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Stabilization of Oscillations of a Controlled Autonomous System

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Abstract—We consider a smooth autonomous system in general form that admits a non-degenerate periodic solution. A global family (with respect to the parameter h) of nondegenerate periodic solutions is constructed, the law of monotonic variation of the period on the family is derived, and the existence of a reduced second-order system is proved. For it, the problem of stabilizing the oscillation of the controlled system, distinguished by the value of the parameter h, is solved. A smooth autonomous control is found, and an attracting cycle is constructed.

Keywords: autonomous system, non-degenerate periodic solution, global family, Lyapunov center theorem, control, attracting cycle, natural stabilization

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1. INTRODUCTION

In 1927 van der Pol proposed an equation describing a linear oscillator affected by a small non-linear force that is linear by velocity and creates dissipation in every point of trajectory (van der Pol dissipation). The equation admits an attracting cycle. Later L.S. Pontryagin in [1] found a sufficient condition for distinguishing a cycle from a family of periodic solutions of a Hamiltonian system: he applied non-Hamiltonian perturbations. Oscillations of a linear oscillator are isochronous, and oscillation period in a non-linear Hamiltonian system depends on an energy constant.

Paper [2] introduced van der Pol type dissipation for a family of periodic solutions where the period is a monotonous function of the family parameter h. For a linear oscillator, we get the van der Pol dissipation. A system is constructed in which to stabilize the oscillation with the parameter h the corresponding value of the parameter in the control is selected. The controlled system remains autonomous and has an asymptotically orbitally stable cycle. These results were further developed in [3–5]. Article [6] proposed a mechatronic stabilization system. In multidimensional systems, a reduced system of a possible lower order was constructed.

Other studies related to stabilization of a desired oscillation regime are different due to the use of explicitly time-variant equations. Let us mention some of them. Paper [7] provides a review using an inversed pendulum case study. Article [8] proposes swinging control. Study [9] solves an orbital stabilization of underactuated non-linear system periodic solutions (a number of independent actuators is one less than a number of degrees of freedom of an uncontrolled conservative system). The synthesized non-linear control law with feedback is time-variant. Paper [10] deals with stabilization of the desired mechanical energy levels using impulse control. Study [11] finds robust stabilizing oscillation control using implicit Lyapunov method.

We can isolate a reduced-order system when the whole set of periodic solutions is described. In non-linear systems, the oscillation period usually varies from one oscillation to another. The set of single-frequency oscillations with monotonic change in period over it depending on a parameter that takes all possible values for the set becomes an unit of such set and is called a global family of periodic solutions. In phase space, the global family fills a connected invariant set. The system may have a finite or countable number of global families.

The global family problem initially arose in numerous applications of A.M. Lyapunov's center theorem (1892). The theorem establishes the existence of a local family of nonlinear periodic motions adjacent to the equilibrium of a conservative system. In specific problems, the question of the limits of applicability of the Lyapunov family always arises. The research of A.A. Zevin (1997) started progress in the boundary question. Paper [12] gives the Hamiltonian conditions that guarantee, in compact set $\Omega \in \mathrm{R}^{2n}$, the continuation of the Lyapunov family to the boundary of $\partial\Omega$. The results are further developed in [13], where strictly starshaped Hamiltonians are found for which the continuation conditions of [12] are satisfied.

It is now understood that the question of the boundary in Lyapunov theorem is resolved by knowing about the global family of periodic solutions, which includes the local Lyapunov family as a component.

Reversible mechanical systems are distinguished by spatial-temporal symmetry, but generally do not allow for a first integral. Let us consider symmetric periodic motions of such systems. Global families in these systems have been studied in [4, 5, 14, 15], the concept of a global family was introduced in [14]. The results on the global family (including the issues of existence, construction, properties, etc.) were used to find a control with a parameter with a changing value in [4–6].

In this paper, we state the problem of constructing a global family of non-degenerate periodic solutions for an autonomous system in general form, unrestricted by additional constraints (e.g., Hamiltonian, inversibility, conservatism, etc.). We also solved it using the Pontryagin approach in ordinary differential equation theory [16] to introduce an inextensible solution. In the second part of the paper, we construct a controlled system for the reduced second-order system. A smooth autonomous control is found with an adjustable parameter that selects a stabilized oscillation.

