal component of the additive criterion in the $\mathcal{M}_1$-problem, therefore representing the "output variable" for the $\mathcal{M}_2$-problem. The $\mathcal{M}_2$-problem participates in constructing the criterion of the $\mathcal{M}_1$-problem (the preliminary problem). Hence, it is possible to solve the $\mathcal{M}_1$-problem (the criterion is completely defined). Now we construct the layers of the Bellman function of this problem and, in particular, the value function defined on X^0. According to [12], this is sufficient to find $\mathbb{V}$ and the optimal starting point. Next, the $\mathcal{M}_1$-optimal FS is standardly built for this point and its finish point is fixed. The $\mathcal{M}_2$-optimal solution is then implemented from this point as a route–track OP and is subsequently glued with the $\mathcal{M}_1$-optimal solution to form the optimal FS of the aggregate problem with a fixed starting point. Complementing the componentwise glued FS with the latter, we finally obtain the optimal RP. This is the general scheme of the study, applied below. Now, let us recall some concepts related to the final problem.

For $j \in \overline{1, \mathbf{n} - N}$, with the megalopolis $M^{(j)}$ we associate a relation $\mathbb{M}^{(j)}$ such that

$$\Big(\mathfrak{M}^{(j)} = \{\mathrm{pr}_1(z) : z \in \mathbb{M}^{(j)}\} \in \mathrm{Fin}(M^{(j)})\Big) \;\&\; \Big(\mathbf{M}^{(j)} = \{\mathrm{pr}_2(z) : z \in \mathbb{M}^{(j)}\} \in \mathrm{Fin}(M^{(j)})\Big) \qquad (3.3)$$

(see (2.13)). The set $\mathcal{A}_2$ consists of all feasible routes in the $\mathcal{M}_2$-problem. Let $\mathfrak{Z}^* \triangleq (X \times X)^{\overline{0, \mathbf{n}-N}}$; for $x \in X^{00}$ and $\beta \in \mathcal{A}_2$,

$$\mathcal{Z}^*_\beta[x] \triangleq \Big\{(z_t)_{t \in \overline{0, \mathbf{n}-N}} \in \mathfrak{Z}^* \mid (z_0 = (x, x))\,\&\,(z_t \in \mathbb{M}^{(\beta(t))} \;\; \forall t \in \overline{1, \mathbf{n} - N})\Big\} \in \mathrm{Fin}(\mathfrak{Z}^*), \qquad (3.4)$$

is the bundle of tracks in the $\mathcal{M}_2$-problem that start from the point x and are consistent with β. From (2.13) and (2.14) it follows that $\mathbf{M}^{(j)} \subset \mathbf{X}$ for $j \in \overline{1, \mathbf{n} - N}$; in addition, $X^{00} \subset \mathbf{X}$ due to (2.14) and (3.2). From (2.13), (2.14), (3.3), and (3.4), we obtain $\mathrm{pr}_1(z_\tau) \in \mathbb{X}$ and $\mathrm{pr}_2(z_\tau) \in \mathbf{X}$ for $x \in X^{00}$, $\beta \in \mathcal{A}_2$, $(z_t)_{t \in \overline{0, \mathbf{n}-N}} \in \mathcal{Z}^*_\beta[x]$, and $\tau \in \overline{1, \mathbf{n} - N}$. Therefore, see (2.13), $\mathbf{c}(\mathrm{pr}_2(z_{\tau-1}), \mathrm{pr}_1(z_\tau), K) \in \mathbb{R}_+$ and $c_{N+\beta(\tau)}(z_\tau, K) \in \mathbb{R}_+$, where $K \in \mathfrak{N}$, are defined for $x \in X^{00}$, $\beta \in \mathcal{A}_2$, $(z_t)_{t \in \overline{0, \mathbf{n}-N}} \in \mathcal{Z}^*_\beta[x]$, and $\tau \in \overline{1, \mathbf{n} - N}$. In view of (3.4), for $x \in X^{00}$,

$$\mathbf{D}^*[x] \triangleq \Big\{(\beta, (z_t)_{t \in \overline{0, \mathbf{n}-N}}) \in \mathcal{A}_2 \times \mathfrak{Z}^* \mid (z_t)_{t \in \overline{0, \mathbf{n}-N}} \in \mathcal{Z}^*_\beta[x]\Big\} \in \mathrm{Fin}(\mathcal{A}_2 \times \mathfrak{Z}^*), \qquad (3.5)$$

is the set of all FSs in the $(\mathcal{M}_2, x)$-problem, i.e., the $\mathcal{M}_2$-problem with the starting point x. Let $\mathfrak{N}^* \triangleq \mathcal{P}'(\overline{1, \mathbf{n} - N})$ (the family of all non-empty subsets of $\overline{1, \mathbf{n} - N}$); for $K \in \mathfrak{N}^*$, we have the list

$$K \oplus N = \{k + N : k \in K\} \in \mathfrak{N} \qquad (3.6)$$

in the original problem. Obviously (see [11, formula (4.9)]), the value

$$\sum_{t=1}^{\mathbf{n}-N} \Big[\mathbf{c}(\mathrm{pr}_2(z_{t-1}), \mathrm{pr}_1(z_t), \beta^1(\overline{t, \mathbf{n} - N}) \oplus N) + c_{N+\beta(t)}(z_t, \beta^1(\overline{t, \mathbf{n} - N}) \oplus N)\Big] + f(\mathrm{pr}_2(z_{\mathbf{n}-N})) \in \mathbb{R}_+ \qquad (3.7)$$

for $x \in X^{00}$, $\beta \in \mathcal{A}_2$, and $(z_t)_{t \in \overline{0, \mathbf{n}-N}} \in \mathcal{Z}^*_\beta[x]$. To derive expressions similar to [14, formula (3.16)] for these values, we introduce a transformation of travel cost functions. To this end, we first define

$$\left(\mathbb{X}^* \triangleq \left(\bigcup_{i=1}^{\mathbf{n}-N} \mathfrak{M}^{(i)}\right) \in \mathrm{Fin}(X)\right) \;\&\; \left(\mathbf{X}^* \triangleq \left(\bigcup_{i=1}^{\mathbf{n}-N} \mathbf{M}^{(i)}\right) \cup X^{00} \in \mathrm{Fin}(X)\right) \qquad (3.8)$$

with the properties $\mathbb{X}^* \subset \mathbb{X}$ and $\mathbf{X}^* \subset \mathbf{X}$ (i.e., consider (2.13), (2.14), and (3.2)). Then, we define $\mathbf{c}^* \in \mathcal{R}_+[\mathbf{X}^* \times \mathbb{X}^* \times \mathfrak{N}^*]$ by the rule

$$\mathbf{c}^*(x, y, K) \triangleq \mathbf{c}(x, y, K \oplus N) \;\; \forall x \in \mathbf{X}^* \;\; \forall y \in \mathbb{X}^* \;\; \forall K \in \mathfrak{N}^* \qquad (3.9)$$

(see (2.15) and (3.6)). Formulas (3.8) and (3.9) determine the travel cost of external movements in the $\mathcal{M}_2$-problem (see (3.7)). For $j \in \overline{1, \mathbf{n} - N}$, we define $c^*_j \in \mathcal{R}_+[\mathbb{M}^{(j)} \times \mathfrak{N}^*]$ by the rule

$$c^*_j(z, K) \triangleq c_{N+j}(z, K \oplus N) \;\; \forall z \in \mathbb{M}^{(j)} \;\; \forall K \in \mathfrak{N}^*. \qquad (3.10)$$

Formula (3.10) determines the travel cost of internal works in the $\mathcal{M}_2$-problem. Finally, the terminal component of the additive criterion coincides with f (see (2.15)). Thus, $(\mathbf{c}^*, c^*_1, \ldots, c^*_{\mathbf{n}-N}, f)$ is the tuple of real-valued travel cost functions in the $\mathcal{M}_2$-problem. In this case, for $x \in X^{00}$, $\beta \in \mathcal{A}_2$, and $(z_t)_{t\in\overline{0,\mathbf{n}-N}} \in \mathcal{Z}^*_\beta[x]$, the expression (3.7) reduces to

$$\mathfrak{C}^*_\beta\left[(z_t)_{t\in\overline{0,\mathbf{n}-N}}\right] \triangleq \sum_{t=1}^{\mathbf{n}-N}\Big[\mathbf{c}^*(\mathrm{pr}_2(z_{t-1}), \mathrm{pr}_1(z_t), \beta^1(\overline{t,\mathbf{n}-N})) + c^*_{\beta(t)}(z_t, \beta^1(\overline{t,\mathbf{n}-N}))\Big] + f(\mathrm{pr}_2(z_{\mathbf{n}-N})) \in \mathbb{R}_+, \quad (3.11)$$

thereby defining the additive criterion of the $\mathcal{M}_2$-problem. For $x \in X^{00}$, we obtain the $(\mathcal{M}_2, x)$-problem

$$\mathfrak{C}^*_\beta\left[(z_t)_{t\in\overline{0,\mathbf{n}-N}}\right] \to \min, \quad \left(\beta, (z_t)_{t\in\overline{0,\mathbf{n}-N}}\right) \in \mathbf{D}^*[x], \quad (3.12)$$

with an optimum $\tilde{V}^*[x] \in \mathbb{R}_+$ and a (non-empty) set $(\mathrm{sol})^*[x]$ of all optimal solutions:

$$\tilde{V}^*[x] \triangleq \min_{(\beta,(z_t)_{t\in\overline{0,\mathbf{n}-N}})\in\mathbf{D}^*[x]} \mathfrak{C}^*_\beta\left[(z_t)_{t\in\overline{0,\mathbf{n}-N}}\right] \in \mathbb{R}_+, \quad (3.13)$$

$$(\mathrm{sol})^*[x] \triangleq \left\{\left(\beta, (z_t)_{t\in\overline{0,\mathbf{n}-N}}\right) \in \mathbf{D}^*[x] \mid \mathfrak{C}^*_\beta\left[(z_t)_{t\in\overline{0,\mathbf{n}-N}}\right] = \tilde{V}^*[x]\right\} \in \mathrm{Fin}(\mathbf{D}^*[x]). \quad (3.14)$$

Formula (3.13) determines an important function $\tilde{V}^*[\cdot]$ of the form

$$x \mapsto \tilde{V}^*[x] : X^{00} \to \mathbb{R}_+. \quad (3.15)$$

The expressions (3.14) and (3.15) can be treated as the resultants of the $\mathcal{M}_2$-problem. Using (3.15), we construct the additive criterion of the $\mathcal{M}_1$-problem.

