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LINEAR SYSTEMS

On the Properties of Orthogonal Projection Method for Reaching Consensus

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Abstract—The article is devoted to an asymptotic behavior of a multi-agent system with information links. We proved that the orthogonal projection method proposed for the regularization of the consensus protocol is characterized by a pseudoinverse matrix for the introduced auxiliary matrix for an arbitrary communication digraph of a multi-agent system. We cosidered the eigenprojection of the Laplacian matrix corresponding to the communication digraph, in which the influences on the fixed agent change proportionally. We obtained a number of results that are of independent importance and can be used in models of multi-agent systems with different protocols.

Keywords: multi-agent system, consensus, eigenprojection, Laplacian matrix, communication digraph, balanced digraph

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1. INTRODUCTION

Multi-agent systems (MAS) with information links (see [1–5]) are represented by a weighted communication digraph, and the protocol for matching characteristics for the continuous case is specified using the Laplacian matrix. Protocols are MAS models for a discrete case, described by stochastic matrices. The conditions for reaching consensus in such models are determined by the algebraic properties of the communication digraph or by the invariants (spectrum, eigenprojection, etc.) of the corresponding matrices. In such models, the existence of a spanning tree is a prerequisite for consensus or for matching characteristics. The asymptotic behavior of the system, as established in [6–8], is determined by the eigenprojection of the communication digraphs' Laplacian matrix. For a discrete model, the consensus also depends on the limit of the sequence of powers of the stochastic matrix. For all protocols, if consensus is reached with any initial vector, then the eigenprojection rank is equal to 1. By definition of regularity [9], a stochastic matrix is regular if the rank of the limit of the sequence of its powers is 1. If the eigenprojection has rank greater than 1, then consensus is not reached for every initial vector. In this case, any method that leads to consensus is called a regularization method. The word "regularization" is related to the fact that the asymptotic behavior of the system is given by a stochastic matrix of rank 1 under the taken measure. And the rank of the limit of the sequence of powers of a stochastic matrix is equal to 1 if it is regular.

The first part of the paper presents a graph interpretation of the regularization method of the consensus protocol—orthogonal projection method. The properties of the projection method are investigated via the pseudoinverse matrix. According to this method, the space of all possible initial opinions is an orthogonal projection, i.e. symmetric idempotent matrix is mapped onto the subspace of the convergence domain of the DeGroot procedure.

In [10] some elements of the graph interpretation of the orthogonal projection method were considered. However, this concerned only the ratio of the weights of outgoing trees sets on the set of vertices of the basic bicomponents (the definition is given in the next section) in the resulting matrix and did not contribute to the justification of the methods application. In [11] a relationship between the method of orthogonal projection and pseudo-inverse according to Moore–Penrose for an auxiliary matrix constructed from the Laplacian matrix was given for a system with separate basic bicomponents (without non-base vertices). In this work, we completely solve the problem posed in [11]. It is shown that the orthogonal projection method is a natural generalization of the process of characteristics matching for a system with repeated zero eigenvalues.

In the second part of the article, we study the eigenprojection of the Laplace matrix of the communication digraph obtained by proportional change of weights of incoming arcs (in general, all vertices) in the original communication digraph. A simple expression for the projection of the modified matrix is derived.

2. NECESSARY TERMS AND AUXILIARY RESULTS

Let $\Gamma = (V, E)$ be a digraph with many vertices V and many arcs E.

Definition 1. A non-empty subset of vertices K of a digraph $\Gamma = (V, E)$ is called a basic bicomponent if all vertices belonging to K are mutually reachable and there are no arcs (i, j), where $j \in K$, $i \in V \setminus K$. The set of vertices of all basic bicomponents will be denoted by $\mathcal{K}$. The set of vertices that do not belong to the basic bicomponents will be denoted by $\bar{\mathcal{K}} = V \setminus \mathcal{K}$, and we will call them non-base ones.

For the digraph in Fig. 1a the sets $\{1, 2\}, \{3, 4, 5\}$ and $\{6, 7\}$ are basic bicomponents.

Consider a MAS with a set of agents $\{1, \ldots, n\}$. Let A be the matrix of links (influences), $A = (a_{ij})$, where a_{ij}—the weight of the influence of the jth agent on ith. We also construct a communication digraph for the system with the set of vertices $V = \{1, \ldots, n\}$, in which each element $a_{ij} > 0$ of the matrix A corresponds to the arc (j, i) with weight a_{ij}.