2. NON-DEGENERATE PERIODIC SOLUTION OF AN AUTONOMOUS SYSTEM

Let us consider a smooth autonomous equation

$$\dot{z} = Z(z), \quad z \in \mathrm{R}^n, \tag{1}$$

and express its solution in terms of $z(z_1^0, \ldots, z_n^0, t)$, where $z^0 = (z_1^0, \ldots, z_n^0)$ is an initial point (at $t = 0$). The necessary and sufficient condition of the existence of T-periodical solution of system (1) is written as an equality

$$f \equiv z(z_1^0, \ldots, z_n^0, T) - z^0 = 0. \tag{2}$$

Suppose that Eq. (2) has a solution $z^0 = z^*$, $T = T^*$ that does not coincide with equilibrium[1] $Z(z^*) \neq 0$. Owing to autonomy of system (1), Eq. (2) together with the mentioned solution always has a family of solutions with respect to the parameter γ, i.e. the shift of point z^0 along the trajectory:

$$z^0 = z^*(\gamma), \quad T = T^*. \tag{3}$$

Let us compute the rank Ra of the functional matrix A_f (Jacobi matrix) for the function f with parameter T at the point z^* while $T = T^*$. This leads to $\mathrm{Ra} \leqslant n - 1$.

[1] reviewer's comment

Definition 1. The case of $\mathrm{Ra} = n - 1$ is called non-degenerate for a periodic solution. When $\mathrm{Ra} = n - 1$, the periodic solution is called non-degenerate; otherwise, it is called degenerate.

Now, let us study the case of $\mathrm{Ra} = n - 1$.

Equation (2) may admit the only solution in the form (3). Then the autonomous Eq. (1) has an isolated periodic solution, which is a cycle with period T^*. Alternatively, there would be a family of solutions, where the period T changes from one solution to another, i.e. is a function of a certain parameter h: $z^0 = z^0(\gamma, h)$, $T = T(h)$.

We derive the following system of linear equalities from equality (2) in the neighbourhood of solution (3):

$$\begin{aligned} &\xi_s \equiv \frac{\partial f_s}{\partial z_1^0} dz_1^0 + \ldots + \frac{\partial f_s}{\partial z_n^0} dz_n^0 + \frac{\partial f_s}{\partial T} dT = 0, \quad s = 1, \ldots, n, \\ &f = (f_1, \ldots, f_n), \end{aligned} \tag{4}$$

partial derivatives in which are computed at $z^0 = z^*$, $T = T^*$. System (4), as well as equality (2), are satisfied identically with respect to shift γ. Due to the derivative

$$\frac{\partial f_s}{\partial T} = \frac{\partial z(z^0, T)}{\partial T} = Z_s(z^0),$$

then it necessarily leads to linear dependency of functions $Z_s(z^0)$, $s = 1, \ldots, n$.

In a cycle situation, the system (4) is satisfied only if $\Delta T = T - T^* = 0$. Matrix A_f has a simple zero eigenvalue.

Next, we consider the case where the matrix A_f contains a Jordan 2 cell of zero-valued eigenvalues. Here, we have a scalar parameter h (see [17]). From equalities (4), considered as a system of linear equations with the matrix A_f, we derive the existence in Eq. (2) of a local family of solutions in dT. Accordingly, Eq. (1) admits a local h-family of periodic solutions. For a T^*-periodic solution: $T^* = T(h^*)$, $T'(h^*) \neq 0$ (where ($'$) denotes differentiation with respect to h). Parameter h belongs to the neighborhood of the number h^*.

In what follows, h is called a parameter of the family of periodic solutions. The function $T(h)$ is monotone, so the period T can be chosen as h.

In a family situation, the solutions of the equations in variations consist of partial derivatives of the periodic solution with respect to both t and h. Differentiating it with respect to h yields a rising along time t solution, that is given in, e.g. [18, p. 416, Eq. (9.9)].

Let us introduce definition 2.

Definition 2. A family of periodic solutions of Eq. (1) is called non-degenerate if the period $T(h)$ in it is a monotonous function of parameter h.

Definition 2 is valid for the local family considered, where h varies in a neighborhood of number h^*. It remains valid for any family of periodic solutions: in it, h ranges inside an interval. Thus, there exists an interval of parameter h values that corresponds to all periodic solutions of the family.

Definition 3. A non-degenerate family of periodic solutions, in which the parameter h takes all possible values for the solutions of the family, is called a global family.