Let $\mathfrak{Z}^\natural \triangleq (X\times X)^{\overline{0,N}}$; for $x \in X^0$ and $\alpha \in \mathcal{A}_1$,

$$\mathcal{Z}^\natural_\alpha[x] \triangleq \left\{(z_t)_{t\in\overline{0,N}} \in \mathfrak{Z}^\natural \mid (z_0 = (x,x))\&(z_\tau \in \mathbb{M}_{\alpha(\tau)} \ \ \forall \tau \in \overline{1,N})\right\} \in \mathrm{Fin}(\mathfrak{Z}^\natural) \quad (3.16)$$

is the bundle of tracks in the $\mathcal{M}_1$-problem that start from the point x and are consistent with the route α. We introduce the sets

$$\left(\mathbb{X}^\natural \triangleq \bigcup_{i=1}^N \mathfrak{M}_i \in \mathrm{Fin}(X)\right) \& \left(\mathbf{X}^\natural \triangleq \left(\bigcup_{i=1}^N \mathbf{M}_i\right) \cup X^0 \in \mathrm{Fin}(X)\right), \quad (3.17)$$

for which $\mathbb{X}^\natural \subset \mathbb{X}$ and $\mathbf{X}^\natural \subset \mathbf{X}$ (see (2.14)). For $x \in X^0$, we define the set of all FSs in the $(\mathcal{M}_1, x)$-problem (the $\mathcal{M}_1$-problem with the starting point x):

$$\mathbf{D}^\natural[x] \triangleq \left\{\left(\alpha, (z_t)_{t\in\overline{0,N}}\right) \in \mathcal{A}_1 \times \mathfrak{Z}^\natural \mid (z_t)_{t\in\overline{0,N}} \in \mathcal{Z}^\natural_\alpha[x]\right\} \in \mathrm{Fin}(\mathcal{A}_1 \times \mathfrak{Z}^\natural). \quad (3.18)$$

Note a quite obvious and important property for "binding" the $\mathcal{M}_2$-problem to the $\mathcal{M}_1$-problem: if $x \in X^0$, $\alpha \in \mathcal{A}_1$, and $(z_t)_{t\in\overline{0,N}} \in \mathcal{Z}^\natural_\alpha[x]$, then

$$\mathrm{pr}_2(z_N) \in X^{00}. \quad (3.19)$$

(For details, see [11, Proposition 3.3].) Now we sequentially introduce the travel cost functions forming the additive criterion in the $\mathcal{M}_1$-problem. We begin with the terminal component, actually identifying it with the function $\tilde{V}^*[\cdot]$ (3.15). Precisely, assuming that $\mathbf{M}^\natural$ is the union of all sets $\mathbf{M}_i$, $i \in \overline{1,N}$, we define $\mathbf{f} \in \mathcal{R}_+[\mathbf{M}^\natural]$ by the rule

$$\left(\mathbf{f}(x) \triangleq \tilde{V}^*[x] \ \ \forall x \in X^{00}\right) \& \left(\mathbf{f}(x) \triangleq 0 \ \ \forall x \in \mathbf{M}^\natural \setminus X^{00}\right). \quad (3.20)$$

The second expression in (3.20) is insignificant; it ensures consistency with [14]. Let $\mathfrak{N}^\natural \triangleq \mathcal{P}'(\overline{1,N})$ (the family of all non-empty subsets of $\overline{1,N}$.) In view of (2.15), we define $\mathbf{c}^\natural \in \mathcal{R}_+[\mathbf{X}^\natural \times \mathbb{X}^\natural \times \mathfrak{N}^\natural]$ by the conditions

$$\mathbf{c}^\natural(z,K) \triangleq \mathbf{c}(z, K \cup \overline{N+1,\mathbf{n}}) \ \ \forall z \in \mathbf{X}^\natural \times \mathbb{X}^\natural \ \ \forall K \in \mathfrak{N}^\natural. \tag{3.21}$$

For $j \in \overline{1,N}$, the function $c_j^\natural \in \mathcal{R}_+[\mathbb{M}_j \times \mathfrak{N}^\natural]$ is given by the rule

$$\mathbf{c}_j^\natural(z,K) \triangleq c_j(z, K \cup \overline{N+1,\mathbf{n}}) \ \ \forall z \in \mathbb{M}_j \ \ \forall K \in \mathfrak{N}^\natural. \tag{3.22}$$

Thus, formulas (3.20)–(3.22) determine the tuple $(\mathbf{c}, c_1^\natural, \ldots, c_N^\natural, \mathbf{f})$ of the real-valued travel cost functions in the $\mathcal{M}_1$-problem. For $x \in X^0$, $\alpha \in \mathcal{A}_1$, and $(z_t)_{t\in\overline{0,N}} \in \mathcal{Z}_\alpha^\natural[x]$, let (see (3.20)–(3.22))

$$\begin{aligned}\mathfrak{C}_\alpha^\natural[(z_t)_{t\in\overline{0,N}}] &\triangleq \sum_{t=1}^{N}\Big[\mathbf{c}^\natural(\mathrm{pr}_2(z_{t-1}), \mathrm{pr}_1(z_t), \alpha^1(\overline{t,N})) + c_{\alpha(t)}^\natural(z_t, \alpha^1(\overline{t,N}))\Big] + \mathbf{f}(\mathrm{pr}_2(z_N)) \\ &= \sum_{t=1}^{N}\Big[\mathbf{c}^\natural(\mathrm{pr}_2(z_{t-1}), \mathrm{pr}_1(z_t), \alpha^1(\overline{t,N})) + c_{\alpha(t)}^\natural(z_t, \alpha^1(\overline{t,N}))\Big] + \tilde{V}^*[\mathrm{pr}_2(z_N)];\end{aligned} \tag{3.23}$$

these expressions take into account (3.19) and (3.20). For $x \in X^0$, we define the $(\mathcal{M}_1, x)$-problem as

$$\mathfrak{C}_\alpha^\natural\left[(z_t)_{t\in\overline{0,N}}\right] \to \min, \ \ \left(\alpha, (z_t)_{t\in\overline{0,N}}\right) \in \mathbf{D}^\natural[x], \tag{3.24}$$

with an optimum $V^\natural[x]$ and a (non-empty) set $(\mathrm{sol})^\natural[x]$ of all optimal solutions:

$$V^\natural[x] \triangleq \min_{\left(\alpha,(z_t)_{t\in\overline{0,N}}\right)\in\mathbf{D}^\natural[x]} \mathfrak{C}_\alpha^\natural\left[(z_t)_{t\in\overline{0,N}}\right] \in \mathbb{R}_+, \tag{3.25}$$

$$(\mathrm{sol})^\natural[x] \triangleq \left\{\left(\alpha, (z_t)_{t\in\overline{0,N}}\right) \in \mathbf{D}^\natural[x] \mid \mathfrak{C}_\alpha^\natural\left[(z_t)_{t\in\overline{0,N}}\right] = V^\natural[x]\right\} \in \mathrm{Fin}(\mathbf{D}^\natural[x]). \tag{3.26}$$

Then $x \mapsto V^\natural[x] : X^0 \to \mathbb{R}_+$ is the value function $V^\natural[\cdot]$ in the $\mathcal{M}_1$-problem, with the starting point as the argument. For separate consideration, we take the problem

$$V^\natural[x] \to \min, \ \ x \in X^0, \tag{3.27}$$

with an optimum $\mathbb{V}^\natural \in \mathbb{R}_+$ and a (non-empty) set $X_{\mathrm{opt}}^\natural \in \mathrm{Fin}(X^0)$ of all optimal starting points:

$$\mathbb{V}^\natural \triangleq \min_{x\in X^0} V^\natural[x] \in \mathbb{R}_+, \tag{3.28}$$

$$X_{\mathrm{opt}}^\natural \triangleq \left\{x \in X^0 \mid V^\natural[x] = \mathbb{V}^\natural\right\} \in \mathrm{Fin}(X^0). \tag{3.29}$$

Thus, we have two interconnected subproblems for the decomposition-based solution of the original problem.

4. SOLUTION STRUCTURE AT THE FUNCTIONAL LEVEL: A SUBSTANTIVE DISCUSSION

This brief section outlines an optimal solution scheme for the (original) $\mathcal{M}$-problem corresponding to the constructs of the previous two sections. We describe the logical chain only (in fact, an algorithm at the functional level).

Stage 1. Using (3.2), determine the set X^{00} of possible starting points in the $\mathcal{M}_2$-problem.

Stage 2. Form the $\mathcal{M}_2$-problem as the system of the $(\mathcal{M}_2, x)$-problems (3.12), where $x \in X^{00}$; determine the value function $\tilde{V}^*[\cdot]$ (3.15), used to construct the terminal component of the additive criterion in the $\mathcal{M}_1$-problem (see (3.20)).

Stage 3. Form the $\mathcal{M}_1$-problem as the system of the $(\mathcal{M}_1, x)$-problems (3.24), where $x \in X^0$; determine the value function $V^\natural[\cdot]$ and the optimum $\mathbb{V}^\natural$ in Problem (3.27) and some (optimal) starting point $x^0 \in X^\natural_{\mathrm{opt}}$; also, determine the optimal solution of the $(\mathcal{M}_1, x^0)$-problem from the set (3.26), where $x = x^0$, calling it the $\mathcal{M}_1$-solution.

Stage 4. Choose the finish point x^{00} of the $\mathcal{M}_1$-solution, an element of X^{00} (see (3.19)), as the starting point in the $\mathcal{M}_2$-problem. Then, construct the optimal solution of the $(\mathcal{M}_2, x^{00})$-problem, thereby obtaining the $\mathcal{M}_2$-solution.

Stage 5. Glue the $\mathcal{M}_1$- and $\mathcal{M}_2$-solutions (separately their routes and tracks), thereby obtaining the optimal RP with x^0.

We emphasize the equality $\mathbb{V} = \mathbb{V}^\natural$, which was established in [11, 12]. It serves for implementing $\mathbb{V}$ easily when seeking the global optimum and the optimal starting point only (i.e., if the optimal RP itself is not so important). In this case, Stages 1 and 2 are retained, and Stage 3 is reduced to finding the value function $V^\natural[\cdot]$. (Minimizing this function on X^0 gives the required value $\mathbb{V}$ and the starting point implementing it.)

Note that the broadly understood DP is the apparatus to implement Stages 1–4.

