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Identification of Periodic Regimes in a Dynamic System

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Abstract—For a dynamic system given by first-order ordinary differential equations, the problem of identification of periodic regimes is investigated. This problem is the establishment the periodicity of an arbitrary solution via the periodicity of the observed value of solution. The conditions under which the problem of identification of periodic regimes is solvable are found. Formulated and proven theorems supplement the well-known results on the observability of dynamic systems.

Keywords: dynamic system, identification of periodic regimes, observed value

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1. INTRODUCTION

Consider a dynamic system given by first-order ordinary differential equations

$$\frac{dx}{dt} = F(t, x), \quad x \in \mathbb{R}^n, \tag{1}$$

where $\mathbb{R}^n$ is the euclidean space of n-dimensional vectors with real coordinates, $n \geqslant 2$, $F(t, y)$: $\mathbb{R}^{1+n} \mapsto \mathbb{R}^n$ is a continuous mapping and ω-periodic in t, $\omega > 0$. We call the periodic regime an arbitrary ω-periodic solution $x(t) = (x_1(t), \dots, x_n(t))$, $x(t + \omega) = x(t)$, $t \in \mathbb{R}$ of the system of Eqs. (1). The behaviour of the solutions of the dynamic system (1) in many cases is related to the existence of periodic regimes. In general, it is difficult to find periodic regimes analytically or numerically. Therefore, it seems relevant to find the periodic regimes of the dynamic system (1) using the so-called observed values $Cx(t)$, where C is a given non-zero matrix of the size $m \times n$. The establishment the ω-periodicity of an arbitrary solution $x(t)$ through the ω-periodicity of the observed value $Cx(t)$ is called the problem of identification of periodic regimes in the dynamic system (1).

In the control theory, the problem of observability, which consists of uniquely determining $x(t)$ from the observed value $Cx(t)$, has been largely studied for linear systems (see, for example, [1, 2]). But the problem of identification of periodic regimes in the linear and non-linear dynamic systems has not been investigated. One can give examples of linear and non-linear systems with no periodic regimes, although the observed values are periodic. In this paper the conditions under which the problem of identification of periodic regimes in the dynamic system (1) is solvable are found. Formulated and proven theorems supplement the well-known results on the observability of dynamic system.

Some papers are devoted the research of the periodic solutions of systems of ordinary differential equations. Among them, one can mention monographs [3, 4] which have similar ideas to the those of authors and present fundamental methods for research bounded and periodic solutions of systems of ordinary differential equations. In [5–7] conditions for the existence of periodic regimes in dynamic models of the control theory were studied.

2. MAIN RESULTS

We investigate the problem of identification of periodic regimes for the system of ordinary differential equations of the form

$$\frac{dx}{dt} = Ax + f(t, Cx), \quad x \in \mathbb{R}^n. \tag{2}$$

Here $n \geqslant 2$, A is a square matrix of order n, C is a matrix of size $m \times n$, $f(t, y) : \mathbb{R}^{1+n} \mapsto \mathbb{R}^n$ is a continuous mapping, ω-periodic in t.

Introduce the matrix

$$B = [C; CA; \ldots; CA^{n-1}], \tag{3}$$

which is composed of rows of matrices $C, CA, \ldots, CA^{n-1}$.

The following is true.

Theorem 1. *Let the rank of the matrix B defined by the formula* (3) *be n:*

$$\operatorname{rank}(B) = n. \tag{4}$$

Then for an arbitrary solution of the system of Eqs. (2) *the ω-periodicity of $Cx(t)$ implies the ω-periodicity of $x(t)$.*

The condition (4) in control theory is called the complete observability condition for a pair of matrices (A, C) [2].

As an example, consider the following system of three ordinary differential equations:

$$\frac{dx_1}{dt} = x_2 + f_1(t, Cx), \quad \frac{dx_2}{dt} = x_3 + f_2(t, Cx), \quad \frac{dx_3}{dt} = f_3(t, Cx), \tag{5}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^\top$, $C = (c_1, c_2, c_3)$, $f(t, y) = (f_1(t, y), f_2(t, y), f_3(t, y))^\top : \mathbb{R}^4 \mapsto \mathbb{R}^3$ is a continuous mapping, ω-periodic in t.