Definition 2. 1) The weight of the digraph G is equal to the weights product of all its arcs: $\varepsilon(G) = \prod_{(i,j)\in E} a_{ji}$. 2) The weight of the set of digraphs $\mathcal{G} = \{G_i\}$ is equal to the sum of the weights of all digraphs in the given set, i.e. $\varepsilon(\mathcal{G}) = \sum_i \varepsilon(G_i)$.

Let us assume that the communication digraph, in addition to the basic bicomponents, also contains non-basic vertices.

Laplacian matrix of the communication digraph plays key role in the theory of multi-agent systems with information links. It is defined as follows: $L = \Delta(A) - A$, where $\Delta(A)$ is a diagonal matrix with ith diagonal element equal to the sum of the weights of the incoming arcs at the vertex i. If $\mathbf{0}_n = (0, \ldots, 0)^T$ and $\mathbf{1}_n = (1, \ldots, 1)^T$—vectors of order n consisting of zeros and ones

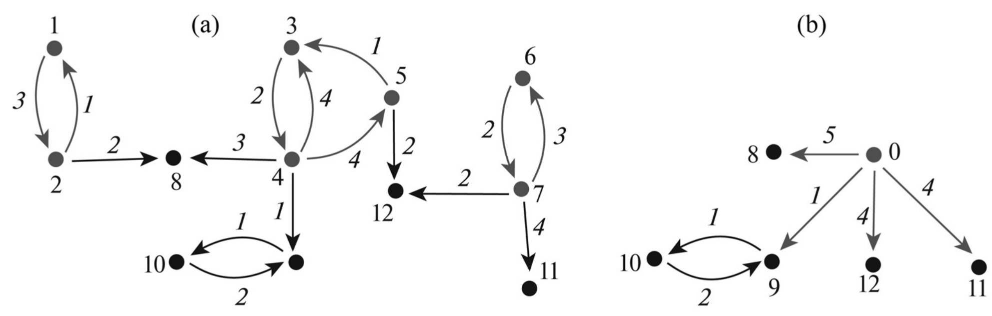

Fig. 1. (a) System with three basic bicomponents $\{1, 2\}$, $\{3, 4, 5\}$, $\{6, 7\}$. (b) "Gluing" basic bicomponents into one vertex 0.

respectively, then $L\mathbf{1}_n = \mathbf{0}_n$, i.e. L is a singular matrix, and the sum of its row elements is equal to zero. If the communication digraph is undirected, then L is symmetric, positive semidefinite.

Definition 3. An eigenprojection (see, for example, [12]) of a square matrix A is a idempotent matrix $A^{\vdash}$, such that $\mathcal{R}(A^{\vdash}) = \mathcal{N}(A^{\nu})$ and $\mathcal{N}(A^{\vdash}) = \mathcal{R}(A^{\nu})$, where ν is the index of L, i.e. is the smallest number for which $\operatorname{rank}(A^{\nu}) = \operatorname{rank}(A^{\nu+1})$ holds.

We note that the eigenprojection $L^{\vdash}$ for the Laplacian matrix L is a non-negative stochastic matrix. In the general case $L^{\vdash}$ is not symmetric for an arbitrary Laplacian matrix, i.e. such a projection is not always orthogonal for a nonsymmetric matrix.

Remark 1. For any Laplacian matrix L $\operatorname{ind} L = 1$ and

$$LL^{\vdash} = L^{\vdash}L = \mathbf{0}_{n\times n}. \tag{1}$$

For any rectangular matrix $A \in \mathbb{R}^{m\times n}$ exists a unique matrix $A^{+} \in \mathbb{R}^{n\times m}$ for which the following four conditions hold: 1) $A^{+}AA^{+} = A^{+}$; 2) $AA^{+}A = A$; 3) $(AA^{+})^{*} = AA^{+}$; 4) $(A^{+}A)^{*} = A^{+}A$. The matrix A^{+} is called *Moore–Penrose pseudoinverse matrix*.

Definition 4. Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are of the same type if the zero elements of these matrices are in the same positions, i.e. $a_{ij} = 0$ if and only if $b_{ij} = 0$.

In matrix notation, we follow the book [13]. For A, we denote by A_{ij} the submatrix obtained by deleting the ith row and jth column of A. Also, for a submatrix formed by rows with numbers from the set $\alpha \subseteq \{1,\dots,n\}$ and columns with numbers from the set $\beta \subseteq \{1,\dots,n\}$, we take notation $A\binom{\alpha}{\beta}$.[1]

Theorem 1. 1) *The eigenprojection $L^{\vdash}$ of the Laplacian matrix L coincides with the normalized matrix of maximum out-forests $Q = (q_{ij})$ of weighted digraph Γ:*

$$l^{\vdash}_{ij} = q_{ij} = \frac{\varepsilon(\mathcal{F}^{j\to i})}{\varepsilon(\mathcal{F})}, \quad i, j = 1,\dots,n,$$

where $\varepsilon(\mathcal{F})$ is the weight of the set of all spanning maximum out-forests of the digraph Γ, $\varepsilon(\mathcal{F}^{j\to i})$ is the weight of the set of those spanning maximum out-forests, where the vertex j is the root of one of the outgoing trees, and i reachable from j.