Let us introduce the natural gluing procedure for tracks. This procedure differs from permutation gluing: it operates tracks, quite different objects. In this regard, we introduce a new designation. First, assume that for $\mathbf{z}' \in \mathfrak{Z}^\natural$ and $\mathbf{z}'' \in \mathfrak{Z}^*$, the tuple $\mathbf{z}'\Box\mathbf{z}'' \in \mathfrak{Z}$ is given by the rule

$$\Big((\mathbf{z}'\Box\mathbf{z}'')(\tau) \triangleq \mathbf{z}'(\tau) \ \ \forall \tau \in \overline{0, N}\Big) \,\&\, \Big((\mathbf{z}'\Box\mathbf{z}'')(\tau) \triangleq \mathbf{z}''(\tau - N) \ \ \forall \tau \in \overline{N+1, \mathbf{n}}\Big). \tag{4.1}$$

In particular, $\mathbf{z}'$ and $\mathbf{z}''$ in (4.1) can be tracks. According to [11, Proposition 6.4], $\forall x \in X^0 \ \ \forall \alpha \in \mathcal{A}_1$ $\forall \beta \in \mathcal{A}_2 \ \ \forall \mathbf{z}' \in \mathcal{Z}^\natural_\alpha[x] \ \ \forall \mathbf{z}'' \in \mathcal{Z}^*_\beta[\mathrm{pr}_2(\mathbf{z}'(N))]$

$$\mathbf{z}'\Box\mathbf{z}'' \in \mathcal{Z}_{\alpha\Diamond\beta}[x]. \tag{4.2}$$

Obviously, see (3.19), $\mathrm{pr}_2(z_N) \in X^{00}$ for $x \in X^0$, $\alpha \in \mathcal{A}_1$, $\beta \in \mathcal{A}_2$, and $(z_t)_{t\in\overline{0,\mathbf{n}}} \in \mathcal{Z}_{\alpha\Diamond\beta}[x]$. In this context, we also recall [11, Proposition 6.5]: if $x \in X^0$, $\alpha \in \mathcal{A}_1$, $\beta \in \mathcal{A}_2$, $(z_t)_{t\in\overline{0,\mathbf{n}}} \in \mathcal{Z}_{\alpha\Diamond\beta}[x]$ and a tuple $(z^*_t)_{t\in\overline{0,\mathbf{n}-N}} \in \mathfrak{Z}^*$ is given by the rule

$$\Big(z^*_0 \triangleq (\mathrm{pr}_2(z_N), \mathrm{pr}_2(z_N))\Big) \,\&\, \Big(z^*_\tau \triangleq z_{\tau+N} \ \ \forall \tau \in \overline{1, \mathbf{n}-N}\Big),$$

then

$$(z^*_t)_{t\in\overline{0,\mathbf{n}-N}} \in \mathcal{Z}^*_\beta[\mathrm{pr}_2(z_N)]. \tag{4.3}$$

In view of (4.2) and (4.3), we establish the following result (see [12, Theorem 1]):

$$\mathbb{V} = \mathbb{V}^\natural. \tag{4.4}$$

Note that (3.28) and [12, formula (6.12)] must be considered in (4.4). Let us present another result concerning the logical chain 1)–5).

Proposition 2.

Assume that $x^0 \in X^\natural_{\mathrm{opt}}$, $(\xi, (y_i)_{i\in\overline{0,N}}) \in (\mathrm{sol})^\natural[x^0]$, *and* $(\eta, (\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}) \in (\mathrm{sol})^*[\mathrm{pr}_2(y_N)]$. *Then the solution*

$$\Big(\xi \diamond \eta, (y_i)_{i\in\overline{0,N}} \,\Box\, (\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}\Big) \in \tilde{\mathbf{D}}[x^0] \tag{4.5}$$

is such that

$$\mathfrak{C}_{\xi\diamond\eta}\left[(y_i)_{i\in\overline{0,N}}\,\square\,(\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}\right] = \mathbb{V}. \tag{4.6}$$

Proposition 2 is proved in the Appendix. In fact, it provides an explicit method for constructing the optimal RP.

Really, let all conditions of Proposition 2 be true (those imposed on x^0, $(\xi, (y_i)_{i\in\overline{0,N}})$, and $(\eta, (\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}})$). In this case, due to (2.12), (4.5), and $x^0 \in X^0$ (see (3.29)), we obtain

$$\left(\xi \diamond \eta, (y_i)_{i\in\overline{0,N}}\,\square\,(\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}, x^0\right) \in \mathbf{D} \tag{4.7}$$

(a feasible RP with the property (4.6)). By (2.22) and (4.6), we have

$$\left(\xi \diamond \eta, (y_i)_{i\in\overline{0,N}}\,\square\,(\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}, x^0\right) \in \mathbf{SOL}, \tag{4.8}$$

i.e., the optimal RP has been built. Note also that $\tilde{V}[x^0] = \mathbb{V}$ due to (2.18), (2.21), and (4.4)–(4.6); consequently, $x^0 \in X^0_{\rm opt}$ (see (2.25)). In the next sections, we discuss the implementation of Stages 1–5 in the light of (4.4) and (4.8).

In addition to the framework [11, 12], we mention several properties similar to the one of Proposition 2 in terms of gluing constructs. These properties are important for the further application of DP.

Proposition 3. *If* $x \in X^0$, $(\xi, (y_i)_{i\in\overline{0,N}}) \in \mathbf{D}^\natural[x]$, *and* $(\eta, (\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}) \in \mathbf{D}^*[\mathrm{pr}_2(y_N)]$, *then*

$$\mathfrak{C}_{\xi\diamond\eta}\left[(y_i)_{i\in\overline{0,N}}\,\square\,(\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}\right] = \mathfrak{C}^\natural_\xi\left[(y_i)_{i\in\overline{0,N}}\right] - \tilde{V}^*\left[\mathrm{pr}_2(y_N)\right] + \mathfrak{C}^*_\eta\left[(\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}\right].$$

In essential part, the proof is common to that of Proposition 2.

Corollary 1. *If* $x \in X^0$, $(\xi, (y_i)_{i\in\overline{0,N}}) \in \mathbf{D}^\natural[x]$, *and* $(\eta, (\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}) \in (\mathrm{sol})^*[\mathrm{pr}_2(y_N)]$, *then*

$$\mathfrak{C}_{\xi\diamond\eta}\left[(y_i)_{i\in\overline{0,N}}\,\square\,(\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}\right] = \mathfrak{C}^\natural_\xi\left[(y_i)_{i\in\overline{0,N}}\right].$$

Corollary 2. *If* $x \in X^0$, $(\xi, (y_i)_{i\in\overline{0,N}}) \in (\mathrm{sol})^\natural[x]$, *and* $(\eta, (\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}) \in (\mathrm{sol})^*[\mathrm{pr}_2(y_N)]$, *then*

$$\mathfrak{C}_{\xi\diamond\eta}[(y_i)_{i\in\overline{0,N}}\,\square\,(\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}] = V^\natural[x].$$

Proposition 4. $X^0_{\rm opt} = X^\natural_{\rm opt}$.

The proof of this result is given in the Appendix. From (4.4) and Proposition 4 it follows that the resulting $\mathcal{M}_1$-problem reproduces the most important elements of the original $\mathcal{M}$-problem: the global optimum (2.21) and the optimal set (2.24). Note another useful property on the coincidence of value functions, which is, however, not employed in further considerations.

Proposition 5. *If* $x \in X^0$, *then* $V^\natural[x] = \tilde{V}[x]$.

It is established by analogy with Proposition 4.

5. THE FINAL PROBLEM

This section discusses the exact implementation of Stages 1–5 using the broadly understood DP; see [13, 14] and other publications. We begin with solving the $\mathcal{M}_2$-problem, following Stage 2 and adhering to the algorithmic presentation similar to [13, 14]. The matter concerns constructing the layers of the Bellman function involving precedence conditions. Note that these constructs have been repeatedly used by us before. Here, they are described in a fairly brief form with the necessary references. First of all, for pending tasks from the list, we introduce the crossing-out operator $\mathbf{I}^*$ acting in $\mathfrak{N}^*$ by the rule: for $K \in \mathfrak{N}^*$ and $\Xi^*[K] \triangleq \{\, z \in \mathbf{K}_2 \mid (\mathrm{pr}_1(z) \in K)\&(\mathrm{pr}_2(z) \in K)\}$, the set $\mathbf{I}^*(K) \in \mathfrak{N}^*$ is

$$\mathbf{I}^*(K) \triangleq K \setminus \{\mathrm{pr}_2(z) :\ z \in \Xi^*[K]\}\,. \tag{5.1}$$

(The definition of $\mathbf{I}^*$ (5.1) agrees with [8, formulas (2.2.27) and (2.2.28)].) The operator $\mathbf{I}^*$ serves to determine substantial lists of pending tasks [13, 14]. Let

$$\mathfrak{S}^* \triangleq \{K \in \mathfrak{N}^* |\ \forall z \in \mathbf{K}_2\ \ (\mathrm{pr}_1(z) \in K) \Longrightarrow (\mathrm{pr}_2(z) \in K)\}\,; \tag{5.2}$$

the sets representing elements of the family (5.2) are called substantial lists in the $\mathcal{M}_2$-problem. For $s \in \overline{1, \mathbf{n} - N}$, let $\mathfrak{S}^*_s \triangleq \{\ K \in \mathfrak{S}^* \mid s = |K|\}$. Then $\mathfrak{S}^*_{\mathbf{n}-N} = \{\overline{1, \mathbf{n} - N}\}$ (singleton) and, for $\hat{\mathbf{K}}_2 \triangleq \{\ \mathrm{pr}_1(z) :\ z \in \mathbf{K}_2\}$,

$$\mathfrak{S}^*_1 = \{\ \{t\} :\ t \in \overline{1, \mathbf{n} - N} \setminus \hat{\mathbf{K}}_2\}.$$

In addition, for $s \in \overline{2, \mathbf{n} - N}$, we have

$$\mathfrak{S}^*_{s-1} = \{\ K \setminus \{t\} :\ K \in \mathfrak{S}^*_s,\ t \in \mathbf{I}^*(K)\} \tag{5.3}$$

(see [14, formula (4.6)]). Formula (5.3) defines the following recurrence procedure:

$$\mathfrak{S}^*_{\mathbf{n}-N} \longrightarrow \mathfrak{S}^*_{\mathbf{n}-N-1} \longrightarrow \ldots \longrightarrow \mathfrak{S}^*_1. \tag{5.4}$$

A regular step of the procedure (5.4) is implemented using (5.3).

The sets $\mathfrak{S}^*_s$, $s \in \overline{1, \mathbf{n} - N}$, being available, we construct the layers of the state space, denoted by $D^*_0, D^*_1, \ldots, D^*_{\mathbf{n}-N}$. Note that

$$D^*_0 \triangleq \{(x, \varnothing) : x \in \hat{\mathcal{M}}^*\}, \tag{5.5}$$

where $\hat{\mathcal{M}}^*$ is identified with the union of all sets $\mathbf{M}^{(j)}$, $j \in \overline{1, \mathbf{n} - N} \setminus \hat{\mathbf{K}}_2$. In addition, let

$$D^*_{\mathbf{n}-N} \triangleq \left\{(x, \overline{1, \mathbf{n} - N}) :\ x \in X^{00}\right\}. \tag{5.6}$$

Consider the construction of D^*_s for $s \in \overline{1, \mathbf{n} - N - 1}$. First, for each $K \in \mathfrak{S}^*_s$, we sequentially construct

$$\mathcal{J}^*_s(K) \triangleq \left\{j \in \overline{1, \mathbf{n} - N} \setminus K \mid \{j\} \cup K \in \mathfrak{S}^*_{s+1}\right\},\ \ \mathcal{M}^*_s[K] \triangleq \bigcup_{j \in \mathcal{J}^*_s(K)} \mathbf{M}^{(j)},$$

$$\mathbb{D}^*_s[K] \triangleq \{(x, K) :\ x \in \mathcal{M}^*_s[K]\}$$

(i.e., implement the procedure $\mathcal{J}^*_s(K) \to \mathcal{M}^*_s[K] \to \mathbb{D}^*_s[K]$). Second, let

$$D^*_s \triangleq \bigcup_{K \in \mathfrak{S}^*_s} \mathbb{D}^*_s[K]. \tag{5.7}$$

Each layer D^*_j, $j \in \overline{0, \mathbf{n} - N}$, is a non-empty set (see [8, Proposition 4.9.3]). Note an important property [11, formula (5.5)] as follows: for $s \in \overline{1, \mathbf{n} - N}$, $(x, K) \in D^*_s$, $j \in \mathbf{I}^*(K)$, and $z \in \mathbb{M}^{(j)}$,

$$(\mathrm{pr}_2(z), K \setminus \{j\}) \in D^*_{s-1}. \tag{5.8}$$

(Also, see [8, Proposition 4.9.4]).