Compose the matrix of coefficients of the system of Eqs. (5):

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

For the matrix $B = [C; CA; CA^2]$ the condition $\operatorname{rank}(B) = 3$ is satisfied only for $c_1 \neq 0$. Hence, according to the Theorem 1, if $c_1 \neq 0$, then for an arbitrary solution of the system of Eqs. (5), the ω-periodicity of the observed value $c_1x_1(t) + c_2x_2(t) + c_3x_3(t)$ implies ω-periodicity of the solution $x(t)$ itself. Existence of ω-periodic solutions depends on the given functions $f_1(t, y)$, $f_2(t, y)$, $f_3(t, y)$. For example, assuming $\omega = 2\pi$, we set

$$f_1(t, y) = -2\cos t\varphi_1(y), \quad f_2(t, y) = -2\sin t\varphi_2(y), \quad f_3(t, y) = \cos t\varphi_3(y),$$

where $\varphi_k(y) = 1$ for $|y| \leqslant |c_1| + |c_2| + |c_3|$, $k = 1, 2, 3$. In this case, the vector function $x^0(t) = (-\sin t, \cos t, \sin t)^\top$ is a 2π-periodic solution of the system of Eqs. (5).

Let's find conditions whereby the system of Eqs. (2) has at least one solution with ω-periodic observed value $Cx(t)$. Obviously, such a solution exists if the system of equations has an ω-periodic solution. From Theorem 13.4, proved in the monograph [8, p. 77–80], it follows that the system of Eqs. (2) has an ω-periodic solution if the matrix A does not have purely imaginary eigenvalues that are multiples of $i2\pi/\omega$, and the mapping $f(t,y)$ satisfies the condition $|y|^{-1}|f(t,y)| \rightrightarrows 0$ for $|y| \to \infty$. Of interest are the cases when there exists a non-ω-periodic solution $x(t)$ with an ω-periodic observed value $Cx(t)$.

Consider the system of linear ordinary differential equations

$$\frac{dx}{dt} = Ax + g(t), \quad x \in \mathbb{R}^n, \tag{6}$$

where the vector-function $g(t)$ is assumed to be given, continuous and ω-periodic.

The following is true.

Theorem 2. *The system of Eqs.* (6) *has a unique solution with ω-periodic observed value $Cx(t)$ if and only if conditions* (4) *and*

$$\det\left(e^{\omega A} - E\right) \neq 0 \tag{7}$$

are satisfied, where $e^{\omega A}$ is matrix exponent, E is the identity matrix of order n.

Note that under the condition (7) the system of equations

$$\frac{dx}{dt} = Ax, \quad x \in \mathbb{R}^n, \tag{8}$$

does not have a non-zero ω-periodic solution [4]. Therefore, from Theorem 2 it follows that if the condition (7) is satisfied and the condition (4) is violated, then the system of Eqs. (8) has a non-ω-periodic solution $x(t)$ with ω-periodic observed value $Cx(t)$.

Now consider the system of the form

$$\frac{dx}{dt} = Ax + G(t,x), \quad x \in \mathbb{R}^n, \tag{9}$$

where the mapping $G(t,y) : \mathbb{R}^{1+n} \mapsto \mathbb{R}^n$ is continous, ω-periodic in t and satisfies Lipschitz condition

$$|G(t,y_1) - G(t,y_2)| \leqslant L|y_1 - y_2|, \quad y_1, y_2 \in \mathbb{R}^n,$$

with the constant $L \geqslant 0$ that does not depend on t, y_1, y_2. From the general properties of solutions to systems of ordinary differential equations [9, Ch. 2, §3] it follows that an arbitrary solution $x(t)$ of the system of Eqs. (9) is defined for all $t \in (-\infty, +\infty)$.

The following is true.

Theorem 3. *Lets condition* (4) *holds. Then*

1) *there exists a number $M > 0$ which depends only on matrices A, C and such that for any vector-function $z(t) \in C^1([0,1];\mathbb{R}^n)$ the inequality*

$$\max_{0\leqslant t\leqslant 1} |z(t)| \leqslant M\left(\max_{0\leqslant t\leqslant 1}\left|\frac{dz(t)}{dt} - Az(t)\right| + \max_{0\leqslant t\leqslant 1}|Cz(t)|\right) \tag{10}$$

holds;

2) *if $LM < 1$, the for an arbitrary solution $x(t)$ of the system of Eq.* (9) *for any $a \in \mathbb{R}$ the following inequality is true*

$$\max_{a\leqslant t\leqslant a+1} |x(t+\omega) - x(t)| \leqslant (1 - LM)^{-1} M \max_{a\leqslant t\leqslant a+1} |Cx(t+\omega) - Cx(t)|. \tag{11}$$

The proofs of the Theorems 1–3 are given in the Appendix.