In phase space, the global family is represented by a connected set of points. The interval of variation of h in the global family can be finite (the pendulum energy from the lower to the upper equilibrium) or unbounded (coordinate unbounded oscillations). Equation (1) may have one, several, or a countable set of global families of non-degenerate periodic solutions. An individual global family will be the subject of the article.

Further, we abbreviate a non-degenerate periodic solution belonging to a family as NPS. Then a global family of non-degenerate periodic solutions will be called a global NPS family and denoted by Σ.

An NPS exists for a system of order $n \geqslant 2$. Oscillations of a mathematical pendulum yield an example of an NPS: $\ddot{x} + \sin x = 0,\ n = 2,\ \mathrm{Ra} = 1$. In addition, local oscillations near the lower equilibrium compose a Lyapunov family of NPS. Its period continuation leads to the whole oscillation family of the pendulum (global NPS family) that connects the lower and upper equilibria. The oscillation period in it, starting at the value of 2π, monotonously approaches infinity along with the oscillation amplitude that approaches π.

A linear oscillator would have a family of degenerate (isochronous) oscillations with a single period: $n = 2,\ \mathrm{Ra} = 0$.

Remark 1. Reversible mechanical systems can have symmetric NPS, paper [4] refines the concept of NPS for such solutions.

3. LOCAL PROPERTIES OF NPS

NPS has the following properties.

1°. Local extendability of NPS.

In system (4), the increment dT is non-zero and varies independently of γ. We find all curves of an NPS family from the system (4) if the increment dT varies continuously and bilaterally. Therefore, Eq. (1) together with NPR, where $h = h^*$, has a family of NPRs with a monotonic dependence of the period $T(h)$ on h. Moreover, the parameter h belongs to a certain neighborhood of the number h^*. Increment dT is constrained by condition $\mathrm{Ra} = n - 1$.

A family of NPS can also be distinguished with an interval of period T variation; the family is denoted as $\Sigma(T)$.

2°. A family $\Sigma(T)$ fills a two-dimensional domain $\hat{\Sigma}(T)$.

The domain $\hat{\Sigma}(T)$ is made up of closed curves, which are NPS of Eq. (1), parameterized with the shift γ. These curves vary along with the family parameter h.

Remark 2. A period is the only parameter in a local family. Additionally, $T'(h^*) \neq 0$ and the period is a function of one parameter h (the law of dependence of the nonlinear oscillation period on one parameter [17]).

4. GLOBAL FAMILY OF NPS

We obtain the global family of NPS (Σ) by the two-sided extension of family $\Sigma(T)$ with respect to period T (with respect to parameter h). Accordingly, the extension of the domain $\hat{\Sigma}(T)$ by the period T in the phase space leads to the global domain $\hat{\Sigma}$ filled with solutions from Σ.

Lemma 1. *The global family Σ exists and fills the global domain $\hat{\Sigma}$. The period on it is a monotonous function of the parameter of the family. For points in domain $\hat{\Sigma}$, the rank is* $\mathrm{Ra} = n-1$, *and the condition* $\mathrm{Ra} = n - 1$ *is not satisfied at its boundary $\partial\hat{\Sigma}$.*

Proof. We consider the NPS of system (1). According to properties 1° and 2°, it belongs to the family $\Sigma(T)$ and fills the domain $\hat{\Sigma}(T)$. As a result, two situations arise. In the first, the domain $\hat{\Sigma}(T)$ does not change, and the global family Σ is constructed. In the second situation, domain $\hat{\Sigma}(T)$ expands in the sense that it contains all the points of the previous domain and also new points. Thus, the family $\Sigma(T)$ has been extended by period T in both the increasing and decreasing directions. The system (4) describes the extension process. The obtained family preserves a monotonous dependence of the period on the parameter of the family.

In the next step, a similar alternative arises as in the previous iteration.

As a result, we obtain the global domain $\hat{\Sigma}$, in which condition $\mathrm{Ra} = n - 1$ is satisfied. The global NPS family fills the domain $\hat{\Sigma}$. On the boundary $\partial\hat{\Sigma}$, the condition $\mathrm{Ra} = n - 1$ is violated.

It takes an infinite number of steps to construct Σ.

Remark 3. We use the Pontryagin approach to the construction of an inextensible solution in the theory of ordinary differential equations to achieve a global family.