The next step is constructing the layers of the Bellman function: $v_0^*, v_1^*, \ldots, v_{\mathbf{n}-N}^*$. We use the Bellman equation (see [23, Theorem 5.1]) and the contractions of the resulting Bellman function to the layers of the state space. These constructs can be reduced to the recurrence procedure

$$v_0^* \to v_1^* \to \ldots \to v_{\mathbf{n}-N}^*. \tag{5.9}$$

In this case, $v_0^* \in \mathcal{R}_+[D_0^*]$ is given by the rule

$$v_0^*(x, \varnothing) \triangleq f(x) \;\; \forall x \in \hat{\mathcal{M}}^*. \tag{5.10}$$

In view of (5.8), a regular step of this procedure is implemented as follows: for $s \in \overline{1, \mathbf{n} - N}$, we determine $v_s^* \in \mathcal{R}_+[D_s^*]$ based on $v_{s-1}^* \in \mathcal{R}_+[D_{s-1}^*]$ according to the rule [11, formula (5.6)], i.e.,

$$v_s^*(x, K) \triangleq \min_{j \in \mathbf{I}^*(K)} \min_{z \in \mathbb{M}^{(j)}} \Big[\mathbf{c}^*(x, \mathrm{pr}_1(z), K) + c_j^*(z, K) + v_{s-1}^*(\mathrm{pr}_2(z), K \setminus \{j\})\Big] \, \forall (x, K) \in D_s^*. \tag{5.11}$$

For the final function $v_{\mathbf{n}-N}^* \in \mathcal{R}_+[D_{\mathbf{n}-N}^*]$, we have

$$\tilde{V}^*[x] = v_{\mathbf{n}-N}^*(x, \overline{1, \mathbf{n} - N}) \;\; \forall x \in X^{00}. \tag{5.12}$$

Remark 2. If $\tilde{V}^*[\cdot]$ needs to be determined without constructing the optimal routing process in the $\mathcal{M}_2$-problem, the procedure (5.9) can be implemented with rewriting the layers. (Note the construct in [24] for a somewhat different problem.) In this case, for $s \in \overline{1, N-1}$, computer's memory contains the layer v_{s-1}^* of the Bellman function; after determining v_s^*, it is eliminated and replaced by the layer v_s^*. If $s \leqslant N - 2$, then the layer v_s^* is used to construct v_{s+1}^*. This simple circumstance follows directly from (5.11) and saves memory resources; see [25].

After constructing $v_{\mathbf{n}-N}^*$, we proceed to solving the $\mathcal{M}_1$-problem using (5.12). So, we pass to Stage 3 and determine the terminal component of the additive criterion based on (3.20).

6. THE PRELIMINARY PROBLEM

Let us find the function $\mathbf{f}$ through (3.20) and (5.12). (Now, it is substantial that

$$\mathbf{f}(x) \triangleq v_{\mathbf{n}-N}^*(x, \overline{1, \mathbf{n} - N}) \; \forall x \in X^{00};$$

hence, $\mathbf{f}$ can be explicitly determined.) Further constructs are similar to those for the $\mathcal{M}_2$-problem and are given in the short form. The matter concerns implementing DP in the spirit of [13, 14] (based on the constructs from [8, § 4.9]).

We introduce the crossing-out operator $\mathbf{I}^\natural$ acting in $\mathfrak{N}^\natural$: for $K \in \mathfrak{N}^\natural$ and

$$\Xi^\natural[K] \triangleq \{\, z \in \mathbf{K}_1 |\; (\mathrm{pr}_1(z) \in K) \& (\mathrm{pr}_2(z) \in K)\},$$

the set $\mathbf{I}^\natural(K)$ is

$$\mathbf{I}^\natural(K) \triangleq K \setminus \{\, \mathrm{pr}_2(z) : \; z \in \Xi^\natural[K]\}. \tag{6.1}$$

Consider the construction of substantial lists of pending tasks in the $\mathcal{M}_1$-problem. Let

$$\mathfrak{S}^\natural \triangleq \{\, K \in \mathfrak{N}^\natural | \; \forall z \in \mathbf{K}_1 \;\; (\mathrm{pr}_1(z) \in K) \Longrightarrow (\mathrm{pr}_2(z) \in K)\}; \tag{6.2}$$

the sets representing elements of the family (6.2) are called substantial lists in the $\mathcal{M}_1$-problem. For $s \in \overline{1, N}$, let

$$\mathfrak{S}_s^\natural \triangleq \{\, K \in \mathfrak{S}^\natural | s = |K|\}.$$

Obviously, $\mathfrak{S}^\natural_N=\{\overline{1,N}\}$ (singleton). For $\hat{\mathbf{K}}_1\triangleq\{\mathrm{pr}_1(z):z\in\mathbf{K}_1\}$, we have $\mathfrak{S}^\natural_1=\{\,\{t\}:t\in\overline{1,N}\setminus\hat{\mathbf{K}}_1\}$. It is evident that, for $s\in\overline{2,N}$,

$$\mathfrak{S}^\natural_{s-1}=\Big\{K\setminus\{j\}:\ K\in\mathfrak{S}^\natural_s,\ j\in\mathbf{I}^\natural(K)\Big\}. \tag{6.3}$$

Formula (6.3) defines the recurrence procedure

$$\mathfrak{S}^\natural_N\longrightarrow\mathfrak{S}^\natural_{N-1}\longrightarrow\ldots\longrightarrow\mathfrak{S}^\natural_1. \tag{6.4}$$

(A regular step of the procedure (6.4) is implemented using (6.3).) The sets $\mathfrak{S}^\natural_s$, $s\in\overline{1,N}$, being available, we construct the layers of the state space, denoted by $D^\natural_0,D^\natural_1,\ldots,D^\natural_N$. Letting

$$\Big(D^\natural_0\triangleq\{\,(x,\varnothing):\ x\in X^{00}\}\Big)\,\&\,\Big(D^\natural_N\triangleq\{\,(x,\overline{1,N}):\ x\in X^0\}\Big), \tag{6.5}$$

yields the extreme layers of the state space. If $s\in\overline{1,N-1}$, then for $K\in\mathfrak{S}^\natural_s$ we first sequentially determine

$$\mathcal{J}^\natural_s(K)\triangleq\{\,j\in\overline{1,N}\setminus K|\ \{j\}\cup K\in\mathfrak{S}^\natural_{s+1}\},\ \ \mathcal{M}^\natural_s(K)\triangleq\bigcup_{j\in\mathcal{J}^\natural_s(K)}\mathbf{M}_j, \tag{6.6}$$
$$\mathbb{D}^\natural_s[K]\triangleq\{(x,K):\ x\in\mathcal{M}^\natural_s(K)\}$$

(i.e., implement the procedure $\mathcal{J}^\natural_s(K)\to\mathcal{M}^\natural_s(K)\to\mathbb{D}^\natural_s[K]$). The layer $D^\natural_s$ is given by the rule

$$D^\natural_s\triangleq\bigcup_{K\in\mathfrak{S}^\natural_s}\mathbb{D}^\natural_s[K]. \tag{6.7}$$

Thus, all layers $D^\natural_0,D^\natural_1,\ldots,D^\natural_N$ have been built. According to [8, Proposition 4.9.3], all these layers are non-empty sets. Note that for $s\in\overline{1,N}$, $(x,K)\in D^\natural_s$, $j\in\mathbf{I}^\natural(K)$, and $z\in\mathbb{M}_j$,

$$(\mathrm{pr}_2(z),K\setminus\{j\})\in D^\natural_{s-1}. \tag{6.8}$$

The construct (6.6)–(6.8) is used for obtaining the layers of the Bellman function: we sequentially determine the functions $v^\natural_0\in\mathcal{R}_+[D^\natural_0]$, $v^\natural_1\in\mathcal{R}_+[D^\natural_1]$, $\ldots$, $v^\natural_N\in\mathcal{R}_+[D^\natural_N]$, coinciding with the contractions of the Bellman function to the layers of the state space. In addition,

$$v^\natural_0(x,\varnothing)\triangleq\tilde{V}^*[x]=v^*_{\mathbf{n}-N}(x,\overline{1,\mathbf{n}-N})\ \ \forall x\in X^{00}. \tag{6.9}$$

If $s\in\overline{1,N}$ and the function $v^\natural_{s-1}$ has been built, then due to (6.8), the function $v^\natural_s\in\mathcal{R}_+[D^\natural_s]$ is such that

$$v^\natural_s(x,K)\triangleq\min_{j\in\mathbf{I}^\natural(K)}\min_{z\in\mathbb{M}_j}\Big[\mathbf{c}^\natural(x,\mathrm{pr}_1(z),K)+c^\natural_j(z,K)+v^\natural_{s-1}(\mathrm{pr}_2(z),K\setminus\{j\})\Big]\ \forall(x,K)\in D_s. \tag{6.10}$$

Thus, we have designed the following recursive procedure:

$$v^\natural_0\longrightarrow v^\natural_1\longrightarrow\ldots\longrightarrow v^\natural_N. \tag{6.11}$$

A regular step of the procedure (6.11) is described by (6.10). In addition,

$$v^\natural_N(x,\overline{1,N})=V^\natural[x]\ \ \forall x\in X^0. \tag{6.12}$$

This property follows from the fact that all functions in (6.11) are the contractions of the Bellman function; also, see (6.5). The function $v_N^\natural$ being available, we determine (see (3.28) and (3.29)) $\mathbb{V}^\natural$ and the point from $X_{\mathrm{opt}}^\natural$. In view of (3.28), (4.4), and (6.12), we have

$$\mathbb{V} = \min_{x \in X^0} v_N^\natural(x, \overline{1,N}), \tag{6.13}$$

whereas the point $x^0 \in X_{\mathrm{opt}}^\natural$ (see (4.4), Proposition 4) is obtained from the following condition: the point $x^0 \in X^0$ is such that

$$\mathbb{V} = v_N^\natural(x^0, \overline{1,N}). \tag{6.14}$$

Formulas (6.13) and (6.14) yield the global optimum and the optimal starting point without constructing the RP. Therefore, a considerable part of Stage 3 is implemented. Note that the logic of Remark 2 fully applies to the procedure (6.11): the variant with overwriting the layers can be used to find $\mathbb{V}$ and x^0. As a matter of fact, we have the uniform glued procedure

$$(v_0^* \longrightarrow v_1^* \longrightarrow \ldots \longrightarrow v_{\mathbf{n}-N}^*) \to (v_0^\natural \longrightarrow v_1^\natural \longrightarrow \ldots \longrightarrow v_N^\natural), \tag{6.15}$$

where gluing is given by (6.9).

