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CONTROL IN SOCIAL ECONOMIC SYSTEMS

Convergence Conditions for the Dynamics of Reflexive Collective Behavior in a Cournot Oligopoly Model under Incomplete Information

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Abstract—This paper considers a Cournot oligopoly model with an arbitrary number of rational agents under incomplete information in the classical case (linear cost and demand functions). Within the dynamic reflexive collective behavior model, at each time instant each agent adjusts his output, taking a step towards the maximum profit under the expected choice of the competitors. Convergence conditions to a Cournot–Nash equilibrium are analyzed using the errors transition matrices of the dynamics. Restrictions on the ranges of agents' steps are imposed and their effect on the convergence properties of the dynamics is demonstrated. Finally, a method is proposed to determine the maximum step ranges ensuring the convergent dynamics of collective behavior for an arbitrary number of agents.

Keywords: Cournot oligopoly, incomplete awareness, reflexive collective behavior, errors transition matrix, step ranges, convergence conditions

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1. INTRODUCTION

A significant number of mathematical works investigating the dynamics of collective behavior in competitive markets were devoted to convergence conditions to Nash equilibrium; for example, see [1–9]. These studies cannot be viewed as complete even for a Cournot oligopoly model with linear cost and demand functions [4, 5, 10]. In what follows, we consider such an oligopoly model.

In the model of reflexive collective behavior, each agent independently of the others adjusts his output by choosing a step towards the current position of his goal [11, 12]. As is known, the dynamics of collective behavior converge to a market equilibrium if each agent adjusts his actions in small steps; for example, see [13–15]. Also, if each agent always takes the maximum admissible step (i.e., chooses his best response to the expected actions of the competitors), the dynamics will converge only for a market consisting of two agents. For markets with more agents, the dynamics of collective behavior diverge [13, 14].

This paper develops an approach based on the norms of the errors transition matrices of collective behavior dynamics to investigate the convergence conditions of the dynamics in oligopoly markets in the class of linear functions [16]. Within this approach, the convergence problem of the dynamics becomes ever more uncertain if the agents can act differently, i.e., choose big steps towards the current positions of their goals or, conversely, small steps.

Below, we pose the following problem: find restrictions on the ranges of the admissible responses of agents in the form of conditions ensuring the convergent dynamics of collective behavior to an equilibrium in a linear Cournot oligopoly model with an arbitrary number of agents.

2. FORMAL PROBLEM STATEMENT

As the basic model, we consider the Cournot oligopoly of agents competing in the outputs of homogeneous products with the goal functions

$$\Pi_i(p(Q), q_i) = p(Q)q_i - \phi(q_i) \to \max_{q_i}, \tag{1}$$

the linear cost functions

$$\phi_i(q_i) = c_i q_i + d_i, \tag{2}$$

and the linear demand function

$$p(Q) = a - bQ. \tag{3}$$

The notations are as follows: $i \in N = \{1, \ldots, n\}$; q_i is the output of agent i; $Q = \sum_{i\in N} q_i$ is the total output of all agents; c_i and d_i are the marginal and constant costs of agent i, respectively; $p(Q)$ is the uniform market price; finally, a and b are the demand parameters. By assumption, everything produced is sold, no capacity constraints are imposed, and no coalitions are allowed. The market state at a time instant t $(t = 0, 1, 2, \ldots)$ is described by the n-dimensional vector $q^t = (q_1^t, \ldots, q_i^t, \ldots, q_n^t)$.

We define a basic process where the change of market states satisfies the axiom of indicator behavior [13] as follows: at each time instant $(t + 1)$, each agent observes the outputs of all agents chosen at the previous time instant t and adjusts his output by taking a step towards the current position of his goal in the iterative procedure

$$q_i^{t+1} = q_i^t + \gamma_i^{t+1}(x_i(q_{-i}^t) - q_i^t), \quad i \in N. \tag{4}$$

Here, the parameter $\gamma_i^{t+1} \in [0, 1]$, independently chosen by each agent i, determines his step towards the current position of his goal. An agent can take a full step with $\gamma_i^{t+1} = 1$ (his best response), remain on the spot with $\gamma_i^{t+1} = 0$, or take a partial step with $\gamma_i^{t+1} \in (0, 1)$.

For agent i, the current position $x_i(q_{-i}^t)$ of his goal is an output maximizing his goal function provided that at the current time instant, the other agents choose the same output as at the previous time instant [13, 17]. Here, $q_{-i}^t = (q_1^t, \ldots, q_{i-1}^t, q_{i+1}^t, \ldots, q_n^t)$ is the opponents' output profile for agent i (the output vector of all agents except agent i) at the time instant t. According to [14–16],

$$x_i(q_{-i}^t) = \frac{h_i - \sum_{j\neq i} q_j^t}{2} = \frac{h_i - Q_{-i}^t}{2}, \tag{5}$$

where $h_i = \frac{a-c_i}{b}$ and $Q_{-i}^t = \sum_{j\neq i} q_j^t$ is the total output of the environment of agent i $(i, j \in N)$.

Each agent precisely knows his costs, goal function, and response function $x_i(q_{-i}^t)$, including the demand parameters a and b, and the previous outputs of other agents. However, each agent has no reliable information about the expected outputs, sets of admissible actions, and cost and goal functions of other agents.

Assume that in the oligopoly model (1)–(3), as in the normal form game, there exists a static Nash equilibrium $q^* = (q_1^*, \ldots, q_i^*, \ldots, q_n^*)$ and all agents are competitive in this equilibrium, i.e., $q_i^* > 0 \ \ \forall i \in N$. In the case of linear cost and demand functions, a static equilibrium exists and is unique.

The equilibrium of the collective behavior dynamics (4), (5) is the static equilibrium q^* in the oligopoly model (1)–(3), but it can be unreachable. Convergence conditions of the dynamics refer to the parameters γ_i^{t+1}, the number of agents in the market, and the initial approximations $q^0 = (q_1^0, \ldots, q_i^0, \ldots, q_n^0)$. Suppose also that $q^0 > 0$.

This paper discusses new aspects of a convergence analysis approach to the models of collective behavior dynamics based on the norm of the errors transition matrix between time instants t and $(t+1)$ in the iterative process (4), (5). For the linear oligopoly model, the approach provides a simple criterion for convergence in norm: the matrix norm must be less than 1 starting from some time instant [16]. When agents independently choose their steps in the range $[0, 1]$, the norm-based criterion cannot be satisfied except for the Cournot duopoly. The main objective of this paper is as follows: for the applications-relevant linear oligopoly model with an arbitrary number of rational agents, find appropriate ranges of their steps in order to satisfy the norm-based criterion. Then, under any initial approximations q^0, the model of collective behavior dynamics (4), (5) will converge to an equilibrium representing a static Nash equilibrium in the oligopoly model (1)–(3). Also, it is of interest to maximize such ranges.

3. INVESTIGATION METHODS

Following [16], we give formal expressions for the norms of the errors transition matrices in the iterative process (4), (5) for the basic oligopoly model and known results on its convergence in norm.