Lemma 2. *The global family of NPS is described by a reduced second-order system.*

Proof. Equalities (4) are identically satisfied in global family Σ for pair (z^0, T). The rank of matrix A_f is $n - 1$, the derivatives with respect to T are linearly dependent and the vector $df/dT \neq 0$. Therefore, the $n - 1$ linearly independent differential forms from (4), connected by the conditions (4), are reduced to a system in which only one of the forms obtained contains dT. The independence of the remaining forms from dT means that solutions of the global family are described by a second-order system.

Remark 4. In the case of a conservative system, a reduced conservative system with one degree of freedom is obtained.

We formulate Theorem 1 based on Lemmas 1 and 2.

Theorem 1. *Assume that Eq.* (1) *admits for an NPS. Then it extends to the global family Σ in respect to period T. The period $T(h)$ in Σ is a monotonous function of the family parameter h. The family Σ fills the global domain $\hat{\Sigma}$, and Σ is described by the reduced second-order system. Rank* $\mathrm{Ra} = n - 1$ *for points in domain $\hat{\Sigma}$. The condition* $\mathrm{Ra} = n - 1$ *is violated at its boundary.*

Remark 5. Approaching the boundary $\partial\hat{\Sigma}$, the derivative $T'(h)$ may tend to zero, infinity, or cease to exist. In the case of $T'(h) \to 0$, the boundary of the family may be a degenerate periodic solution. Alternatively, a center can be located on the boundary (Examples 1 and 2, $\kappa < 1$). The family can also become unbounded in coordinate (Example 2, $\kappa > 1$). The case of $T'(h) \to \infty$ is realized for a center (Example 2, $\kappa > 1$), a saddle (Example 1), as well as for an unbounded global family (Example 2, $\kappa < 1$) (see [19]).

Corollary 1. *A Lyapunov family extends onto a global family of NPS (global Lyapunov center theorem).*

Proof. The period on a local Lyapunov family depends monotonically on the constant energy [18]. Therefore, the family consists of NPS. According to Theorem 1, the global family exists and is obtained by extending any NPS. Therefore, the Lyapunov center theorem has a global character.

Remark 6. Description of the global domain $\hat{\Sigma}$ and its boundary provides a complete solution to the boundary problem in the Lyapunov center theorem.

Remark 7. A special case of Theorem 1 yields results of [4, 14, 15].

The examples illustrate Theorem 1 and Remark 5. In particular, they show the movements realized on the $\partial\hat{\Sigma}$.

Example 1. $\ddot{x} + \sin x = 0$. The oscillations in the mathematical pendulum form a global family of non-degenerate oscillations. Its boundaries are the equilibria (the center and the saddle) and the separatrix. The period monotonously rises from the center to the saddle in the family.

Example 2. $\ddot{x} + x^\kappa = 0$, $\kappa > 0$. A family of periodic solutions surrounds the equilibrium $x = 0$, $\dot{x} = 0$ on the phase plane. The equation allows for the first integral

$$\dot{x}^2/2 + x^{\kappa+1}/(\kappa + 1) = h(\mathrm{const}).$$

Let us calculate the period of the oscillation with amplitude $x_{\max}$:

$$T = 4 \int\limits_0^{x_{\max}} \frac{dx}{\sqrt{2(h - x^{\kappa+1}/\kappa + 1)}}.$$

Assume $x = h^{1/(\kappa+1)}y$. Then

$$T = 4h^{\frac{1-\kappa}{2(\kappa+1)}} \int\limits_0^{y_{\max}} \frac{dy}{\sqrt{2(1 - y^{\kappa+1}/(\kappa+1))}},$$

where $x_{\max} = h^{1/(\kappa+1)}y_{\max}$. It follows that if $\kappa < 1$ then the period monotonously rises along with h (the amplitude of the oscillation), if $\kappa = 1$ (a case of a linear oscillator) then the period of the oscillation does not depend on h, if $\kappa > 1$ then the period monotonously decays to zero as the amplitude increases. Therefore, if $\kappa \neq 1$ then the equation has a family of non-degenerate unbounded oscillations, while its one boundary is the zero equilibrium, and another boundary approaches infinity.

5. CONTROLLED REDUCED SYSTEM

Assume that Eq. (1) admits for NPS. According to Theorem 1 it belongs to the global family Σ that fills the global domain $\hat{\Sigma}$. We suggest that $\hat{\Sigma}$ is an attracting domain.