Now let us construct the optimal RP under the assumption that all functions in (6.15) are available. (In other words, the procedure (6.15) has been implemented with all these functions stored in computer's memory.) Following 3), we first determine the $\mathcal{M}_1$-solution. Well, $\mathbb{V} \in \mathbb{R}_+$ and $x^0 \in X_{\mathrm{opt}}^\natural$ (see (6.13) and (6.14)). Considering (4.4) and Proposition 4, let $y_0 \triangleq (x^0, x^0)$; by the choice of x^0,

$$(\mathrm{pr}_2(y_0), \overline{1,N}) = (x^0, \overline{1,N}) \in D_N^\natural \tag{6.16}$$

(see (6.5)). From (6.10) and (6.14) it follows that

$$\mathbb{V} = \min_{j \in \mathbf{I}^\natural(\overline{1,N})} \min_{z \in \mathbb{M}_j} \Big[\mathbf{c}^\natural(x^0, \mathrm{pr}_1(z), \overline{1,N}) + c_j^\natural(z, \overline{1,N}) + v_{N-1}^\natural(\mathrm{pr}_2(z), \overline{1,N} \setminus \{j\})\Big]. \tag{6.17}$$

In view of (6.16) and (6.17), we choose $\xi_1 \in \mathbf{I}^\natural(\overline{1,N})$ and $y_1 \in \mathbb{M}_{\xi_1}$ so that

$$\mathbb{V} = \mathbf{c}^\natural(x^0, \mathrm{pr}_1(y_1), \overline{1,N}) + c_{\xi_1}^\natural(y_1, \overline{1,N}) + v_{N-1}^\natural(\mathrm{pr}_2(y_1), \overline{1,N} \setminus \{\xi_1\}). \tag{6.18}$$

The relations (6.8) and (6.16) imply $(\mathrm{pr}_2(y_1), \overline{1,N} \setminus \{\xi_1\}) \in D_{N-1}^\natural$. According to (6.10), we therefore have the equality

$$\begin{aligned} v_{N-1}^\natural(\mathrm{pr}_2(y_1), \overline{1,N} \setminus \{\xi_1\}) = \min_{j \in \mathbf{I}^\natural(\overline{1,N} \setminus \{\xi_1\})} \min_{z \in \mathbb{M}_j} \Big[&\mathbf{c}^\natural(\mathrm{pr}_2(y_1), \mathrm{pr}_1(z), \overline{1,N} \setminus \{\xi_1\}) \\ &+ c_j^\natural(z, \overline{1,N} \setminus \{\xi_1\}) + v_{N-2}^\natural(\mathrm{pr}_2(z), \overline{1,N} \setminus \{\xi_1; j\})\Big]. \end{aligned} \tag{6.19}$$

Considering (6.19), we choose $\xi_2 \in \mathbf{I}^\natural(\overline{1,N} \setminus \{\xi_1\})$ and $y_2 \in \mathbb{M}_{\xi_2}$ so that

$$\begin{aligned} v_{N-1}^\natural(\mathrm{pr}_2(y_1), \overline{1,N} \setminus \{\xi_1\}) &= \mathbf{c}^\natural(\mathrm{pr}_2(y_1), \mathrm{pr}_1(y_2), \overline{1,N} \setminus \{\xi_1\}) \\ &+ c_{\xi_2}^\natural(y_2, \overline{1,N} \setminus \{\xi_1\}) + v_{N-2}^\natural(\mathrm{pr}_2(y_2), \overline{1,N} \setminus \{\xi_1; \xi_2\}); \end{aligned} \tag{6.20}$$

in this case, by (6.8), $(\mathrm{pr}_2(y_2), \overline{1,N} \setminus \{\xi_1; \xi_2\}) \in D_{N-2}^\natural$. Due to (6.18) and (6.20), we have the equality

$$\begin{aligned} \mathbb{V} = \mathbf{c}^\natural(x^0, \mathrm{pr}_1(y_1), \overline{1,N}) + \mathbf{c}^\natural(\mathrm{pr}_2(y_1), \mathrm{pr}_1(y_2), \overline{1,N} \setminus \{\xi_1\}) \\ + c_{\xi_1}^\natural(y_1, \overline{1,N}) + c_{\xi_2}^\natural(y_2, \overline{1,N} \setminus \{\xi_1\}) + v_{N-2}^\natural(\mathrm{pr}_2(y_2), \overline{1,N} \setminus \{\xi_1; \xi_2\}). \end{aligned} \tag{6.21}$$

Remark 3. For $N = 2$, the optimality of the OP $((\xi_i)_{i\in\overline{1,2}}, (y_i)_{i\in\overline{0,2}})$ in the $(\mathcal{M}_1, x^0)$-problem is easily derived from (6.21).

In the general case $N \geqslant 2$, the procedures similar to (6.18) and (6.20) should be continued until exhausting $\overline{1,N}$. As a result, we will construct the tuples $\xi \triangleq (\xi_i)_{i\in\overline{1,N}} \in \mathcal{A}_1$ and $(y_i)_{i\in\overline{0,N}} \in \mathcal{Z}_\xi^\natural[x^0]$ with the property

$$\mathfrak{C}_\xi^\natural[(y_i)_{i\in\overline{0,N}}] = \mathbb{V}. \tag{6.22}$$

Then, see (3.18), $(\xi, (y_i)_{i\in\overline{0,N}}) \in \mathbf{D}^\natural[x^0]$. Furthermore, according to (3.25), (3.28), (4.4), and (6.22),

$$\mathbb{V}^\natural \leqslant V^\natural[x^0] \leqslant \mathfrak{C}_\xi^\natural[(y_i)_{i\in\overline{0,N}}] = \mathbb{V} = \mathbb{V}^\natural,$$

and consequently, $\mathfrak{C}_\xi^\natural[(y_i)_{i\in\overline{0,N}}] = V^\natural[x^0] = \mathbb{V} = \mathbb{V}^\natural$. From (3.26) we obtain the property

$$(\xi, (y_i)_{i\in\overline{0,N}}) \in (\mathrm{sol})^\natural[x^0]. \tag{6.23}$$

On the other hand, since $V^\natural[x^0] = \mathbb{V}^\natural$, from (3.29) it follows that $x^0 \in X_{\mathrm{opt}}^\natural$. Hence (see (6.23)),

$$x^0 \in X_{\mathrm{opt}}^\natural : (\xi, (y_i)_{i\in\overline{0,N}}) \in (\mathrm{sol})^\natural[x^0]. \tag{6.24}$$

Thus, the $\mathcal{M}_1$-solution (6.23) has been built and Stage 3 is complete.

7. THE COMPOSITION SOLUTION OF THE AGGREGATE PROBLEM: THE OPTIMAL ROUTING PROCESS

This section finalizes Stages 4 and 5. Recall that $\mathrm{pr}_2(y_N) \in X^{00}$ by (3.19). In view of this fact, letting $x^{00} \triangleq \mathrm{pr}_2(y_N)$ yields

$$x^{00} \triangleq \mathrm{pr}_2(y_N) \in X^{00}. \tag{7.1}$$

Note that the functions $v_0^*, v_1^*, \ldots, v_{\mathbf{n}-N}^*$ are known; also, we emphasize (5.12). Due to (7.1),

$$\hat{y}_0 \triangleq (x^{00}, x^{00}) = (\mathrm{pr}_2(y_N), \mathrm{pr}_2(y_N));$$

$\hat{y}_0 \in X^{00} \times X^{00}$. In addition, according to (5.6),

$$(x^{00}, \overline{1, \mathbf{n}-N}) = (\mathrm{pr}_2(\hat{y}_0), \overline{1, \mathbf{n}-N}) \in D_{\mathbf{n}-N}^*. \tag{7.2}$$

Considering (5.11) and (7.2), we obtain the equality

$$v_{\mathbf{n}-N}^*(x^{00}, \overline{1, \mathbf{n}-N}) = \min_{j\in\mathbf{I}^*(\overline{1,\mathbf{n}-N})} \min_{z\in\mathbb{M}^{(j)}} \Big[\mathbf{c}^*(x^{00}, \mathrm{pr}_1(z), \overline{1, \mathbf{n}-N}) + c_j^*(z, \overline{1, \mathbf{n}-N}) + v_{\mathbf{n}-N-1}^*(\mathrm{pr}_2(z), \overline{1, \mathbf{n}-N} \setminus \{j\})\Big]. \tag{7.3}$$

In view of (7.3), we choose $\eta_1 \in \mathbf{I}^*(\overline{1, \mathbf{n}-N})$ and $\hat{y}_1 \in \mathbb{M}^{(\eta_1)}$ so that

$$v_{\mathbf{n}-N}^*(x^{00}, \overline{1, \mathbf{n}-N}) = \mathbf{c}^*(x^{00}, \mathrm{pr}_1(\hat{y}_1), \overline{1, \mathbf{n}-N}) + c_{\eta_1}^*(\hat{y}_1, \overline{1, \mathbf{n}-N}) + v_{\mathbf{n}-N-1}^*(\mathrm{pr}_2(\hat{y}_1), \overline{1, \mathbf{n}-N} \setminus \{\eta_1\}), \tag{7.4}$$

where $(\mathrm{pr}_2(\hat{y}_1), \overline{1, \mathbf{n} - N} \setminus \{\eta_1\}) \in D^*_{\mathbf{n}-N-1}$ by (5.8). From (5.11) we obtain the equality

$$\begin{aligned} &v^*_{\mathbf{n}-N-1}(\mathrm{pr}_2(\hat{y}_1), \overline{1, \mathbf{n} - N} \setminus \{\eta_1\}) \\ &\quad = \min_{j \in \mathbf{I}^*(\overline{1,\mathbf{n}-N} \setminus \{\eta_1\})} \min_{z \in \mathbb{M}^{(j)}} \Big[\mathbf{c}^*(\mathrm{pr}_2(\hat{y}_1), \mathrm{pr}_1(z), \overline{1, \mathbf{n} - N} \setminus \{\eta_1\}) \\ &\qquad + c^*_j(z, \overline{1, \mathbf{n} - N} \setminus \{\eta_1\}) + v^*_{\mathbf{n}-N-2}(\mathrm{pr}_2(z), \overline{1, \mathbf{n} - N} \setminus \{\eta_1; j\})\Big]. \end{aligned}$$