The error of the iterative process, $\varepsilon^{t+1} = (\varepsilon_1^{t+1}, \varepsilon_2^{t+1}, \ldots, \varepsilon_n^{t+1})^T = (q_1^{t+1} - q_1^*, q_2^{t+1} - q_2^*, \ldots, q_n^{t+1} - q_n^*)^T$, is given by $\varepsilon^{t+1} = B^{t+1}\varepsilon^t$ $(t = 0, 1, 2, \ldots)$, where B^{t+1} is the errors transition (recalculation, or transformation) matrix between time instants t and $(t+1)$. This matrix has the form

$$B^{t+1} = B(\gamma^{t+1}) = \begin{pmatrix} 1-\gamma_1^{t+1} & -\gamma_1^{t+1}/2 & \ldots & -\gamma_1^{t+1}/2 \\ -\gamma_2^{t+1}/2 & 1-\gamma_2^{t+1} & \ldots & -\gamma_2^{t+1}/2 \\ \ldots & \ldots & \ldots & \ldots \\ -\gamma_n^{t+1}/2 & -\gamma_n^{t+1}/2 & \ldots & 1-\gamma_n^{t+1} \end{pmatrix}, \tag{6}$$

where $\gamma^{t+1} = (\gamma_1^{t+1}, \ldots, \gamma_i^{t+1}, \ldots, \gamma_n^{t+1})$.

The convergence of the iterative process (4), (5) means that $\varepsilon^t \to 0$ in the Euclidean norm as $t \to \infty$ and is completely defined by the matrix B^{t+1}. The Euclidean norm of the vector ε is given by $\|\varepsilon\| = \sqrt{\sum_{j=1}^n \varepsilon_j^2}$. Let $q^t \to q^*$ indicate that the vector sequence $\{q^t\}_{t=0}^\infty$ converges in norm to the equilibrium q^* as $t \to \infty$. The norm of a real matrix B containing n rows and n columns is subordinate to the Euclidean vector norm, being defined as $||B|| = \max\limits_{\|\varepsilon\|=1} \|B\varepsilon\|$. By the definition of this norm, we have $\|B\varepsilon\| < \|B\|\|\varepsilon\|$ for all B and ε, or $\|B\varepsilon\| < \|B\|$ for all B and $\|\varepsilon\| = 1$ [18, 19].

In this case [16],

$$\|B^{t+1}\| = \max_{\|\varepsilon\|=1} \|B(\gamma^{t+1})\varepsilon\| = \max_{\|\varepsilon\|=1} \sqrt{\sum_{i\in N} \left[\varepsilon_i - \frac{\gamma_i^{t+1}}{2}\left(\varepsilon_i + \sum_{j\in N} \varepsilon_j\right)\right]^2}, \tag{7}$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_i, \ldots, \varepsilon_n)$ is an arbitrary unit vector. In formula (7), we omit the superscript "t" for the components of the vector ε as the one not affecting the result.

Recall the following results [16] on the convergence of the iterative process in terms of the norm of the matrix B^{t+1}.

Lemma 1. *For the process* (4), (5) *to converge to an equilibrium under any initial approximation* q^0 *it is sufficient to have*

$$\|B^{t+1}\| < 1 \tag{8}$$

starting from some time instant t_0.

The nonnegativity requirement for the current outputs of agents (e.g., dictated by economic considerations) can be implemented in the process

$$q_i^{t+1} = \begin{cases} q_i^t + \gamma_i^{t+1}(x_i(q_{-i}^t) - q_i^t), & x_i(q_{-i}^t) > 0; \\ 0, & x_i(q_{-i}^t) \leqslant 0. \end{cases} \tag{9}$$

Proposition 1. *If* $\|B^{t+1}\| < 1$ *starting from some time instant* t_0, *then the process* (9), (5) *converges under any initial approximation* q^0.

According to this proposition, if the norm of the errors transition matrix in the process (4), (5) is less than 1, the process (9), (5) with nonnegative current outputs will converge as well.

We denote by $f(\gamma^{t+1})$ the radicand in (7), i.e.,

$$f(\gamma^{t+1}) = \sum_{i\in N}\left[\varepsilon_i - \frac{\gamma_i^{t+1}}{2}\left(\varepsilon_i + \sum_{j\in N}\varepsilon_j\right)\right]^2. \tag{10}$$

Proposition 2. *Let the vectors* γ^{t+1}, $\overrightarrow{\gamma}^{t+1}$, $\overleftarrow{\gamma}^{t+1}$ *be such that*

$$\gamma_i^{t+1} \in [\overrightarrow{\gamma}_i^{t+1}, \overleftarrow{\gamma}_i^{t+1}], \quad [\overrightarrow{\gamma}_i^{t+1}; \overleftarrow{\gamma}_i^{t+1}] \subseteq [0,1] \ \ \textit{and} \ \ \gamma_i^{t+1} = \alpha_i^{t+1}\overrightarrow{\gamma}_i^{t+1} + \beta_i^{t+1}\overleftarrow{\gamma}_i^{t+1},$$

where $\alpha_i^{t+1}, \beta_i^{t+1} \in [0,1]$, $\alpha_i^{t+1} + \beta_i^{t+1} = 1$, $i \in N$.

Then the function $f(\gamma^{t+1})$ *satisfies the inequalities*

$$\begin{aligned} &f(\alpha_1^{t+1}\overrightarrow{\gamma}_1^{t+1} + \beta_1^{t+1}\overleftarrow{\gamma}_1^{t+1}, \ldots, \alpha_i^{t+1}\overrightarrow{\gamma}_i^{t+1} + \beta_i^{t+1}\overleftarrow{\gamma}_i^{t+1}, \ldots, \alpha_n^{t+1}\overrightarrow{\gamma}_n^{t+1} + \beta_n^{t+1}\overleftarrow{\gamma}_n^{t+1}) \\ \leqslant &\sum_{y_1\in\{\alpha_1^{t+1},\beta_1^{t+1}\}} \cdots \sum_{y_i\in\{\alpha_i^{t+1},\beta_i^{t+1}\}} \cdots \sum_{y_n\in\{\alpha_n^{t+1},\beta_n^{t+1}\}} y_1\cdot\ldots y_i\cdot\ldots y_n \cdot f(z_1^{t+1},\ldots,z_i^{t+1},\ldots,z_n^{t+1}), \end{aligned} \tag{11}$$

where $\alpha_i^{t+1}, \beta_i^{t+1} \in [0,1]$, $\alpha_i^{t+1} + \beta_i^{t+1} = 1$, $z_i^{t+1} = \begin{cases} \overrightarrow{\gamma}_i^{t+1}, & y_i = \alpha_i^{t+1}; \\ \overleftarrow{\gamma}_i^{t+1}, & y_i = \beta_i^{t+1}. \end{cases}$

$$\begin{aligned} &f(\alpha_1^{t+1}\overrightarrow{\gamma}_1^{t+1} + \beta_1^{t+1}\overleftarrow{\gamma}_1^{t+1}, \ldots, \alpha_i^{t+1}\overrightarrow{\gamma}_i^{t+1} + \beta_i^{t+1}\overleftarrow{\gamma}_i^{t+1}, \ldots, \alpha_n^{t+1}\overrightarrow{\gamma}_n^{t+1} + \beta_n^{t+1}\overleftarrow{\gamma}_n^{t+1}) \\ &\leqslant \sum_{i\in N}\left\{\alpha_i^{t+1}\left[\varepsilon_i - \frac{\overrightarrow{\gamma}_i^{t+1}}{2}\left(\varepsilon_i + \sum_{j\in N}\varepsilon_j\right)\right]^2 + \beta_i^{t+1}\left[\varepsilon_i - \frac{\overleftarrow{\gamma}_i^{t+1}}{2}\left(\varepsilon_i + \sum_{j\in N}\varepsilon_j\right)\right]^2\right\}. \end{aligned} \tag{12}$$