The family Σ is described by a reduced second-order system. For this system, we state the problem of stabilizing the NPR from the family Σ. We consider a smooth system.

Hence, we solve the problem of finding smooth controls F and G with small controller gain ε in the system

$$\dot{x} = X(x, y) + \varepsilon F, \quad \dot{y} = Y(x, y) + \varepsilon G, \tag{5}$$

such that (5) would realize an attracting cycle that is close to the specified NPS from the family Σ. We suggest that the family Σ, i.e. the solutions of the system (5) at $\varepsilon = 0$, are described as

$$x = \varphi(h, t), \quad y = \psi(h, t). \tag{6}$$

We find controls that solve the stabilization problem for any NPS of the family Σ: the parameter value is $h = h^*$ for the specified NPS.

For small ε we set down a non-homogeneous linear system

$$\begin{aligned} \delta\dot{x} &= a_{11}\delta x + a_{12}\delta y + \varepsilon F, \\ \delta\dot{y} &= a_{21}\delta x + a_{22}\delta y + \varepsilon G, \\ \delta x &= x - \varphi(h, t), \quad \delta y = y - \psi(h, t), \end{aligned} \tag{7}$$

where the homogeneous part coincides with the variation equations for the solution (6). Expression (7) accounts only for linear terms by $\delta x, \delta y$. The addition of non-linear terms does not change the qualitative conclusions obtained, while the quantitative difference would be $o(\varepsilon)$.

Lemma 3. *The system* (7) *is transformed to the form*

$$\begin{aligned} \dot{u} &= \varepsilon[\dot{\psi}(h^*, t)F - \dot{\varphi}(h^*, t)G]/\Delta, \\ \dot{v} &= \frac{T'(h^*)}{T^*}u + \varepsilon[\eta(h^*, t)F - \xi(h^*, t)G]/\Delta, \end{aligned} \tag{8}$$

where T^*-periodic functions $\xi(h^*, t)$ and $\eta(h^*, t)$ are derivatives of corresponding functions $\varphi(h, (T/T^*)t)$ and $\psi(h, (T/T^*)t)$ with respect to h at $h = h^*$.

Proof. We transform the homogeneous part of (7) to a system with fixed coefficients using the Lyapunov transform. Then we set down the full system as (8).

Remark 8. We provide the explicit form of the transform in the Appendix. Formulas (8) are correct for NPS corresponding to the parameter value $h = h^*$.

6. ATTRACTING CYCLE OF A CONTROLLED SYSTEM

The necessary condition for the existence of a T^*-periodic solution of the controlled system is derived from the first equation of system (8):

$$\int_0^{T^*} [\dot{\psi}(h^*, t)F - \dot{\varphi}(h^*, t)G]/\Delta(h^*, t)d\tau = 0. \tag{9}$$

If condition (9) is satisfied, integrating the second equation in (8) yields a formula for the variable v. Thus, the uniqueness of an isolated T^*-periodic solution, i.e., the existence of a cycle, is deduced from the analysis of (9).

The condition (9) can be considered as an equation to determine the value of the parameter h^* for NPS. Therefore, after substituting h into functions φ and ψ, (9) yields the amplitude equation

$$I(h) \equiv \int_0^{T^*} [\dot{\psi}(h, t)F - \dot{\varphi}(h, t)G]/\Delta(h, t)d\tau = 0. \tag{10}$$

The system (8) defines the mapping $t : 0 \to T$ with period T^*. The inequality $I'(h^*) \neq 0$ guarantees the uniqueness of the fixed point of the mapping, and the inequality $I'(h^*) < 0$ guarantees the belonging of the eigenvalue to the left half-plane. It is calculated as in [2].

Each simple root $h = h^*$ of the Eq. (10) corresponds to a unique T^*-periodic solution of the Eq. (8), i.e. the cycle of the controlled system (5). The inequality $I'(h^*) < 0$ gives the corresponding condition for the attraction of solutions of the controlled system to the cycle.

Thus, Lemma 4 holds.

Lemma 4. *The simple root of the amplitude Eq.* (10) *corresponds to the cycle of the controlled system* (5). *If the inequality $I'(h^*) < 0$ is true, then the cycle is attracting.*

7. CONTROL SEARCH

The basis for the control search is the amplitude Eq. (10). According to lemma 4, the problem of stabilizing the oscillations of the controlled system (5) is solved by controls F and G for which the amplitude Eq. (10) admits a simple root h^*: $I'(h^*) < 0$.