2) *If i and j belong to the same basic bicomponent, then the corresponding columns of the eigenprojection are proportional.*

Theorem 2 (matrix tree theorem). *The cofactor of any element of the ith row of the Laplacian matrix is equal to the total weight of the spanning trees outgoing from the ith vertex.*

If any column in the Laplacian matrix of a digraph that is replaced by a column of ones, then the determinant of the resulting matrix will be equal to the weight of the set of all spanning outgoing trees.

3. INTERPRETATION OF THE ORTHOGONAL PROJECTION METHOD USING A PSEUDOINVERSE MATRIX

We consider the basic differential model

$$\dot{x}(t) = -Lx(t), \tag{2}$$

where $x_i(t)$ is the characteristic of the ith agent.

[1] When listing row or column numbers a comma is not put between the indices. In some papers, $A\binom{\alpha}{\beta}$ denotes the minor of a submatrix.

Note that the protocol (2) has been studied by many authors (see, for example, [3–5]). It is known that if 0 is a simple eigenvalue of the Laplacian matrix L, then for any vector of initial characteristics $x(0)$ the asymptotic consensus exists and is equal to the limit [6]

$$\lim_{t\to\infty} x(t) = L^{\vdash} x(0).$$

Otherwise for an arbitrary vector of initial values the consensus may not be reached. Further the question arises: how to change the protocol to obtain consensus for any vector of initial values? Such a regularization problem arises not only in multi-agent systems, but also in clustering problems on a disconnected digraph. In this case, after some changes in the original stochastic matrix, the stationary distribution vector is used to weight the clusters in the spectral clustering problem is uniquely determined up to a factor.

A few papers on multi-agent systems with a disconnected communication digraph include [7, 10]. In [7] were explored several latent consensus protocols. These protocols are based on the addition of additional arcs that lead to consensus for any vector of initial characteristics of agents. These methods are similar to those used in PageRank to rank pages on the Internet. For example, in the background connected method, a complete graph with small weights is added to the digraph. Such a protocol has the following representation:

$$\dot{x}(t) = -(L + \delta D)x(t), \tag{3}$$

where $\delta > 0, \quad D = I - \mathbf{1}v^T, \quad v_i > 0, \quad \sum_{i=1}^n v_i = 1.$

In [7] in particular, it is proved that if $x(t)$ is the solution of the system (3), then

$$\lim_{\delta\to+0}\lim_{t\to\infty} x(t) = \mathbf{1}v^T L^{\vdash} x(0).$$

If $v = \frac{1}{n}\mathbf{1}$, then

$$\lim_{\delta\to+0}\lim_{t\to\infty} x(t) = EL^{\vdash} x(0),$$

where E is the matrix with elements n^{-1}.

Using the regularization by principles similar to PageRank leads to the averaging of the rows of its eigenprojection.

Another regularization method is the orthogonal projection method, was proposed in [10]. It can also be applied for the DeGroot $x_k = Px_{k-1}$ model and the continuous protocol. According to this method, the space of all possible initial opinions is an orthogonal projection, i.e. symmetric idempotent matrix S is mapped onto the subspace Q_L — the region of convergence of the DeGroot procedure. The image $\mathcal{R}(S)$ of the matrix S matches with the linear span of the vectors consisting of the linearly independent columns of the matrix $I - P$ and the vector of ones. If x_0 is the vector of initial opinions, and x_0' is a transformed vector, then $|x_0' - x_0|$ will be minimal, because matrix S is an orthogonal projection. Some coordinates of the transformed vector may have negative signs, even if the original vector of initial characteristics was positive. However, $P^\infty S$ is not only a stochastic matrix, but also a matrix of 1 rank. Therefore, if the vector of initial values x_0 has only positive coordinates, then the resulting vector $P^\infty S x_0$ will also be positive.

The orthogonal projection S onto the subspace $Q_L = \mathcal{R}(L) \oplus \mathrm{Span}(\mathbf{1})$ is represented as

$$S = UU^+ = U(U^TU)^{-1}U^T, \tag{4}$$

where U is the matrix of full column rank r and is obtained from L by deleting one column corresponding to any vertex from each basic bicomponent of the digraph and adding the column $\mathbf{1}_n$ as first.