The left-hand side of inequality (11) incorporates the value of the function $f(\gamma^{t+1})$ at an inner point of the n-dimensional rectangular parallelepiped $[\overrightarrow{\gamma}_1^{t+1}, \overleftarrow{\gamma}_1^{t+1}; \ldots; \overrightarrow{\gamma}_i^{t+1}, \overleftarrow{\gamma}_i^{t+1}; \ldots; \overrightarrow{\gamma}_n^{t+1}, \overleftarrow{\gamma}_n^{t+1}]$; its right-hand side, a special linear combination of the values of the function $f(\gamma^{t+1})$ at extreme points of this parallelepiped. In particular, for $n = 3$, the inequality takes the form

$$
\begin{aligned}
&f\big(\alpha_1^{t+1}\overrightarrow{\gamma}_1^{t+1} + \beta_1^{t+1}\overleftarrow{\gamma}_1^{t+1}, \alpha_2^{t+1}\overrightarrow{\gamma}_2^{t+1} + \beta_2^{t+1}\overleftarrow{\gamma}_2^{t+1}, \alpha_3^{t+1}\overrightarrow{\gamma}_3^{t+1} + \beta_3^{t+1}\overleftarrow{\gamma}_3^{t+1}\big)\\
&\leqslant \alpha_1^{t+1}\alpha_2^{t+1}\alpha_3^{t+1} f\big(\overrightarrow{\gamma}_1^{t+1}, \overrightarrow{\gamma}_2^{t+1}, \overrightarrow{\gamma}_3^{t+1}\big) + \beta_1^{t+1}\alpha_2^{t+1}\alpha_3^{t+1} f\big(\overleftarrow{\gamma}_1^{t+1}, \overrightarrow{\gamma}_2^{t+1}, \overrightarrow{\gamma}_3^{t+1}\big)\\
&+ \alpha_1^{t+1}\beta_2^{t+1}\alpha_3^{t+1} f\big(\overrightarrow{\gamma}_1^{t+1}, \overleftarrow{\gamma}_2^{t+1}, \overrightarrow{\gamma}_3^{t+1}\big) + \alpha_1^{t+1}\alpha_2^{t+1}\beta_3^{t+1} f\big(\overrightarrow{\gamma}_1^{t+1}, \overrightarrow{\gamma}_2^{t+1}, \overleftarrow{\gamma}_3^{t+1}\big)\\
&+ \beta_1^{t+1}\beta_2^{t+1}\alpha_3^{t+1} f\big(\overleftarrow{\gamma}_1^{t+1}, \overleftarrow{\gamma}_2^{t+1}, \overrightarrow{\gamma}_3^{t+1}\big) + \beta_1^{t+1}\alpha_2^{t+1}\beta_3^{t+1} f\big(\overleftarrow{\gamma}_1^{t+1}, \overrightarrow{\gamma}_2^{t+1}, \overleftarrow{\gamma}_3^{t+1}\big)\\
&+ \alpha_1^{t+1}\beta_2^{t+1}\beta_3^{t+1} f\big(\overrightarrow{\gamma}_1^{t+1}, \overleftarrow{\gamma}_2^{t+1}, \overleftarrow{\gamma}_3^{t+1}\big) + \beta_1^{t+1}\beta_2^{t+1}\beta_3^{t+1} f\big(\overleftarrow{\gamma}_1^{t+1}, \overleftarrow{\gamma}_2^{t+1}, \overleftarrow{\gamma}_3^{t+1}\big).
\end{aligned}
$$

4. INVESTIGATION RESULTS AND THEIR DISCUSSION

We introduce the notations

$$
\begin{aligned}
&\gamma^{t+1}(\beta^{t+1}) = \big(\gamma_1^{t+1}(\beta_1^{t+1}), \ldots, \gamma_i^{t+1}(\beta_i^{t+1}), \ldots, \gamma_n^{t+1}(\beta_n^{t+1})\big),\\
&\gamma^{t+1}(\beta^{t+1}) = (1 - \beta_i^{t+1})\overrightarrow{\gamma}_i^{t+1} + \beta_i^{t+1}\overleftarrow{\gamma}_i^{t+1} \ (i \in N) \text{ and}\\
&f^{t+1} = f\big(\gamma^{t+1}(\beta^{t+1})\big).
\end{aligned}
$$

The results of Section 3 can be applied to the convergence problem as follows.

For an extreme point of the parallelepiped $[\overrightarrow{\gamma}_1^{t+1}, \overleftarrow{\gamma}_1^{t+1}; \ldots; \overrightarrow{\gamma}_i^{t+1}, \overleftarrow{\gamma}_i^{t+1}; \ldots; \overrightarrow{\gamma}_n^{t+1}, \overleftarrow{\gamma}_n^{t+1}]$, we reduce formula (12) to $f^{t+1} \leqslant 1 - \varepsilon^T F^{t+1}\varepsilon$, where F^{t+1} is a symmetric matrix of dimensions $n \times n$. If F^{t+1} is positive definite (i.e., $\varepsilon^T F^{t+1}\varepsilon > 0$ for each set of real numbers $\varepsilon_1, \ldots, \varepsilon_2, \ldots, \varepsilon_n$ with at least one nonzero number), then $f^{t+1} < 1$ for this extreme point. If $f^{t+1} < 1$ for all extreme points, then we have $f^{t+1} < 1$ for any inner point of the parallelepiped due to inequality (11) and, consequently, $\|B^{t+1}\| \leqslant \max\limits_{\|\varepsilon\|=1} \sqrt{1 - \varepsilon^T F^{t+1}\varepsilon} < 1$ for this point due to (7) and (10). In other words, the convergence criterion (8) for the process (4), (5) holds for all points of the parallelepiped $[\overrightarrow{\gamma}_1^{t+1}, \overleftarrow{\gamma}_1^{t+1}; \ldots; \overrightarrow{\gamma}_i^{t+1}, \overleftarrow{\gamma}_i^{t+1}; \ldots; \overrightarrow{\gamma}_n^{t+1}, \overleftarrow{\gamma}_n^{t+1}]$. If all agents have the same left and right bounds of the ranges, then the computational complexity of solving this problem is significantly decreased by reducing the number of extreme points analyzed: from 2^n to $(n-1)$. Furthermore, modern computer mathematics packages provide necessary tools to check the positive definiteness of matrices of almost any size.

In addition, the results of Section 3 can be adopted to maximize the ranges of agents' steps (find a parallelepiped of the maximum volume) ensuring the convergent dynamics of collective behavior to an equilibrium. This is another significant result of the paper.

Let us begin with the premises for restricting the ranges of agents' steps. First, we consider the classical range $[\overrightarrow{\gamma}_i^{t+1}, \overleftarrow{\gamma}_i^{t+1}] = [0, 1]$ $(\forall i \in N; t = 0, 1, 2, \ldots)$.