By using the results from the book [1, ch. 4], the given theorems can be generalized under the assumptions that the matrices A and C continuously depend on t and are ω-periodic.

APPENDIX

Let us verify the validity of the following lemma.

Lemma 1. *For an arbitrary vector $u \in \mathbb{R}^n$ the identity $Ce^{tA}u \equiv 0$, $t \in (t_1, t_2)$ is equivalent to*

$$Cu = 0, \quad CAu = 0, \quad \ldots, \quad CA^{n-1}u = 0. \tag{A.1}$$

Proof of Lemma. Let the identity $Ce^{tA}u \equiv 0$, $t \in (t_1, t_2)$ holds. Let's check that $Ce^{tA}u \equiv 0$, $t \in R$. To do this, it suffices to show that for any $v \in \mathbb{R}^m$ the function $\varphi(t) = \langle Ce^{tA}u, v\rangle$ is identically equal to zero on $\mathbb{R}$.

Let's find the derivatives of the function $\varphi(t)$: $\varphi^{(k)}(t) = \langle CA^k e^{tA}u, v\rangle$, $k = 1, 2, \ldots$. Next, we use the fact that according to the Hamilton–Cayley theorem [10, p. 93] matrix A satisfies its characteristic equation

$$A^n + q_1 A^{n-1} + \ldots + q_{n-1}A + q_n E = O,$$

where

$$\lambda^n + q_1\lambda^{n-1} + \ldots + q_{n-1}\lambda + q_n \equiv \det(\lambda E - A).$$

From here it follows that function $\varphi(t)$ satisfies the linear homogeneous differential equation

$$y^{(n)}(t) + q_1 y^{(n-1)}(t) + \ldots + q_{n-1}y'(t) + q_n y(t) = 0, \quad t \in R^n.$$

For this equation, only the zero solution can vanish identically on some interval. Since according to the condition $\varphi(t) \equiv 0$, $t \in (t_1, t_2)$, so $\varphi(t) \equiv 0$, $t \in R$. Therefore, the identity $Ce^{tA}u \equiv 0$, $t \in R$ holds. Differentiating this identity k times and setting $k = 0, 1, \ldots, n-1$, $t = 0$, we obtain equalities (A.1).

Conversely, if equality (A.1) holds, then from the Hamilton–Cayley theorem follows that $CA^k u = 0$ for any integer $k \geqslant 0$. Hence, by the definition of the matrix exponent, we derive $Ce^{tA}u \equiv 0$, $t \in \mathbb{R}$. The lemma is proven.

Proof of Theorem 1. Let's condition (4) holds and $x(t)$ be a solution of the system of Eq. (2) satisfying the conditions

$$Cx(t+\omega) = Cx(t), \quad t \in (-\infty, +\infty). \tag{A.2}$$

We solve the system of Eqs. (2) with respect to $x(t)$, assuming that the vector-function $f(t, Cx(t))$ is given:

$$x(t) = e^{tA}\left(x(0) + \int_0^t e^{-sA} f(s, Cx(s))ds\right). \tag{A.3}$$

Given this equality, condition (A.2) takes the following form:

$$Ce^{tA}\left(\left(e^{\omega A} - E\right)x(0) + \int_0^{t+\omega} e^{(\omega-s)A} f(s, Cx(s))ds - \int_0^t e^{-sA} f(s, Cx(s))ds\right) = 0.$$

It is easy to verify that

$$\frac{d}{dt}\left(\int_0^{t+\omega} e^{(\omega-s)A} f(s, Cx(s))ds - \int_0^t e^{-sA} f(s, Cx(s))ds\right)$$
$$= f(t+\omega, Cx(t+\omega)) - f(t, Cx(t)) = 0, \quad t \in (-\infty, +\infty).$$

Consequently

$$\int_0^{t+\omega} e^{(\omega-s)A} f(s, Cx(s))ds - \int_0^t e^{-sA} f(s, Cx(s))ds \equiv \int_0^{\omega} e^{(\omega-s)A} f(s, Cx(s))ds,$$

and we obtain the equality

$$Ce^{tA}\left(\left(e^{\omega A} - E\right)x(0) + \int_0^{\omega} e^{(\omega-s)A} f(s, Cx(s))ds\right) = 0, \quad t \in (-\infty, +\infty).$$

From here by virtue of the Lemma we obtain:

$$B\left(\left(e^{\omega A} - E\right)x(0) + \int_0^{\omega} e^{(\omega-s)A} f(s, Cx(s))ds\right) = 0. \tag{A.4}$$

Thus, for the solution $x(t)$ of the system (2) from (A.2) follows (A.4) and

$$f(t+\omega, Cx(t+\omega)) = f(t, Cx(t)), \quad t \in (-\infty, +\infty). \tag{A.5}$$

The converse is also true, if (A.4) and (A.5) hold for the solution $x(t)$ of the system of Eqs. (2), then (A.2) holds.