The variables x and y and the acting controls F and G enter to the system (5) in equally right. The amplitude Eq. (10) preserves this equality of importance. It follows that one can stabilize the distinguished oscillation by any pair of controls $(F, 0)$ or $(0, G)$ and, in general, one can choose $F \neq 0,\ G \neq 0$. Besides, for a periodic solution, the variables x and y determine the position and velocity of the point on the trajectory. This is well demonstrated in the reduced system (8), where the variable u determines the position of the point and v is its velocity. It can also be seen that when the pair $(F, 0)$ is chosen, dissipation along the variable x also leads to dissipation along y.

According to the problem statement in Section 5, we find a control that stabilizes any NPS of the family Σ distinguished by the value of the parameter h. Therefore, the control contains a certain characteristic K such that, when substituting the function $K(h)$ into the expression for the control instead of the number K, the amplitude Eq. (10) would be fulfilled identically, i.e.,

$$\int_0^{T(h)} \Phi(K(h), \varphi(h, t), \psi(h, t), \dot{\varphi}(h, t), \dot{\psi}(h, t))d\tau \equiv 0, \qquad \Phi = (\dot{\psi}F - \dot{\varphi}G)/\Delta. \tag{11}$$

Without loss of generality, the function Φ is assumed to be linear in K. Then, the identity (11) is satisfied with the function

$$\Phi = [1 - Ka(h,t)]b(h,t). \tag{12}$$

We obtain explicit expressions for $a(h,t)$ and $b(h,t)$ by substituting functions $\varphi(h,t)$ and $\psi(h,t)$ along with their derivatives into Φ.

When differentiating the identities (11) along h with the function (12) substituted, we obtain

$$[1 - Ka(h,T)]b(h,T)\frac{dT(h)}{dh} + \int_0^{T(h)} \frac{d\Phi(h)}{dh} d\tau \equiv 0. \tag{13}$$

One of the derivatives $\dot{\varphi}(h,t)$ or $\dot{\psi}(h,t)$ in an NPS at time points of $t = 0, T$ equals zero. For this reason, we search for controls under condition that $b(h,0) = b(h,T) = 0$, and we derive the following equality from (13)

$$\frac{dI(h^*)}{dh} = \frac{dK(h^*)}{dh}\nu, \quad \nu = \int_0^{T^*} a(h^*,t)b(h^*,t)d\tau. \tag{14}$$

We calculate the number K for the given h^* based on the amplitude Eq. (11):

$$K(h^*) = \frac{\int_0^{T(h^*)} b(h^*,t)d\tau}{\int_0^{T(h^*)} a(h^*,t)b(h^*,t)d\tau}. \tag{15}$$

The function $K(h)$ is a characteristic of the control (and, simultaneously, of the family Σ).

Therefore, we formulate a general Theorem 2 based on Lemma 4.

Theorem 2. *Any NPS of the global family Σ that corresponds to the parameter value $h = h^*$ is stabilized by controls with the function* (12) *if the inequality $K'(h^*)\nu < 0$ is satisfied for the characteristic $K(h)$ at $h = h^*$.*

Remark 9. We use an adaptive control with an adjustable parameter h.

8. SPECIFIC CONTROLS

Theorem 2 identifies a class of controls. In the particular case of mechanical systems, this class includes a control (see [2–4]), which naturally stabilizes an oscillation without the use of any other controls. It applies a non-linear force, linearly dependent on velocity, acting at the current point on the trajectory and leading to dissipation. Other controls in the identified class also possess this property.

Below are some specific controls.

1. We consider controls

$$F = \alpha(1 - Kx^2)\dot{x}, \quad G = -\beta(1 - Ky^2)\dot{y}, \quad \alpha, \beta \text{ — const}, \tag{16}$$

with characteristic $K = K(h)$. Then, we derive the amplitude equation from (11):

$$I(h) \equiv \int_0^{T^*} [1 - K(\alpha x^2 + \beta y^2)]\dot{\varphi}(h,t)\dot{\psi}(h,t)/\Delta(h,t)d\tau = 0.$$