Then $\gamma_i^{t+1} = (1 - \beta_i^{t+1}) \cdot 0 + \beta_i^{t+1} \cdot 1 = \beta_i^{t+1}$, and inequality (12) takes the form

$$
f^{t+1} = f(\gamma^{t+1}(\beta^{t+1})) \leqslant 1 - \sum_{i \in N} \beta_i^{t+1}(\varepsilon_i)^2 + \frac{1}{4}\sum_{i \in N} \beta_i^{t+1}\left(\sum_{j \in N\setminus\{i\}} \varepsilon_j\right)^2. \tag{13}
$$

If the symmetric quadratic form

$$
\sum_{i \in N} \beta_i^{t+1}(\varepsilon_i)^2 + \frac{1}{4}\sum_{i \in N} \beta_i^{t+1}\left(\sum_{j \in N\setminus\{i\}} \varepsilon_j\right)^2
$$

is positive definite for each set of real numbers $\varepsilon_1, \ldots, \varepsilon_i, \ldots, \varepsilon_n$, then equivalently, it is positive definite for each set of real numbers such that $\sum\limits_{i \in N} (\varepsilon_i)^2 = 1$. In this case, $f^{t+1} < 1$ and $\|B^{t+1}\| < 1$ due to (7) and (10). Note that $f^{t+1} \geqslant 0$ since $\sum\limits_{i \in N} \beta_i^{t+1}(\varepsilon_i)^2 \leqslant 1$.

The symmetric matrix corresponding to this quadratic form is given by

$$F^{t+1} = F(\gamma^{t+1}(\beta^{t+1}))$$

$$= \begin{pmatrix} \beta_1^{t+1} - \frac{1}{4}\sum\limits_{i\in N\setminus\{1\}}\beta_i^{t+1} & -\frac{1}{4}\sum\limits_{i\in N\setminus\{1,2\}}\beta_i^{t+1} & \dots & -\frac{1}{4}\sum\limits_{i\in N\setminus\{1,n\}}\beta_i^{t+1} \\ -\frac{1}{4}\sum\limits_{i\in N\setminus\{2,1\}}\beta_i^{t+1} & \beta_2^{t+1} - \frac{1}{4}\sum\limits_{i\in N\setminus\{2\}}\beta_i^{t+1} & \dots & -\frac{1}{4}\sum\limits_{i\in N\setminus\{2,n\}}\beta_i^{t+1} \\ \dots & \dots & \dots & \dots \\ -\frac{1}{4}\sum\limits_{i\in N\setminus\{n,1\}}\beta_i^{t+1} & -\frac{1}{4}\sum\limits_{i\in N\setminus\{n,2\}}\beta_i^{t+1} & \dots & \beta_n^{t+1} - \frac{1}{4}\sum\limits_{i\in N\setminus\{n\}}\beta_i^{t+1} \end{pmatrix}. \tag{14}$$

We utilize the following well-known result (for example, see [19]): a real quadratic form is positive definite if and only if the determinants of all principal minors of the corresponding matrix are positive (or equivalently, if the matrix is positive definite).

Let us consider in detail special cases of the market: $n = 2, 3, 4$.

The classical case is the Cournot duopoly. Here, inequality (13) and the matrix (14) take the form

$$f^{t+1} \leqslant 1 - \beta_1^{t+1}(\varepsilon_1)^2 + \frac{\beta_1^{t+1}}{4}(\varepsilon_2)^2 - \beta_2^{t+1}(\varepsilon_2)^2 + \frac{\beta_2^{t+1}}{4}(\varepsilon_1)^2,$$

$$F^{t+1} = \begin{pmatrix} \beta_1^{t+1} - \beta_2^{t+1}/4 & 0 \\ 0 & \beta_2^{t+1} - \beta_1^{t+1}/4 \end{pmatrix}.$$

If $4\beta_1^{t+1} - \beta_2^{t+1} > 0$ and, by symmetry, $4\beta_2^{t+1} - \beta_1^{t+1} > 0$, then all principal minors of the matrix F^{t+1} have positive determinants. The process converges when these inequalities are satisfied (i.e., each agent's step exceeds one-fourth of the other) starting from some time instant t. In particular, they hold if $\beta_1^{t+1} = \beta_2^{t+1} = 1$, meaning that each agent chooses the best response to the expected actions of the competitor. When the agents act differently and these inequalities are false, convergence in norm remains an open question, although the process is known to converge for $\beta_1^{t+1},\ \beta_2^{t+1} \in (0, 1]$.

In the case of the Cournot oligopoly model with three agents, inequality (13) and the matrix (14) take the form

$$f^{t+1} \leqslant 1 - \beta_1^{t+1}(\varepsilon_1)^2 - \beta_2^{t+1}(\varepsilon_2)^2 - \beta_3^{t+1}(\varepsilon_3)^2 + \frac{\beta_1^{t+1}}{4}(\varepsilon_2+\varepsilon_3)^2 + \frac{\beta_2^{t+1}}{4}(\varepsilon_1+\varepsilon_3)^2 + \frac{\beta_3^{t+1}}{4}(\varepsilon_1+\varepsilon_2)^2,$$

$$F^{t+1} = \begin{pmatrix} \beta_1^{t+1} - \frac{\beta_2^{t+1}}{4} - \frac{\beta_3^{t+1}}{4} & -\frac{\beta_3^{t+1}}{4} & -\frac{\beta_2^{t+1}}{4} \\ -\frac{\beta_3^{t+1}}{4} & \beta_2^{t+1} - \frac{\beta_1^{t+1}}{4} - \frac{\beta_3^{t+1}}{4} & -\frac{\beta_1^{t+1}}{4} \\ -\frac{\beta_2^{t+1}}{4} & -\frac{\beta_1^{t+1}}{4} & \beta_3^{t+1} - \beta_1^{t+1} - \frac{\beta_2^{t+1}}{4} \end{pmatrix}.$$

As is easily verified, for each set of real numbers β_1^{t+1}, β_2^{t+1}, and β_3^{t+1}, the determinant is nonpositive. Therefore, the convergence of the process cannot be established based on inequality (13). This is a rather unexpected conclusion since the process is known to converge for many sets of the parameters β_i^{t+1}. This conclusion is due to a significant difference in the values $\overrightarrow{\gamma}_i^{t+1} = 0$ and $\overleftarrow{\gamma}_i^{t+1} = 1$: agents can choose any steps, from close to 0 (almost remaining on the spot) to the maximum possible (profit-optimal) ones.

For $n > 3$ and the classical step range $[\overrightarrow{\gamma}_i^{t+1}, \overleftarrow{\gamma}_i^{t+1}] = [0, 1]$, we also cannot confirm the convergent process for any of the parameter sets β_i^{t+1}. Note that if all agents have equal parameters β_i^{t+1}, the matrix F^{t+1} for $n > 2$ is not positive definite. It suffices to check this fact for one parameter value. For $n = 4$ and equal parameters, the determinant of the second principal minor of the matrix becomes negative.

To confirm the convergence hypothesis, we restrict the ranges of agents' steps. When determining the bounds of the ranges, we proceed from the assumption that large steps are often preferable to small steps for the agents. Hence, it is desirable to have the right bounds as close to 1 as possible.

The following propositions and lemmas will be fruitful for further considerations. Their proofs are given in the Appendix.