In view of this relation, we choose $\eta_2 \in \mathbf{I}^*(\overline{1, \mathbf{n} - N} \setminus \{\eta_1\})$ and $\hat{y}_2 \in \mathbb{M}^{(\eta_2)}$ so that

$$\begin{aligned} v^*_{\mathbf{n}-N-1}(\mathrm{pr}_2(\hat{y}_1), \overline{1, \mathbf{n} - N} \setminus \{\eta_1\}) &= \mathbf{c}^*(\mathrm{pr}_2(\hat{y}_1), \mathrm{pr}_1(\hat{y}_2), \overline{1, \mathbf{n} - N} \setminus \{\eta_1\}) \\ &+ c^*_{\eta_2}(\hat{y}_2, \overline{1, \mathbf{n} - N} \setminus \{\eta_1\}) + v^*_{\mathbf{n}-N-2}(\mathrm{pr}_2(\hat{y}_2), \overline{1, \mathbf{n} - N} \setminus \{\eta_1; \eta_2\}), \end{aligned} \tag{7.5}$$

where $(\mathrm{pr}_2(\hat{y}_2), \overline{1, \mathbf{n} - N} \setminus \{\eta_1; \eta_2\}) \in D^*_{\mathbf{n}-N-2}$ by (5.8). Note that conditions (7.4) and (7.5) imply

$$\begin{aligned} v^*_{\mathbf{n}-N}(x^{00}, \overline{1, \mathbf{n} - N}) &= \mathbf{c}^*(x^{00}, \mathrm{pr}_1(\hat{y}_1), \overline{1, \mathbf{n} - N}) + \mathbf{c}^*(\mathrm{pr}_2(\hat{y}_1), \mathrm{pr}_1(\hat{y}_2), \overline{1, \mathbf{n} - N} \setminus \{\eta_1\}) \\ &+ c^*_{\eta_1}(\hat{y}_1, \overline{1, \mathbf{n} - N}) + c^*_{\eta_2}(\hat{y}_2, \overline{1, \mathbf{n} - N} \setminus \{\eta_1\}) + v^*_{\mathbf{n}-N-2}(\mathrm{pr}_2(\hat{y}_2), \overline{1, \mathbf{n} - N} \setminus \{\eta_1; \eta_2\}). \end{aligned} \tag{7.6}$$

(Obviously, for $\mathbf{n} = N + 2$, formula (7.6) ensures the optimality of the solution $((\eta_i)_{i \in \overline{1,2}}, (\hat{y}_i)_{i \in \overline{0,2}})$ in the $(\mathcal{M}_2, x^{00})$-problem; there is an analogy with Remark 3.) In the general case $N \in \overline{2, \mathbf{n} - 2}$, the choice procedures (precisely, the ones for solving the local optimization problems) similar to (7.4) and (7.5) should be continued until exhausting $\overline{1, \mathbf{n} - N}$. As a result, we will construct the tuples $\eta \triangleq (\eta_i)_{i \in \overline{1,\mathbf{n}-N}} \in \mathcal{A}_2$ and $(\hat{y}_i)_{i \in \overline{0,\mathbf{n}-N}} \in \mathcal{Z}^*_\eta[x^{00}]$ for which (see (5.12))

$$\mathfrak{C}^*_\eta[(\hat{y}_i)_{i \in \overline{0,\mathbf{n}-N}}] = v^*_{\mathbf{n}-N}(x^{00}, \overline{1, \mathbf{n} - N}) = \tilde{V}^*[x^{00}]. \tag{7.7}$$

Then $(\eta, (\hat{y}_i)_{i \in \overline{0,\mathbf{n}-N}}) \in \mathbf{D}^*[x^{00}]$ (see (3.5)), and the relations (3.14) and (7.7) imply

$$(\eta, (\hat{y}_i)_{i \in \overline{0,\mathbf{n}-N}}) \in (\mathrm{sol})^*[x^{00}]. \tag{7.8}$$

In view of (7.1) and (7.8), the natural property is

$$(\eta, (\hat{y}_i)_{i \in \overline{0,\mathbf{n}-N}}) \in (\mathrm{sol})^*[\mathrm{pr}_2(y_N)]. \tag{7.9}$$

Considering (6.24) and (7.9) and Proposition 2, we obtain

$$(\xi \diamond \eta, (y_i)_{i \in \overline{0,N}} \,\square\, (\hat{y}_i)_{i \in \overline{0,\mathbf{n}-N}}) \in \tilde{\mathbf{D}}[x^0]$$

and (see (4.6)) $\mathfrak{C}_{\xi \diamond \eta}[(y_i)_{i \in \overline{0,N}} \,\square\, (\hat{y}_i)_{i \in \overline{0,\mathbf{n}-N}}] = \mathbb{V}$. Due to (2.12), (3.29), and (6.24), the triplet

$$(\xi \diamond \eta, (y_i)_{i \in \overline{0,N}} \,\square\, (\hat{y}_i)_{i \in \overline{0,\mathbf{n}-N}}, x^0) \in \mathbf{D}$$

is the optimal RP, i.e., (4.8) holds. Thus, implementing the two-stage procedure

$$[x^0 \to (\xi_1, y_1) \to \ldots \to (\xi_N, y_N)] \to [x^{00} = \mathrm{pr}_2(y_N) \to (\eta_1, \hat{y}_1) \to \ldots \to (\eta_{\mathbf{n}-N}, \hat{y}_{\mathbf{n}-N})]$$

gives the optimal RP (4.8). In addition, $x^0 \in X^0_{\mathrm{opt}}$ by Proposition 4, i.e., x^0 is the optimal starting point in the sense of (2.24).

8. A COMPUTATIONAL EXPERIMENT (THE PRE-CUTTING OF LONG PARTS)

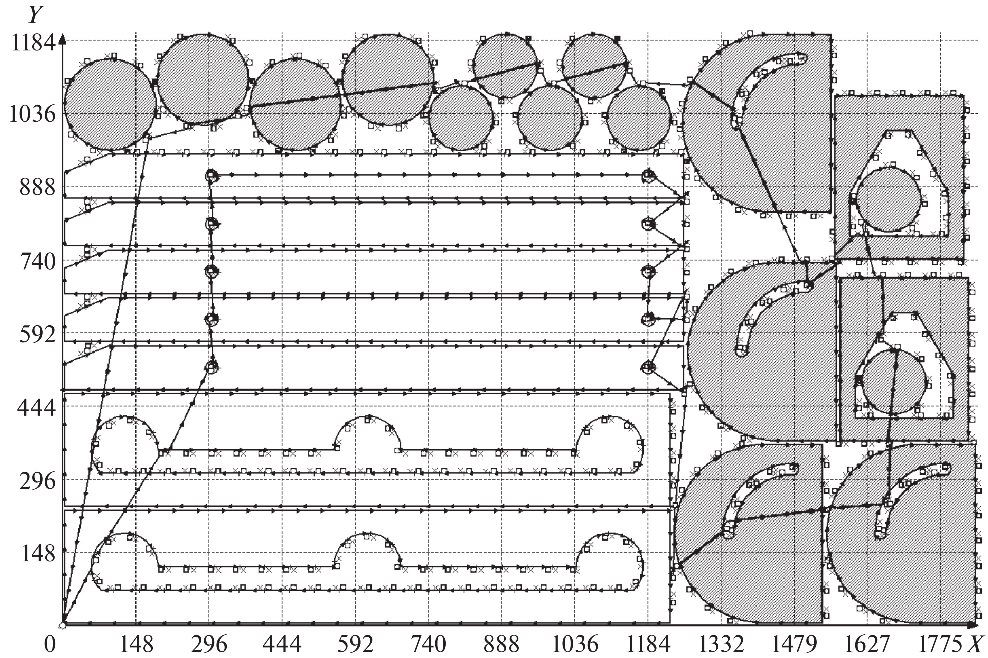
We consider an illustrative example: the sheet cutting of parts on CNC machines. In this section, $X = [0, a] \times [0, b]$, where $a > 0$ and $b > 0$ are two given values. By assumption, a cutting plan is specified for the sheet X. There are $\mathbf{n}$ pairwise disjunct contours to be cut. Megapolises are assigned to contours according to the standard rule: for each contour, an equidistant is specified, on which possible cut-in points and the corresponding tool switch-off points are located. External movements (between megalopolises and from the starting point to megalopolises) are performed in the non-cutting (idle) mode, i.e., "fast." Movements from a cut-in point to the starting point on a contour and, after the cut finish, to the tool switch-off point are implemented in the operating (work stroke) mode, i.e., "slowly." We optimize the total time, excluding the cutting time of the contours, which is the same for all variants of solving the problem. However, we consider possible thermal deformations through penalties for violating the requirements for efficient heat rejection; see [14, Secs. 5, 6]. Such an approach leads (see [14, Sec. 6]) to the travel cost functions that depend on the list of pending tasks.

The precedence conditions in the aggregate problem are determined by considerations related to the nesting of contours and parts. For example, if a part has inner contours, they must be cut before an outer contour; a similar requirement applies to the cutting of nested parts (see [10, § 1.3.2]). Thus, the set $\tilde{\mathbf{K}}$ of aggregate address pairs is defined (Remark 1). However, we suppose that the family $\mathcal{M}$ of megalopolises is divided into the disjunct sum of the subfamilies $\mathcal{M}_1$ and $\mathcal{M}_2$ (Section 2) provided that $\mathcal{M}_1$ includes the megalopolises corresponding to the contours of long parts (Section 1). Note that there are other ways to select contours for priority cutting; for example, see [10, § 1.3.3]. By assumption, the constraint $\tilde{\mathbf{K}}$ is reduced to a variant defined by the sets $\mathbf{K}_1$ and $\tilde{\mathbf{K}}_2$ of Remark 1, where $\tilde{\mathbf{K}}_2$ is due to a rational choice of $\mathbf{K}_2$. These considerations can easily be implemented based on nesting rules (roughly speaking, using the distribution over the volumetric parts and their contours). Of course, constructing $\mathcal{M}_1$ and $\mathcal{M}_2$ requires preliminary work to implement the precedence conditions corresponding to the $\mathcal{M}_1$- and $\mathcal{M}_2$-problems. Let this (rather uncomplicated) stage of the problem formulation be complete.

We allow the choice of different starting points in the interest of optimizing the additive criterion, specifying a required final set X^0 only. As a rule, X^0 is a subset of the boundary of the sheet X. This set is simultaneously the set of possible starting points in the $\mathcal{M}_1$-problem. The set X^{00}, which plays the same role for the $\mathcal{M}_2$-problem, is generated by the algorithm in Stage 1.

Computational experiment. The computations were performed on a personal computer with an Intel i5-11300H processor, 8GB RAM, and Windows 11 (64-bit) OS. The program was developed in the C++ language with the MinGW compiler and the Qt interface library. Let us present the experimental data: 42 contours, 24 parts, and 20 address pairs. (Due to limited space, we omit here the coordinates of contour points, cut-in points, the starting points of contour cut, and tool switch-off points.) All contours were divided into two clusters. The first included the contours of 7 long parts (19 contours); the second one, the contours of 17 compact parts (23 contours). For the sake of simplicity, let X^0 be a singleton corresponding to the origin of coordinates. The values of f correspond to the calculated non-cutting stroke time when the tool returns to the origin (the more comprehensible case of the closed problem). In the example, the thermal constraints described in [14, Secs. 5, 6] were taken into account. The matter concerns the formation of special cut completion domains with obtaining a threshold that characterizes the share of solid metal in each such domain. Precisely, a cut completion domain has a length of 100 mm and a width of 25 mm. The threshold for using the penalty is equal to 0.5 of the area of the cut completion domain.