Since rank $(B) = n$, therefore (A.4) is possible if only under the condition

$$\left(e^{\omega A} - E\right)x(0) + \int_0^{\omega} e^{(\omega-s)A} f(s, Cx(s))ds = 0. \tag{A.6}$$

From (A.3) and (A.6) it follows ω-periodicity of $x(t)$.

Theorem 1 is proven.

Proof of Theorem 2. Above it was shown that for the solution $x(t)$ of the system of Eqs. (2), condition (A.2) is equivalent to the conditions (A.4) and (A.5). Assuming $f(s, Cx(s)) \equiv g(s)$ in these conditions, it follows that the system of Eqs. (6) has a unique solution with ω-periodic observed value $Cx(t)$ if and only if the system of algebraic equations

$$B\left(\left(e^{\omega A} - E\right)x(0) + \int_0^{\omega} e^{(\omega-s)A} g(s)ds\right) = 0$$

has a unique solution with unknown $x(0) \in \mathbb{R}^n$. But this is possible only under the condition

$$\operatorname{rank}\left(B\left(e^{\omega A} - E\right)\right) = n.$$

This condition, according to the definition and the general properties of the rank of a matrix, is equivalent to the conditions (4) and (7).

Theorem 2 is proven.

Proof of Theorem 3. Suppose the inequality (10) doesn't hold. Then there is an infinite sequence of vector-functions $z_j(t) \in C^1([0,1]; \mathbb{R}^n)$, $j = 1, 2, \ldots$ such that

$$\max_{0\leqslant t\leqslant 1} |z_j(t)| > j\left(\max_{0\leqslant t\leqslant 1}\left|\frac{dz_j(t)}{dt} - Az_j(t)\right| + \max_{0\leqslant t\leqslant 1}|Cz_j(t)|\right), \quad j = 1, 2, \ldots.$$

Consider the vector-functions

$$v_j(t) = r_j^{-1} z_j(t), \quad t \in [0,1], \quad j = 1, 2, \ldots,$$

where r_j is the maximum of the function $|z_j(t)|$ on the interval $[0,1]$. For these vector-functions we have:

$$1 = \max_{0 \leqslant t \leqslant 1} |v_j(t)| > j \left(\max_{0 \leqslant t \leqslant 1} |v_j'(t) - Av_j(t)| + \max_{0 \leqslant t \leqslant 1} |Cv_j(t)| \right), \quad j = 1, 2, \ldots.$$

Passing to the limit along a uniformly convergent subsequence of the vector-functions $v_{j_1}(t)$, $v_{j_2}(t), \ldots$, as a limit we obtain the function $v(t) \in C^1([0,1]; \mathbb{R}^n)$ such that

$$\max_{0 \leqslant t \leqslant 1} |v(t)| = 1, \quad v'(t) - Av(t) \equiv 0, \quad Cv(t) \equiv 0.$$

From here it follows that

$$v(t) \equiv e^{tA} v(0), \quad v(0) \neq 0, \quad Ce^{tA} v(0) \equiv 0.$$

By virtue of the Lemma from the last identity it follows that the system of Eqs. (A.1) has a non-zero solution, which contradicts the condition $\operatorname{rank}(B) = n$. The inequality (10) is proven.

Let $LM<1$ and $x(t)$ be an arbitrary solution of the system of Eqs. (9). Substituting $x(t+a+\omega) - x(t+a)$ in (10) instead of $z(t)$, we get

$$\max_{a \leqslant t \leqslant a+1} |x(t+\omega) - x(t)|$$
$$\leqslant M \left(\max_{a \leqslant t \leqslant a+1} |G(t, x(t+\omega)) - G(t, x(t))| + \max_{a \leqslant t \leqslant a+1} |Cx(t+\omega) - Cx(t)| \right).$$

Further, by using the Lipschitz condition

$$\max_{a \leqslant t \leqslant a+1} |G(t, x(t+\omega)) - G(t, x(t))| \leqslant L \max_{a \leqslant t \leqslant a+1} |x(t+\omega) - x(t)|,$$

we obtain the inequality (11).

Theorem 3 is proven.

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