Proposition 3. *In the Cournot oligopoly model, the process* (4), (5) *converges for the step ranges* $[\overrightarrow{\gamma}_i^{t+1}, \overleftarrow{\gamma}_i^{t+1}]$ *if all matrices* F^{t+1} *with the entries*

$$\begin{gathered} f_{ii}^{t+1} = 3(\mu_i^{t+1} - \eta_i^{t+1}) + \mu_i^{t+1} - \sum_{k \in N} \eta_k^{t+1}, \\ f_{ij}^{t+1} = (\mu_i^{t+1} - \eta_i^{t+1}) + (\mu_j^{t+1} - \eta_j^{t+1}) - \sum_{k \in N} \eta_k^{t+1}, \quad i, j \in N, \quad i \neq j, \end{gathered} \tag{15}$$

are positive definite starting from some time instant t_0. *Here,*

$$\begin{gathered} \mu_i^{t+1} = [\overrightarrow{\gamma}_i^{t+1} + \beta_i^{t+1}(\overleftarrow{\gamma}_i^{t+1} - \overrightarrow{\gamma}_i^{t+1})]/2, \\ \eta_i^{t+1} = [(\overrightarrow{\gamma}_i^{t+1})^2 + \beta_i^{t+1}((\overleftarrow{\gamma}_i^{t+1})^2 - (\overrightarrow{\gamma}_i^{t+1}))^2]/4, \quad \beta_i^{t+1} \in [0, 1]. \end{gathered}$$

Lemma 2. *Let* A *be a square matrix of dimensions* $m \times m$ *with entries* $a_{ii} = a$ *and* $a_{ij} = b$, $i, j \in M = \{1 \ldots, m\}$, $i \neq j$. *Then* $\det(A) = (a - b)^{m-1}[a + b(m - 1)]$.

Lemma 3. *The matrix* F^{t+1} *with the entries* (15) *is positive definite for* $\gamma_i^{t+1} = \overleftarrow{\gamma}^{t+1} < \frac{4}{1+n}$, $i \in N$.

By Lemma 3, in the case $n = 2$, we have $\overleftarrow{\gamma}^{t+1} < \frac{4}{3}$; hence, the process will reach equilibrium when all agents choose the maximum steps (equal to 1). In the case $n = 3$, the requirement becomes $\overleftarrow{\gamma}^{t+1} < 1$, so the process will not converge when all agents choose the maximum steps. In the case $n = 4$, we can confirm convergence only if the agents' steps do not exceed the right bounds of the range equal to 0.8.

Now, let us change the range bounds in the Cournot duopoly model: $\overrightarrow{\gamma}_i^{t+1} = 0.2$ and $\overleftarrow{\gamma}_i^{t+1} = 1$. In other words, agents cannot choose the smallest steps. Then the terms of the entries (15) are $\mu_i^{t+1} = 0.1 + \beta_i^{t+1} 0.4$, and $\eta_i^{t+1} = 0.01 + \beta_i^{t+1} 0.24$, and consequently, $F^{t+1} = \begin{pmatrix} 0.35 + 0.64\beta_1^{t+1} - 0.24\beta_2^{t+1} & 0.16 - 0.08\beta_1^{t+1} - 0.08\beta_2^{t+1} \\ 0.16 - 0.08\beta_1^{t+1} - 0.08\beta_2^{t+1} & 0.35 + 0.64\beta_2^{t+1} - 0.24\beta_1^{t+1} \end{pmatrix}$. The matrix F^{t+1} is positive definite for β_1^{t+1}, $\beta_1^{t+1} \in [0, 1]$. Obviously, the first principal minor has a positive determinant. The same property of the second principal minor follows from the fact that each diagonal element is greater than each off-diagonal counterpart. Therefore, if the agents choose their steps in the range $[\overrightarrow{\gamma}_i^{t+1}, \overleftarrow{\gamma}_i^{t+1}] = [0.2, 1]$, the process (4), (5) will converge for β_1^{t+1}, $\beta_1^{t+1} \in [0, 1]$. The next proposition shows that the left bound of the range can be reduced in the case $n = 2$. Its proof is postponed to the Appendix.

Proposition 4. *Consider the Cournot duopoly model under the assumption that the agents choose their steps in the range* γ_1^{t+1}, $\gamma_2^{t+1} \in [0.136, 1]$ *starting from some time instant* t_0. *Then the errors transition matrices* F^{t+1} *are positive definite,* $\|B^{t+1}\| < 1$, *and the process* (4), (5) *converges.*

These conclusions agreed with known results obtained by other methods [3–5, 14, 15], including experiments.

Let us revert to the Cournot oligopoly model with three agents. Recall that for the range $[0,1]$, the convergence hypothesis has not been confirmed above for any of the parameter sets β. We change the step range by assigning $\overrightarrow{\gamma}_i^{t+1} = 0.6$ and $\overleftarrow{\gamma}_i^{t+1} = 1$. Suppose that the agents choose unequal steps, e.g., $\beta_1^{t+1} = 0.4$, $\beta_2^{t+1} = 0.5$, and $\beta_3^{t+1} = 0.6$. Then the terms of the entries (15) are $\mu_1^{t+1} = 0.38$, $\eta_1^{t+1} = 0.154$, $\mu_2^{t+1} = 0.4$, $\eta_2^{t+1} = 0.17$, $\mu_3^{t+1} = 0.42$, and $\eta_3^{t+1} = 0.186$. The corresponding matrix is $F^{t+1} = \begin{pmatrix} 0.548 & -0.054 & -0.05 \\ -0.054 & 0.58 & -0.56 \\ -0.05 & -0.56 & 0.612 \end{pmatrix}$. This matrix is positive definite (with a large margin): the convergence condition of the process certainly holds for $\gamma_1^{t+1} = 0.76$, $\gamma_2^{t+1} = 0.8$, and $\gamma_3^{t+1} = 0.84$.

The following result characterizes the maximum step range ensuring the convergence of the process.

Proposition 5. *Consider the Cournot oligopoly model under the assumption that the agents choose their steps in the range* $\gamma_1^{t+1}, \gamma_2^{t+1}, \gamma_3^{t+1} \in [0.334, 1]$ *starting from some time instant* t_0. *Then the errors transition matrices* F^{t+1} *are positive definite,* $\|B^{t+1}\| < 1$, *and the process* (4), (5) *converges.*

For the Cournot oligopoly model with four or more agents, the maximum step ranges ensuring the convergent dynamics of collective behavior can be obtained using the same method as in the proof of Propositions 4 and 5 in the cases $n = 2$ and $n = 3$. This method includes the following stages.

1. A uniform right bound $\overleftarrow{\gamma}^*$ of the step ranges is determined for all agents and all time instants t. By Lemma 2, the desired maximum bound can be calculated from the condition $\overleftarrow{\gamma}^* = \min\left\{\frac{4}{1+n}; 1\right\}$. For $n = 2$, we have $\overleftarrow{\gamma}^* = 1$; for $n = 3$, $\overleftarrow{\gamma}^* = 1$ (in the case of three agents, 1 is excluded from the range); for $n = 4$, $\overleftarrow{\gamma}^* = 0.8$; for $n = 5$, $\overleftarrow{\gamma}^* = 2/3$ and so on.

For n agents in (10), we have $f(\overleftarrow{\gamma}^*, \overleftarrow{\gamma}^*, \ldots, \overleftarrow{\gamma}^*) < 1$ for each set of real numbers $\varepsilon_1, \ldots, \varepsilon_n$ such that $\sum\limits_{i \in N} (\varepsilon_i)^2 = 1$. (Note that this inequality involves an n-dimensional vector as the argument.)

2. A uniform left bound $\overrightarrow{\gamma}^*$ of the step ranges is determined for all agents and all time instants t. For this purpose, $(n-1)$ positive definiteness problems are solved for the matrices F.