The figure shows the placement of the parts to be cut on the sheet and the process track obtained during the computational experiment. The rhombus at the origin of the coordinates is the starting



The result yielded by the algorithm.

and finishing point. Square boxes are cut-in points. Crosses are tool switch-off points. The pluses on the contours are the starting and finishing points of cutting. The non-cutting stroke track is a set of separate lines: the line from the starting point to the cut-in point for the first contour, the lines from the tool switch-off point for the previous contour to the cut-in point for the next contour, and the line from the tool switch-off point for the last contour to the movement finish. The cutting track is a set of lines, each consisting of a line from the cut-in point to the point where the contour begins to be cut, the contour itself, and the line from the cut-out point to the tool switch-off point.

The statement of this particular problem was described in detail in [14, Secs. 5, 6]; here, we briefly recall the key aspects. As already noted, the total time is optimized (measured in seconds). All penalty constants coincide with 1 000 000. The result obtained, a value of 97.724 for $\mathbb{V}$, is significantly less than this constant. Hence, all heat rejection constraints were satisfied. (See the general considerations on this topic at the beginning of the section.) The computing time was 7 min 8 s, which is quite acceptable from the practical point of view; furthermore, 42 contours correspond to the problem of significant dimension. We emphasize that for problems of smaller dimension, DP without decomposition requires significantly higher time for computations: for $N = 30$ and $|\mathbf{K}| = 20$, the computing time in [26, Sec. 5] was 51 min 58 s (the standard cutting version). Thus, the approach adopted in this paper considers technological constraints and can be used in applications.

Note finally that research related to sheet metal cutting on CNC machines was comprehensively surveyed in [26, Introduction].

9. CONCLUSIONS

This paper has developed a method for solving an optimal routing problem with a dedicated system of first-priority tasks. The study is based on decomposing the original problem into the preliminarily and final optimal subproblems and using the broadly understood dynamic programming in each of them. This approach allows solving routing problems of significant dimensions in a reasonable time. A possible application of the method is tool control during the shaped sheet cutting of parts with zoning on CNC machines.

APPENDIX

Proof of Proposition 2. In view of (4.2),

$$(\omega_t)_{t\in\overline{0,\mathbf{n}}} \triangleq (y_t)_{t\in\overline{0,N}} \,\square\, (\hat{y}_t)_{t\in\overline{0,\mathbf{n}-N}} \in \mathcal{Z}_{\xi\diamond\eta}[x^0].$$

Since $\xi \diamond \eta \in \mathbf{P}$, we obtain (4.5), i.e.,

$$(\xi \diamond \eta, (\omega_t)_{t\in\overline{0,\mathbf{n}}}) \in \tilde{\mathbf{D}}[x^0]. \tag{A.1}$$

According to (3.26) and (3.21), $\mathfrak{C}^{\natural}_{\xi}[(y_t)_{t\in\overline{0,N}}] = V^{\natural}[x^0] = \mathbb{V}^{\natural}$, and furthermore, $\mathrm{pr}_2(y_N) \in X^{00}$ due to (3.19); $\tilde{V}^*[\mathrm{pr}_2(y_N)] \in \mathbb{R}_+$. In addition,

$$\mathfrak{C}^*_{\eta}[(\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}] = \tilde{V}^*[\mathrm{pr}_2(y_N)]. \tag{A.2}$$

Note that by (3.9)–(3.11), however, it follows that

$$\mathfrak{C}^*_{\eta}[(\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}] = \sum_{t=1}^{\mathbf{n}-N} \Big[\mathbf{c}(\mathrm{pr}_2(\hat{y}_{t-1}), \mathrm{pr}_1(\hat{y}_t), \eta^1(\overline{t, \mathbf{n}-N}) \oplus N) \\ + c_{N+\eta(t)}(\hat{y}_t, \eta^1(\overline{t, \mathbf{n}-N}) \oplus N)\Big] + f(\mathrm{pr}_2(\hat{y}_{\mathbf{n}-N})). \tag{A.3}$$

Based on the choice of ξ and $(y_t)_{t\in\overline{0,N}}$, we have

$$\mathfrak{C}^{\natural}_{\xi}[(y_t)_{t\in\overline{0,N}}] = \sum_{t=1}^{N} \Big[\mathbf{c}(\mathrm{pr}_2(y_{t-1}), \mathrm{pr}_1(y_t), \xi^1(\overline{t,N}) \cup \overline{N+1,\mathbf{n}}) \\ + c_{\xi(t)}(y_t, \xi^1(\overline{t,N}) \cup \overline{N+1,\mathbf{n}})\Big] + \tilde{V}^*[\mathrm{pr}_2(y_N)] \tag{A.4}$$

(see (3.21) and (3.22)). Finally, considering (2.16) and (A.1) leads to the equality

$$\mathfrak{C}_{\xi\diamond\eta}[(\omega_t)_{t\in\overline{0,\mathbf{n}}}] = \sum_{t=1}^{N} \Big[\mathbf{c}(\mathrm{pr}_2(\omega_{t-1}), \mathrm{pr}_1(\omega_t), (\xi\diamond\eta)^1(\overline{t,\mathbf{n}})) + c_{(\xi\diamond\eta)(t)}(\omega_t, (\xi\diamond\eta)^1(\overline{t,\mathbf{n}}))\Big] \\ + \sum_{t=N+1}^{\mathbf{n}} \Big[\mathbf{c}(\mathrm{pr}_2(\omega_{t-1}), \mathrm{pr}_1(\omega_t), (\xi\diamond\eta)^1(\overline{t,\mathbf{n}})) + c_{(\xi\diamond\eta)(t)}(\omega_t, (\xi\diamond\eta)^1(\overline{t,\mathbf{n}}))\Big] + f(\mathrm{pr}_2(\omega_{\mathbf{n}})). \tag{A.5}$$

Due to (4.1), (A.5), and the definition of $(\omega_t)_{t\in\overline{0,\mathbf{n}}}$, we obtain

$$\mathfrak{C}_{\xi\diamond\eta}[(\omega_t)_{t\in\overline{0,\mathbf{n}}}] = \sum_{t=1}^{N} \Big[\mathbf{c}(\mathrm{pr}_2(y_{t-1}), \mathrm{pr}_1(y_t), \xi^1(\overline{t,N}) \cup \overline{N+1,\mathbf{n}}) + c_{\xi(t)}(y_t, \xi^1(\overline{t,N}) \cup \overline{N+1,\mathbf{n}})\Big] \\ + \sum_{t=N+1}^{\mathbf{n}} \Big[\mathbf{c}(\mathrm{pr}_2(\hat{y}_{t-N-1}), \mathrm{pr}_1(\hat{y}_{t-N}), \eta^1(\overline{t-N, \mathbf{n}-N}) \oplus N) \\ + c_{\eta(t-N)+N}(\hat{y}_{t-N}, \eta^1(\overline{t-N, \mathbf{n}-N}) \oplus N)\Big] + f(\mathrm{pr}_2(\hat{y}_{\mathbf{n}-N})). \tag{A.6}$$

This expression involves the chain of equalities $\mathrm{pr}_2(\omega_N) = \mathrm{pr}_2(y_N) = \mathrm{pr}_2(\hat{y}_0)$ (see (3.4), (3.5), and (3.14)). From (A.3), (A.4), and (A.6) it follows that

$$\mathfrak{C}_{\xi\diamond\eta}[(\omega_t)_{t\in\overline{0,\mathbf{n}}}] = \mathfrak{C}^{\natural}_{\xi}[(y_t)_{t\in\overline{0,N}}] - \tilde{V}^*[\mathrm{pr}_2(y_N)]$$
$$+ \sum_{\tau=1}^{\mathbf{n}-N} \Big[\mathbf{c}(\mathrm{pr}_2(\hat{y}_{\tau-1}), \mathrm{pr}_1(\hat{y}_\tau), \eta^1(\overline{\tau, \mathbf{n}-N}) \oplus N) + c_{\eta(\tau)+N}(\hat{y}_\tau, \eta^1(\overline{\tau, \mathbf{n}-N}) \oplus N)\Big]$$
$$+ f(\mathrm{pr}_2(\hat{y}_{\mathbf{n}-N})) = \mathfrak{C}^{\natural}_{\xi}[(y_t)_{t\in\overline{0,N}}] - \tilde{V}^*[\mathrm{pr}_2(y_N)] + \mathfrak{C}^*_{\eta}[(\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}].$$

Hence, considering (A.2), we can write

$$\mathfrak{C}_{\xi\diamond\eta}[(\omega_t)_{t\in\overline{0,\mathbf{n}}}] = \mathfrak{C}^{\natural}_{\xi}[(y_t)_{t\in\overline{0,N}}], \tag{A.7}$$

where $\mathfrak{C}^{\natural}_{\xi}[(y_t)_{t\in\overline{0,N}}] = V^{\natural}[x^0] = \mathbb{V}^{\natural}$ by the choice of $(\xi, (y_t)_{t\in\overline{0,N}})$ and (3.26). Using (4.4) and (A.7), we derive the chain of equalities

$$\mathfrak{C}_{\xi\diamond\eta}[(\omega_t)_{t\in\overline{0,\mathbf{n}}}] = \mathbb{V}^{\natural} = \mathbb{V};$$

according to the definition of $(\omega_t)_{t\in\overline{0,\mathbf{n}}}$, the property (4.6) is the case, where (4.5) holds due to (A.1).