This leads to the function

$$K(h)=\frac{\int\limits_0^{T(h)}\dot\varphi(h,t)\dot\psi(h,t)/\Delta(h,t)d\tau}{\int\limits_0^{T(h)}[\alpha\varphi^2(h,t)+\beta\psi^2(h,t)]\dot\varphi(h,t)\dot\psi(h,t)/\Delta(h,t)d\tau}.$$

Finally, the formula (14) for stabilization of NPS with parameter value $h=h^*$ takes the form

$$\frac{dI(h^*)}{dh}=\frac{dK(h^*)}{dh}\nu,\quad \nu=\int\limits_0^{T^*}(\alpha x^2+\beta y^2)\dot\varphi(h^*,t)\dot\psi(h^*,t)/\Delta(h^*,t)d\tau.$$

Remark 10. For a single second-order equation $x=\dot y$, for this reason, in (16) we assume $\alpha=1$, $\beta=0$.

Remark 11. The results for a mechanical system affected by positional forces, in particular potential forces, are known (see [2]). For the system $\Delta(h,t)=1$.

Remark 12. If $Y(x,0)\equiv 0$, we obtain a reduced reversible second-order mechanical system. For the system $\Delta(h,t)=1$. We apply controls (16) to it [4].

2. We select controls

$$F=(1-Kx^2)\dot y,\quad G\equiv 0.$$

Then the following formula provides the characteristic:

$$K(h)=\frac{\int\limits_0^{T(h)}\dot\psi^2(h,t)/\Delta(h,t)d\tau}{\int\limits_0^{T(h)}\varphi^2(h,t)\dot\psi^2(h,t)/\Delta(h,t)d\tau}.$$

The case in consideration in (14) leads to

$$\frac{dI(h^*)}{dh}=\frac{dK(h^*)}{dh}\nu,\quad \nu=\int\limits_0^{T^*}\varphi^2(h^*,t)\dot\psi^2(h^*,t)/\Delta(h^*,t)d\tau.$$

3. Assume that we apply controls

$$F\equiv 0,\quad G=-(1-Ky^2)\dot x.$$

We compute the following characteristic:

$$K(h)=\frac{\int\limits_0^{T(h)}\dot\varphi^2(h,t)/\Delta(h,t)d\tau}{\int\limits_0^{T(h)}\psi^2(h,t)\dot\varphi^2(h,t)/\Delta(h,t)d\tau},$$

and set down the formula (14) as

$$\frac{dI(h^*)}{dh}=\frac{dK(h^*)}{dh}\nu,\quad \nu=\int\limits_0^{T^*}\psi^2(h^*,t)\dot\varphi^2(h^*,t)/\Delta(h^*,t)d\tau.$$

Remark 13. The papers [2–4, 20] contain case studies of NPS stabilization for specific systems. In the following, we provide a case study of stabilization of an oscillation that belongs to a degenerate family.

9. DEGENERATE FAMILY

In the NPS family, the parameter h reflects the monotonous change in the period T along with h: in the family $T'(h) \neq 0$. The inequality is not satisfied in the degenerate family. For such a family, Theorem 1 on a global family is not applicable. However, the approach developed in the article is applicable to the degenerate family in a two-dimensional manifold.

For the degenerate family, we can propose another characteristic of the family of solutions instead of the period, namely the oscillation amplitude. Like the period, the amplitude can monotonously change in the family. The solutions of the family are parameterized by the amplitude of oscillations.

Below, we give an interesting example of stabilization (in the large) of a cycle born from a degenerate family of periodic solutions of an autonomous system.

Example 3. Consider the controlled system

$$\dot{x} = y, \quad \dot{y} + x = \varepsilon(1 - Kx^2)y, \tag{17}$$

where K is a constant. We apply the control (16) to the linear oscillator. In this control, we use numbers $\alpha = 1$, $\beta = 0$ with an accuracy to denotations of variables.

At $\varepsilon = 0$, the oscillations are given by the formula $x = A\cos(t + \gamma)$, where γ is a time shift on the trajectory. The amplitude equation is written as

$$\int_0^{2\pi} (1 - KA^2\cos^2(t+\gamma))A^2\sin^2(t+\gamma)d\tau = 0.$$

Two roots are found: $A = 0$ and $A = 2/\sqrt{K}$. The first root corresponds to the origin, and the second root corresponds to a cycle close to the oscillation of a linear oscillator with an amplitude of $2/\sqrt{K}$. The amplitude A is chosen as a parameter of the family of oscillations of the linear oscillator, while the amplitude of the cycle of the system (17) is determined by the parameter K in the control.