For the parameter set $(\overrightarrow{\gamma}, \overleftarrow{\gamma}^*, \overleftarrow{\gamma}^*, \ldots, \overleftarrow{\gamma}^*)$, assigning $\beta_1^{t+1} = 0$ and $\beta_2^{t+1} = \beta_3^{t+1} = \ldots = \beta_n^{t+1} = 1$, we determine the matrix by (15). Let $\overrightarrow{\gamma}_{(1)}$ be the minimum value of the parameter $\overrightarrow{\gamma}$ under which this matrix remains positive definite. We have $f(\overrightarrow{\gamma}_{(1)}, \overleftarrow{\gamma}^*, \overleftarrow{\gamma}^*, \ldots, \overleftarrow{\gamma}^*) < 1$ if $\sum\limits_{i \in N} (\varepsilon_i)^2 = 1$. In addition, $f(\overleftarrow{\gamma}^*, \overrightarrow{\gamma}_{(1)}, \overleftarrow{\gamma}^*, \ldots, \overleftarrow{\gamma}^*) < 1$, $f(\overleftarrow{\gamma}^*, \overleftarrow{\gamma}^*, \overrightarrow{\gamma}_{(1)}, \overleftarrow{\gamma}^*, \ldots, \overleftarrow{\gamma}^*) < 1$, $\ldots$, $f(\overleftarrow{\gamma}^*, \overleftarrow{\gamma}^*, \overleftarrow{\gamma}^*, \ldots, \overleftarrow{\gamma}^*, \overrightarrow{\gamma}_{(1)}) < 1$.

For the parameter set $(\overrightarrow{\gamma}, \overrightarrow{\gamma}, \overleftarrow{\gamma}^*, \overleftarrow{\gamma}^*, \ldots, \overleftarrow{\gamma}^*)$, assigning $\beta_1^{t+1} = \beta_2^{t+1} = 0$ and $\beta_3^{t+1} = \ldots = \beta_n^{t+1} = 1$, we determine the matrix by (15). Let $\overrightarrow{\gamma}_{(2)}$ denote the minimum value of the parameter $\overrightarrow{\gamma}$ under which this matrix remains positive definite. We have $f(\overrightarrow{\gamma}_{(2)}, f\overrightarrow{\gamma}_{(2)}, \overleftarrow{\gamma}^*, \ldots, \overleftarrow{\gamma}^*) < 1$ if $\sum\limits_{i \in N} (\varepsilon_i)^2 = 1$. In addition, $f(\overrightarrow{\gamma}_{(2)}, \overleftarrow{\gamma}^*, \overrightarrow{\gamma}_{(2)}, \overleftarrow{\gamma}^*, \ldots, \overleftarrow{\gamma}^*) < 1$, $f(\overleftarrow{\gamma}^*, \overrightarrow{\gamma}_{(2)}, \overrightarrow{\gamma}_{(2)}, \overleftarrow{\gamma}^*, \ldots, \overleftarrow{\gamma}^*) < 1$, $\ldots$, $f(\overleftarrow{\gamma}^*, \overleftarrow{\gamma}^*, \ldots, \overleftarrow{\gamma}^*, \overrightarrow{\gamma}_{(2)}, \overrightarrow{\gamma}_{(2)}) < 1$.

The values $\overrightarrow{\gamma}_{(3)}, \overrightarrow{\gamma}_{(4)}, \ldots, \overrightarrow{\gamma}_{(n-1)}$ are obtained by analogy.

When determining the left bound $\overrightarrow{\gamma}_{(n)}$, we have $\det(F) = \overrightarrow{\gamma}^n (1+n)(2-\overrightarrow{\gamma})^{n-1}(2-(1+n)\overrightarrow{\gamma})$ for the parameter set $(\overrightarrow{\gamma}, \overrightarrow{\gamma}, \ldots, \overrightarrow{\gamma})$, and the matrix F is positive definite if $\overrightarrow{\gamma} \neq 0$.

The uniform left bound of the step ranges is calculated from the condition $\overrightarrow{\gamma}^* = \max\limits_i \{\overrightarrow{\gamma}_{(i)}; i = \overline{1, (n-1)}\}$.

5. CONCLUSIONS

A possible approach to the analytical investigation of convergence conditions for the dynamic processes of collective behavior in the linear oligopoly model is based on the errors transition matrices of the processes between time instants t and $(t+1)$. Using this approach, we have attempted to analyze convergence conditions for the Cournot model by restricting the ranges of agents' steps towards their current goals. We have proposed a method for determining the maximum step ranges ensuring convergent processes for an arbitrary number of agents. The resulting ranges are independent of the parameters of the market and agents but depend on the number of agents in the market. For the markets with two and three agents, the maximum step ranges have been calculated by this method.

According to Proposition 1, if the process (4), (5) converges in norm, the process (9), (5) will also converge in these ranges provided that the agents' competitiveness conditions are satisfied for all time instants starting from some time instant.

The problems of finding equilibria or convergence conditions are formulated and solved within other models of reflexive dynamics; for example, see [20]. The approach proposed in this paper seems promising for such models if they yield sequences of the form $\varepsilon^{t+1} \to 0$ and $\varepsilon^{t+1} = B^{t+1}\varepsilon^t$.

APPENDIX

Proof of Proposition 3. This result is established by transforming the right-hand side of (12):

$$\sum_{i\in N}\left\{\left(1-\beta_i^{t+1}\right)\left[\varepsilon_i - \frac{\overrightarrow{\gamma}_i^{t+1}}{2}\left(\varepsilon_i + \sum_{j\in N}\varepsilon_j\right)\right]^2 + \beta_i^{t+1}\left[\varepsilon_i - \frac{\overleftarrow{\gamma}_i^{t+1}}{2}\left(\varepsilon_i + \sum_{j\in N}\varepsilon_j\right)\right]^2\right\}$$

$$= \sum_{i\in N}(\varepsilon_i)^2 - \sum_{i\in N}\varepsilon_i\overrightarrow{\gamma}_i^{t+1}\left(\varepsilon_i + \sum_{j\in N}\varepsilon_j\right) + \left(\frac{\overleftarrow{\gamma}_i^{t+1}}{2}\right)^2\left(\varepsilon_i + \sum_{j\in N}\varepsilon_j\right)^2$$

$$- \sum_{i\in N}\beta_i^{t+1}\left\{\left[\varepsilon_i - \frac{\overrightarrow{\gamma}_i^{t+1}}{2}\left(\varepsilon_i + \sum_{j\in N}\varepsilon_j\right)\right]^2 - \left[\varepsilon_i - \frac{\overleftarrow{\gamma}_i^{t+1}}{2}\left(\varepsilon_i + \sum_{j\in N}\varepsilon_j\right)\right]^2\right\}$$

$$= 1 - \sum_{i\in N}\varepsilon_i\overrightarrow{\gamma}_i^{t+1}\left(\varepsilon_i + \sum_{j\in N}\varepsilon_j\right) + \left(\frac{\overrightarrow{\gamma}_i^{t+1}}{2}\right)^2\sum_{i\in N}\left(\varepsilon_i + \sum_{j\in N}\varepsilon_j\right)^2$$

$$- \sum_{i\in N}\beta_i^{t+1}\left[2\varepsilon_i - \left(\frac{\overleftarrow{\gamma}_i^{t+1}}{2} + \frac{\overrightarrow{\gamma}_i^{t+1}}{2}\right)\left(\varepsilon_i + \sum_{j\in N}\varepsilon_j\right)\right]\left(\frac{\overleftarrow{\gamma}_i^{t+1}}{2} - \frac{\overrightarrow{\gamma}_i^{t+1}}{2}\right)\left(\varepsilon_i + \sum_{j\in N}\varepsilon_j\right)$$