Proof of Proposition 4. Let $x_* \in X^{\natural}_{\mathrm{opt}}$, i.e., $x_* \in X^0$ and $V^{\natural}[x_*] = \mathbb{V}$ (see (4.4)). Using (3.27), we choose

$$(\xi, (y_i)_{i\in\overline{0,N}}) \in (\mathrm{sol})^{\natural}[x_*],$$

obtaining $(\xi, (y_i)_{i\in\overline{0,N}}) \in \mathbf{D}^{\natural}[x_*]$ with the property $\mathfrak{C}^{\natural}_{\xi}[(y_i)_{i\in\overline{0,N}}] = V^{\natural}[x_*]$ (the optimal FS in the $(\mathcal{M}_1, x_*)$-problem). Then, see (4.4), $\mathfrak{C}^{\natural}_{\xi}[(y_i)_{i\in\overline{0,N}}] = \mathbb{V}$. In view of (3.15), we choose

$$(\eta, (\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}) \in (\mathrm{sol})^*[\mathrm{pr}_2(y_N)],$$

obtaining $(\eta, (\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}) \in \mathbf{D}^*[\mathrm{pr}_2(y_N)]$ with the property

$$\mathfrak{C}^*_{\eta}[(\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}] = \tilde{V}^*[\mathrm{pr}_2(y_N)].$$

By Proposition 2, $(\xi \diamond \eta, (y_i)_{i\in\overline{0,N}} \,\square\, (\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}) \in \tilde{\mathbf{D}}[x_*]$ is such that (4.6) holds. Considering (2.18), we have the inequality

$$\tilde{V}[x_*] \leqslant \mathfrak{C}_{\xi\diamond\eta}[(y_i)_{i\in\overline{0,N}} \,\square\, (\hat{y}_i)_{i\in\overline{0,\mathbf{n}-N}}] = \mathbb{V},$$

where $\mathbb{V} \leqslant \tilde{V}[x_*]$ due to (2.21). As a result, $\tilde{V}[x_*] = \mathbb{V}$ and consequently, $x_* \in X^0_{\mathrm{opt}}$ (see (2.25)). Thus,

$$X^{\natural}_{\mathrm{opt}} \subset X^0_{\mathrm{opt}}. \tag{A.8}$$

Let $x^* \in X^0_{\mathrm{opt}}$, i.e., $x^* \in X^0$ and $\tilde{V}[x^*] = \mathbb{V}$. In view of (2.19), we choose the optimal FS

$$(\alpha, (z_i)_{i\in\overline{0,\mathbf{n}}}) \in (\mathrm{sol})[x^*];$$

then $\mathfrak{C}_{\alpha}[(z_i)_{i\in\overline{0,\mathbf{n}}}] = \tilde{V}[x^*] = \mathbb{V}$. In addition, $\alpha \in \mathbf{P}$, which implies $\alpha = \alpha_1 \diamond \alpha_2$, where $\alpha_1 \in \mathcal{A}_1$ and $\alpha_2 \in \mathcal{A}_2$. Therefore, see (2.11), $(z_i)_{i\in\overline{0,\mathbf{n}}} \in \mathcal{Z}_{\alpha_1\diamond\alpha_2}[x^*]$. Then $(z_i)_{i\in\overline{0,N}} \in \mathcal{Z}^{\natural}_{\alpha_1}[x^*]$ and consequently,

$$(\alpha_1, (z_i)_{i\in\overline{0,N}}) \in \mathbf{D}^{\natural}[x^*]$$

(see (3.18)). We introduce a tuple $(\tilde{z}_i)_{i\in\overline{0,\mathbf{n}-N}}$ in $X \times X$ by the rule

$$(\tilde{z}_0 \triangleq (\mathrm{pr}_2(z_N), \mathrm{pr}_2(z_N)))\&(\tilde{z}_t \triangleq z_{N+t} \ \ \forall t \in \overline{1, \mathbf{n} - N}).$$

Obviously, $(\tilde{z}_t)_{t\in\overline{0,\mathbf{n}-N}} \in \mathcal{Z}^*_{\alpha_2}[\mathrm{pr}_2(z_N)]$ (see (4.3)). Therefore, see (3.5), we have

$$(\alpha_2, (\tilde{z}_t)_{t\in\overline{0,\mathbf{n}-N}}) \in \mathbf{D}^*[\mathrm{pr}_2(z_N)].$$

In addition, $(z_t)_{t\in\overline{0,\mathbf{n}}} = (z_t)_{t\in\overline{0,N}} \,\square\, (\tilde{z}_t)_{t\in\overline{0,\mathbf{n}-N}}$. Hence, according to Proposition 3 and (4.4),

$$\mathbb{V}^\natural = \mathfrak{C}_\alpha[(z_t)_{t\in\overline{0,\mathbf{n}}}] = \mathfrak{C}^\natural_{\alpha_1}[(z_t)_{t\in\overline{0,N}}] - \tilde{V}^*[\mathrm{pr}_2(z_N)] + \mathfrak{C}^*_{\alpha_2}[(\tilde{z}_t)_{t\in\overline{0,\mathbf{n}-N}}],$$

where $\tilde{V}^*[\mathrm{pr}_2(z_N)] \leqslant \mathfrak{C}^*_{\alpha_2}[(\tilde{z}_t)_{t\in\overline{0,\mathbf{n}-N}}]$ (see (3.14)). Now, we obtain

$$\mathfrak{C}^\natural_{\alpha_1}[(z_t)_{t\in\overline{0,N}}] = \mathbb{V}^\natural - \mathfrak{C}^*_{\alpha_2}[(\tilde{z}_t)_{t\in\overline{0,\mathbf{n}-N}}] + \tilde{V}^*[\mathrm{pr}_2(z_N)] \leqslant \mathbb{V}^\natural.$$

Then $\mathbb{V}^\natural \leqslant \mathfrak{C}^\natural_{\alpha_1}[(z_t)_{t\in\overline{0,N}}] \leqslant \mathbb{V}^\natural$. As a result, $\mathfrak{C}^\natural_{\alpha_1}[(z_t)_{t\in\overline{0,N}}] = \mathbb{V}^\natural[x^*] = \mathbb{V}^\natural$ and consequently, see (3.29), $x^* \in X^\natural_{\mathrm{opt}}$. This finally verifies the property $X^0_{\mathrm{opt}} \subset X^\natural_{\mathrm{opt}}$ and, see (A.8), the equality $X^0_{\mathrm{opt}} = X^\natural_{\mathrm{opt}}$ as well. $\square$

REFERENCES

1. Gutin, G. and Punnen, A., *The Traveling Salesman Problem and Its Variations*, Berlin: Springer, 2002.
2. Cook, W.J., *In Pursuit of the Traveling Salesman. Mathematics at the Limits of Computation*, Princeton: Princeton University Press, 2012.
3. Gimadi, E.Kh. and Khachai, M.Yu., *Ekstremal'nye zadachi na mnozhestvakh perestanovok* (Extremal Problems on Permutation Sets), Yekaterinburg: UPI Training Center, 2016.
4. Melamed, I.I., Sergeev, S.I., and Sigal, I.Kh., The Traveling Salesman Problem, *Autom. Remote Control*, 1989, vol. 50, no. 9, pp. 1147–1173; no. 10, pp. 1303–1324; no. 11, pp. 1459–1479.
5. Little, J.D., Murty, K.G., Sweeney, D.W., and Karel, C., An Algorithm for the Traveling Salesman Problem, *Oper. Res.*, 1963, no. 11(6), pp. 972–989.
6. Bellman, R., Dynamic Programming Treatment of the Travelling Salesman Problem, *J. ACM (JACM)*, 1962, vol. 9, no. 1, pp. 61–63.
7. Held, M. and Karp, R., A Dynamic Programming Approach to Sequencing Problems, *J. SIAM*, 1962, vol. 10, no. 1, pp. 196–210.
8. Chentsov, A.G., *Ekstremal'nye zadachi marshrutizatsii i raspredeleniya zadanii: voprosy teorii* (Extremal Problems of Routing and Task Distribution: Theoretical Fundamentals), Moscow–Izhevsk: Regulyarnaya i Khaoticheskaya Dinamika, 2008.
9. Chentsov, A.G., Chentsov, A.A., and Sesekin, A.N., *Zadachi marshrutizatsii peremeshchenii s neadditivnym agregirovaniem zatrat* (Movement Routing Problems with Non-additive Cost Aggregation), Moscow: Lenand, 2021.
10. Petunin, A.A., Chentsov, A.G., and Chentsov, P.A., *Optimal'naya marshrutizatsiya instrumenta mashin figurnoi listovoi rezki s chislovym programmnym upravleniem. Matematicheskie modeli i algoritmy* (Optimal Tool Routing of Shaped Sheet Cutting Machines with Computer Numerical Control. Mathematical Models and Algorithms), Yekaterinburg: Ural Federal University, 2020.
11. Chentsov, A.G. and Chentsov, P.A., Dynamic Programming in the Routing Problem: Decomposition Variant, *Russian Universities Reports. Mathematics*, 2022, vol. 27, no. 137, pp. 95–124.

12. Chentsov, A.G. and Chentsov, P.A., An Extremal Two-Stage Routing Problem and Procedures Based on Dynamic Programming, *Trudy Inst. Mat. i Mekh. Ural. Otd. Ross. Akad. Nauk*, 2022, vol. 28, no. 2, pp. 215–248.

13. Chentsov, A.G., Problem of Successive Megalopolis Traversal with the Precedence Conditions, *Autom. Remote Control*, 2014, vol. 75, no. 4, pp. 728–744.

14. Chentsov, A.G. and Chentsov, P.A., Routing under Constraints: Problem of Visit to Megalopolises, *Autom. Remote Control*, 2016, vol. 77, no. 11, pp. 1957–1974.

15. Petunin, A.A., On Some Strategies for Constructing the Tool's Route in the Development of Controlling Programs for Thermal Cutting Machines, *Vestn. UGATU, Ser. Upravlen., Vychisl. Tekh. Informatika*, 2009, vol. 13, no. 2(35), pp. 280–286.

16. Frolovskii, V.D., Automating the Design of Controlling Programs for Heat Metal Cutting on Equipment with Digital Program Control, *Inform. Tekhnol. Proektirovan. Proizvod.*, 2005, no. 4, pp. 63–66.

17. Wang, G.G. and Xie, S.Q., Optimal Process Planning for a Combined Punch-and-Laser Cutting Machine Using Ant Colony Optimization, *Int. J. Product. Res.*, 2005, vol. 43, no. 11, pp. 2195–2216.

18. Lee, M.-K. and Kwon, K.-B., Cutting Path Optimization in CNC Cutting Processes Using a Two-Step Genetic Algorithm, *Int. J. Product. Res.*, 2006, no. 44, pp. 5307–5326.

19. Kuratowski, K. and Mostowski, A., *Set Theory*, Amsterdam: North-Holland, 1967.

20. Dieudonné, J., *Foundations of Modern Analysis*, New York: Academic, 1960.

21. Cormen, T.H., Leiserson, C.E., and Rivest, R.L., *Introduction to Algorithms*, Cambridge: MIT Press, 1990.

22. Warga, J., *Optimal Control of Differential and Functional Equations*, New York: Academic, 1972.

23. Chentsov, A.G., On Routing Complexes of Jobs, *Vestn. UdGU, Mat. Mekh. Komp'yut. Nauki*, 2013, no. 1, pp. 58–82.

24. Lawler, E.L., Efficient Implementation of Dynamic Programming Algorithms for Sequencing Problems, Report BW106, Amsterdam: Mathematisch Centrum, 1979, pp. 1–16.

25. Chentsov, A.G. and Chentsov, A.A., On Finding the Value of the Routing Problem with Constraints, *Probl. Upravlen. Informat.*, 2016, no. 1, pp. 41–54.

26. Petunin, A.A., Chentsov, A.G., and Chentsov, P.A., Optimal Routing in Problems of Sequential Traversal of Megapolises in the Presence of Constraints, *Chelyabinsk Physical and Mathematical Journal*, 2022, vol. 7, no. 2, pp. 209–233.

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