The formula $K = 4/A^2$ gives us the function $K(A)$, the derivative is $K'(A) < 0$, therefore, the cycle stabilizes in the small.

Consequently, we generalize the approach to stabilization of an oscillation belonging to a NPS family to the family of degenerate periodic solutions.

Remark 14. Essentially, the system (17) contains the van der Pol equation. The cycle of the system (17) is stabilized globally.

10. CONCLUSION

A non-degenerate periodic solution of an autonomous system can be a cycle or belong to a family. On the NPS, the period changes monotonically with the family parameter. The NPS continues over the period for the global NPS family. A global NPS family fills an invariant manifold and is described by a reduced second-order system. Monotonically changes of the period on the global family NPS is preserved.

We solve the problem of NPS stabilization for a global NPS family that fills a global domain that is assumed to be attracting. An approach is being developed in which the control is built for the entire global family of NPS, and to stabilize the selected NPS, the parameter of the family is fixed. A smooth autonomous control is applied, as a result of which an attracting cycle is realized, which is close to the distinguished NRS of the global family. The approach is also applicable to a degenerate family of periodic solutions.

APPENDIX

Reduction of a homogeneous system.

We set down the variation equations for the solution (6) as

$$\begin{aligned} \delta\dot{x} &= \frac{\partial X(x,y)}{\partial x}\delta x + \frac{\partial X(x,y)}{\partial y}\delta y, \\ \delta\dot{y} &= \frac{\partial Y(x,y)}{\partial x}\delta x + \frac{\partial Y(x,y)}{\partial y}\delta y, \end{aligned} \tag{A.1}$$

where the partial derivatives are calculated for $x = \varphi(h,t),\ y = \psi(h,t)$.

The following matrix gives the fundamental system of solutions in (Π.1):

$$\left\| \begin{matrix} \dfrac{\partial\varphi(h,t)}{\partial t} & \dfrac{\partial\varphi(h,t)}{\partial h} \\ \dfrac{\partial\psi(h,t)}{\partial t} & \dfrac{\partial\psi(h,t)}{\partial h} \end{matrix} \right\| \tag{A.2}$$

with the determinant

$$\Delta(h,t) = \Delta(h,0)\exp\int\limits_0^t \left(\frac{\partial X(x,y)}{\partial x} + \frac{\partial Y(x,y)}{\partial y}\right) d\tau.$$

We set down the general solution of the system in (A.1)

$$\begin{aligned} \delta x &= c_1\dot{\varphi}(h,t) + c_2\varphi'(h,t), \\ \delta y &= c_1\dot{\psi}(h,t) + c_2\psi'(h,t) \end{aligned} \tag{A.3}$$

with constants c_1 and c_2. We use resolution of the system (A.3) with regard to c_1 and c_2 to transition to new variables u, v:

$$u = -(\dot{\psi}\delta x - \dot{\varphi}\delta y)/\Delta, \quad v = [\eta(h,t)\delta x - \xi(h,t)\delta y]/\Delta. \tag{A.4}$$

At the same time we assume

$$\begin{aligned} \xi(h,t) &= \frac{T'(h)}{T^*}t\dot{\varphi}(h,t) + \frac{\partial\varphi(h,t)}{\partial h}, \\ \eta(h,t) &= \frac{T'(h)}{T^*}t\dot{\psi}(h,t) + \frac{\partial\psi(h,t)}{\partial h}, \end{aligned}$$

where the functions $\xi(h^*,t),\ \eta(h^*,t)$ would be T^*-periodic. Finally, we obtain the following from the expression for v in (A.4):

$$v = \frac{T'(h)}{T^*}tu + (\psi'(h,t)\delta x - \varphi'(h,t)\delta y)/\Delta. \tag{A.5}$$

We substitute the first solution into formulas (A.4) and (A.5) and calculate $u = 0$, $v = 1$. Correspondingly, we set down the following for the second solution:

$$u = -(\dot{\psi}\varphi' - \dot{\varphi}\psi')/\Delta = 1, \quad v = \frac{T'(h^*)}{T^*}tu + u.$$

Therefore, we transition the system (A.1) to the following form:

$$\dot{u} = 0, \quad \dot{v} = \frac{T'(h^*)}{T^*}u.$$

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