$$= 1 - \sum_{i\in N}2\varepsilon_i\left(\varepsilon_i + \sum_{j\in N}\varepsilon_j\right)\left[\frac{\overleftarrow{\gamma}_i^{t+1}}{2} + \beta_i^{t+1}\left(\frac{\overleftarrow{\gamma}_i^{t+1}}{2} - \frac{\overrightarrow{\gamma}_i^{t+1}}{2}\right)\right]$$

$$+ \sum_{i\in N}\left(\varepsilon_i + \sum_{j\in N}\varepsilon_j\right)^2\left[\left(\frac{\overrightarrow{\gamma}_i^{t+1}}{2}\right)^2 + \beta_i^{t+1}\left(\left(\frac{\overleftarrow{\gamma}_i^{t+1}}{2}\right)^2 - \left(\frac{\overrightarrow{\gamma}_i^{t+1}}{2}\right)^2\right)\right]$$

$$= 1 - \sum_{i\in N}2\varepsilon_i\left(\varepsilon_i + \sum_{j\in N}\varepsilon_j\right)\mu_i^{t+1} + \sum_{i\in N}\left(\varepsilon_i + \sum_{j\in N}\varepsilon_j\right)^2\eta_i^{t+1}$$

$$= 1 - 2\sum_{i\in N}(\varepsilon_i)^2\mu_i^{t+1} - \sum_{i\in N}2\varepsilon_i\mu_i^{t+1}\sum_{j\in N}\varepsilon_j + \sum_{i\in N}(\varepsilon_i)^2\eta_i^{t+1} + \sum_{i\in N}2\varepsilon_i\eta_i^{t+1}\sum_{j\in N}\varepsilon_j + \sum_{i\in N}\left(\sum_{j\in N}\varepsilon_j\right)^2\eta_i^{t+1}$$

$$= 1 - \sum_{i\in N} (\varepsilon_i)^2 \left(4\mu_i^{t+1} - 3\eta_i^{t+1} - \sum_{k\in N} \eta_k^{t+1} \right)$$

$$- \sum_{i\in N} \sum_{j\in N\setminus\{i\}} \varepsilon_i \varepsilon_j \left[\left(\mu_i^{t+1} - \eta_i^{t+1}\right) + \left(\mu_j^{t+1} - \eta_j^{t+1}\right) - \sum_{k\in N} \eta_k^{t+1} \right].$$

If the principal minors of the matrix (15) have positive determinants, then the quadratic form $\sum\limits_{i\in N} (\varepsilon_i)^2 \left(4\mu_i^{t+1} - 3\eta_i^{t+1} - \sum\limits_{k\in N} \eta_k^{t+1} \right) + \sum\limits_{i\in N} \sum\limits_{i\in N\setminus\{i\}} \varepsilon_i \varepsilon_j \left[(\mu_i^{t+1} - \eta_i^{t+1}) + (\mu_j^{t+1} - \eta_j^{t+1}) - \sum\limits_{k\in N} \eta_k^{t+1} \right]$ is positive definite and $\|B^{t+1}\| < 1$ due to (7) and (10). In other words, the process (4), (5) converges for the parameter values $\overrightarrow{\gamma}_i^{t+1}, \overleftarrow{\gamma}_i^{t+1}, \beta_i^{t+1}$.

The proof of Proposition 3 is complete.

Proof of Lemma 2. We take advantage of the following properties of determinants:

1) The determinant of a matrix does not change when multiplying any row by an arbitrary number and adding the result to any other row.

2) The determinant of a triangular square matrix is equal to the product of its diagonal elements.

Adding to each row the next row multiplied by (-1), we obtain

$$\det(A) = \begin{vmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ \dots & \dots & \dots & \dots & \dots \\ b & b & b & \dots & a \end{vmatrix} = \begin{vmatrix} a-b & b-a & 0 & \dots & 0 \\ b & a & b & \dots & b \\ \dots & \dots & \dots & \dots & \dots \\ b & b & b & \dots & a \end{vmatrix} = \begin{vmatrix} a-b & b-a & 0 & \dots & 0 \\ 0 & a-b & b-a & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ b & b & b & \dots & a \end{vmatrix}.$$

Using the row expansion for the last row and the triangular determinant formula, we finally arrive at $\det(A) = a(a-b)^{m-1} + (m-1)b(a-b)^{m-1} = (a-b)^{m-1}[a + (m-1)b]$.

The proof of Lemma 2 is complete.

Proof of Lemma 3. We have $\gamma_i^{t+1} = \overleftarrow{\gamma}_i^{t+1}$; in view of (15), for $\beta_i^{t+1} = 1$,

$$\mu_i^{t+1} = \frac{\overleftarrow{\gamma}_i^{t+1}}{2}, \quad \eta_i^{t+1} = \left(\frac{\overleftarrow{\gamma}_i^{t+1}}{2}\right)^2,$$

$$f_{ii}^{t+1} = 2\overleftarrow{\gamma}_i^{t+1} - (3+n)\left(\frac{\overleftarrow{\gamma}_i^{t+1}}{2}\right)^2, \quad f_{ij}^{t+1} = \overleftarrow{\gamma}_i^{t+1} - (2+n)\left(\frac{\overleftarrow{\gamma}_i^{t+1}}{2}\right)^2, \quad i,j\in N,\ i\neq j.$$

According to Lemma 2, $\det(F^{t+1}) = (1+n)\left(\frac{\overleftarrow{\gamma}_i^{t+1}}{2}\right)^n \left(2 - \frac{\overleftarrow{\gamma}_i^{t+1}}{2}\right)^{n-1} \left[2 - (1+n)\frac{\overleftarrow{\gamma}_i^{t+1}}{2}\right]$. The determinant is positive for $\overleftarrow{\gamma}_i^{t+1} < \frac{4}{1+n}$. Under this inequality, the kth principal minor of the matrix F^{t+1} $(k<n)$ has a positive determinant as well. Therefore, $a + (k-1)b = \overleftarrow{\gamma}_i^{t+1}\Big[2(1+k) - (1+2k+nk)\overleftarrow{\gamma}_i^{t+1}/2\Big]/2$.

For $\overleftarrow{\gamma}_i^{t+1} = \frac{4}{1+n}$, it follows that $a + (k-1)b = \frac{4(n-k)}{(1+n)^2} > 0$. For $\overleftarrow{\gamma}_i^{t+1} < \frac{4}{1+n}$, the positivity of the determinant of the kth principal minor is obvious.

The proof of Lemma 3 is complete.

Proof of Proposition 4. When restricting the step ranges, it is desirable to have their right bounds as close to 1 as possible. Let all agents have the same step range at all time instants. Therefore, we omit the superscript $(t+1)$ for f and F.

Based on Lemma 2, for the Cournot duopoly, the right bound of $\overleftarrow{\gamma}$ of the range should be taken equal to 1 and $f(1,1) < 1$.

Consider $f(\overrightarrow{\gamma},1)$. The matrix (15) corresponding to this quadratic form is defined for $\beta_1 = 0$ and $\beta_2 = 1 : F = \begin{pmatrix} 2\overrightarrow{\gamma} - \overrightarrow{\gamma}^2 - 0.25 & \overrightarrow{\gamma}/2 - \overrightarrow{\gamma}^2/2 \\ \overrightarrow{\gamma}/2 - \overrightarrow{\gamma}^2/2 & 1 - \overrightarrow{\gamma}^2/4 \end{pmatrix}$. For the convenience of calculations, the determinant of this matrix can be simplified to $\det(F) = \begin{vmatrix} 3.75 & \overrightarrow{\gamma}/2 - 2 \\ \overrightarrow{\gamma}/2 - 2 & 1 - \overrightarrow{\gamma}^2/4 \end{vmatrix}$. The matrix is positive definite if $\overrightarrow{\gamma} \geqslant 0.136$. Therefore, $f(0.136, 1) < 1 \ \ \forall \varepsilon_1, \varepsilon_2$.

Naturally, $f(0.136, 1) = f(1, 0.136) < 1$.

Consider $f(\overrightarrow{\gamma},\overrightarrow{\gamma})$. The matrix (15) corresponding to this quadratic form is defined for $\beta_1 = \beta_2 = 0 \ : \ F = \begin{pmatrix} 2\overrightarrow{\gamma} - 5\overrightarrow{\gamma}^2/4 & \overrightarrow{\gamma} - \overrightarrow{\gamma}^2 \\ \overrightarrow{\gamma} - \overrightarrow{\gamma}^2 & 2\overrightarrow{\gamma} - 5\overrightarrow{\gamma}^2/4 \end{pmatrix}$. The matrix is positive definite if $\overrightarrow{\gamma} \neq 0$. Therefore, $f(0.136, 0.136) < 1 \ \ \forall \varepsilon_1, \varepsilon_2$.

The desired result follows from (11), and the proof of Proposition 4 is complete.

Proof of Proposition 5.

Consider $f(1,1,\overrightarrow{\gamma})$. The matrix (15) corresponding to this quadratic form is defined for $\beta_1 = \beta_2 = 1$ and $\beta_3 = 0$: $F = \begin{pmatrix} 0.75 - \overrightarrow{\gamma}^2/4 & -\overrightarrow{\gamma}^2/4 & -0.25 + \overrightarrow{\gamma}/2 - \overrightarrow{\gamma}^2/2 \\ -\overrightarrow{\gamma}^2/4 & 0.75 - \overrightarrow{\gamma}^2/4 & -0.25 + \overrightarrow{\gamma}/2 - \overrightarrow{\gamma}^2/2 \\ -0.25 + \overrightarrow{\gamma}/2 - \overrightarrow{\gamma}^2/2 & -0.25 + \overrightarrow{\gamma}/2 - \overrightarrow{\gamma}^2/2 & -0.5 + 2\overrightarrow{\gamma} - \overrightarrow{\gamma}^2 \end{pmatrix}$.

For the convenience of calculations, the determinant of this matrix can be simplified to $\det(F) = \begin{vmatrix} 0.75 & -\overrightarrow{\gamma}^2/4 & -0.25 + \overrightarrow{\gamma}/2 - \overrightarrow{\gamma}^2/2 \\ 0 & 0.75 - \overrightarrow{\gamma}^2/2 & -0.25 + \overrightarrow{\gamma}/2 - \overrightarrow{\gamma}^2 \\ 0 & -0.25 + \overrightarrow{\gamma}/2 - \overrightarrow{\gamma}^2/2 & -0.5 + 2\overrightarrow{\gamma} - \overrightarrow{\gamma}^2 \end{vmatrix}$.

The minimum value $\overrightarrow{\gamma}$ under which the matrix F remains positive definite is 0.334. Hence, $f(1, 1, 0.334) < 1 \quad \forall \varepsilon_1, \varepsilon_2$.

In addition, we have $f(1, 1, 0.334) = f(1, 0.334, 1) = f(0.334, 1, 1) < 1 \ \forall \varepsilon_1, \varepsilon_2$.

Consider $f(1,\overrightarrow{\gamma},\overrightarrow{\gamma})$. The matrix (15) corresponding to this quadratic form is defined for $\beta_1 = 1$ and $\beta_2 = \beta_3 = 0 \ : \ F = \begin{pmatrix} 1 - \overrightarrow{\gamma}^2/2 & \overrightarrow{\gamma}/2 - 3\overrightarrow{\gamma}^2/4 & \overrightarrow{\gamma}/2 - 3\overrightarrow{\gamma}^2/4 \\ \overrightarrow{\gamma}/2 - 3\overrightarrow{\gamma}^2/4 & -0.25 + 2\overrightarrow{\gamma} - 5\overrightarrow{\gamma}^2/4 & -0.25 + \overrightarrow{\gamma} - \overrightarrow{\gamma}^2 \\ \overrightarrow{\gamma}/2 - 3\overrightarrow{\gamma}^2/4 & -0.25 + \overrightarrow{\gamma} - \overrightarrow{\gamma}^2 & -0.25 + 2\overrightarrow{\gamma} - 5\overrightarrow{\gamma}^2/4 \end{pmatrix}$. For the convenience of calculations, the determinant of this matrix can be simplified to $\det(F) = \begin{Vmatrix} 1 - \overrightarrow{\gamma}^2/2 & \overrightarrow{\gamma} - 3\overrightarrow{\gamma}^2/2 & 1 - 3\overrightarrow{\gamma}^2/4 \\ \overrightarrow{\gamma}/2 - 3\overrightarrow{\gamma}^2/4 & -0.5 + 3\overrightarrow{\gamma} - 9\overrightarrow{\gamma}^2/4 & -0.25 + \overrightarrow{\gamma} - \overrightarrow{\gamma}^2 \\ 0 & 0 & \overrightarrow{\gamma} - \overrightarrow{\gamma}^2/4 \end{Vmatrix}$.

The minimum value $\overrightarrow{\gamma}$ under which the matrix F remains positive definite is 0.22. Hence, $f(1, 0.22, 0.22) < 1 \ \forall \varepsilon_1, \varepsilon_2$.

Also, $f(0.22, 1, 0.22) = f(0.22, 0.22, 1) = f(1, 0.22, 0.22) < 1 \ \forall \varepsilon_1, \varepsilon_2$.

Consider $f(\overrightarrow{\gamma},\overrightarrow{\gamma},\overrightarrow{\gamma})$. The matrix (15) corresponding to this quadratic form is defined for $\beta_1 = \beta_2 = \beta_3 = 0 : F = \begin{pmatrix} 2\overrightarrow{\gamma} - 3\overrightarrow{\gamma}^2/2 & \overrightarrow{\gamma} - 5\overrightarrow{\gamma}^2/4 & \overrightarrow{\gamma} - 5\overrightarrow{\gamma}^2/4 \\ \overrightarrow{\gamma} - 5\overrightarrow{\gamma}^2/4 & 2\overrightarrow{\gamma} - 3\overrightarrow{\gamma}^2/2 & \overrightarrow{\gamma} - 5\overrightarrow{\gamma}^2/4 \\ \overrightarrow{\gamma} - 5\overrightarrow{\gamma}^2/4 & \overrightarrow{\gamma} - 5\overrightarrow{\gamma}^2/4 & 2\overrightarrow{\gamma} - 3\overrightarrow{\gamma}^2/2 \end{pmatrix}$.

By Lemma 2, $\det(F) = \overrightarrow{\gamma}^3(2 - \overrightarrow{\gamma}/2)^2(1 - \overrightarrow{\gamma})$. The matrix is positive definite if $\overrightarrow{\gamma} \neq 0.1$. From these lower bounds we choose the maximum value, i.e., 0.334.

Based on Lemma 2, for the Cournot oligopoly model with three agents, the right bound $\overleftarrow{\gamma}$ of the step range should be chosen less than 1.

The desired result follows from (11), and the proof of Proposition 5 is complete.

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