3.1. Relationship between the Generalized Inverse Matrix for U and the Orthogonal Projection Method

A graph interpretation of the orthogonal projection method was partially given in [10] using the matrices X and Z (see item 3 of Theorem 3 in [10]). In [11] a connection between the generalized inverse matrix for U and the orthogonal projection method for the class of digraphs without non-basic vertices was given. In this section, we consider a more general case, assuming that the communication digraph, in addition to individual basic bicomponents, also contains vertices that do not belong to the set $\mathcal{K}$. Let us show that the orthogonal projection method is a natural generalization of the consensus protocol.

Let E_{10} be a square matrix of order n. The first column of which consists of ones and all other elements are equal to zero.

Let's first assume that $\operatorname{rank}(L) = n-1$. In this case, there is no need for regularization and for any vector of initial values consensus is reached, and $L^{\vdash} = E_{10}U^{-1}$.

If $\operatorname{rank}(L) < n-1$, then

$$L^{\vdash}S = L^{\vdash}UU^{+} = E_{10}U^{+}, \tag{5}$$

i.e. in both cases the consensus is uniquely determined by the first row of the generalized inverse matrix for U: in the first case U^{-1}, and in the second U^{+}. Thus, the following proposition holds.

Proposition 1.

1) *If* $\operatorname{rank}(L) = n-1$, *then* $L^{\vdash}S = L^{\vdash}I = E_{10}U^{-1}$.

2) *If* $\operatorname{rank}(L) < n-1$, *then* $L^{\vdash}S = E_{10}U^{+}$.

The Proposition 1 postulates that if a consensus is reached in the system, then it is uniquely determined by the normalized weights of the set of outgoing spanning trees of the communication digraph. In turn, these weights, according to the matrix tree Theorem 2, are uniquely determined by the first row of the matrix U^{-1}. By virtue of the Proposition 1, the orthogonal projection method is a natural generalization of matching characteristics and is defined in the general case by the first row of the Moore–Penrose pseudoinverse matrix U. It is known that (see, for example, Appendix A in [14]) elements of a pseudoinverse matrix, as well as for a nonsingular matrix, can be represented using minors of the original matrix as follows:

$$u^{+}_{1i_1} = \frac{\sum\limits_{i_2<\ldots<i_r} \det U\begin{pmatrix} i_2\ldots i_r \\ 2\ldots r\end{pmatrix} \det U\begin{pmatrix} i_1\ldots i_r \\ 1\ldots r\end{pmatrix}}{\sum\limits_{k_1<\ldots<k_r} \left(\det U\begin{pmatrix} k_1\ldots k_r \\ 1\ldots r\end{pmatrix}\right)^2}. \tag{6}$$

Using (6), we further characterize the orthogonal projection method and the elements of the matrix U^{+} using the forest structure of the communication digraph. To do this, we need the following assertions, assuming that the matrix L has form (7).

$$L = \begin{pmatrix} L_1 & 0 & \cdots & 0 & 0 \\ 0 & L_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & L_v & 0 \\ * & * & \cdots & * & L_R \end{pmatrix}, \tag{7}$$

where v is the number of basic bicomponents in the corresponding digraph, $*$ — blocks, which in the general case are nonzero, L_R is a submatrix of L, rows and columns which correspond to all non-basic vertices. It is important to note that $\det L_R$ is equal to the weight of the set of outgoing spanning trees of the Γ_ξ digraph obtained from Γ by "gluing" of all vertices from $\mathcal{K}$ into one vertex ξ (see [15]). In Fig. 1b there is a digraph by "gluing" of base vertices of the digraph from Fig. 1a.

Proposition 2. 1) *The minors* $\det U\binom{i_1\dots i_r}{1\dots r}$ *and* $\det U\binom{i_2\dots i_r}{2\dots r}$ *are equal to zero if they are obtained by deleting at least one row corresponding to the vertex from* $\bar{\mathcal{K}}$.

2) *The minor* $\det U\binom{i_2\dots i_r}{2\dots r}$ *is equal to zero if the set* $\{i_2,\dots,i_r\}$ *contains all vertices of one basic bicomponent.*

3) *The absolute value of the non-zero minor* $\det U\binom{i_2\dots i_r}{2\dots r}$ *is equal to the product of* $\det L_R$ *and the weight of the set of all outgoing forests on* $\mathcal{K}$ *with vertex roots* $\{1,\dots,n\}\setminus\{i_2,\dots,i_r\}$.

Proposition 3. *Let* $\{j_1,\dots,j_{m_s}\}$*—the set of all vertices of some basic bicomponent s contained in* $\{i_1,\dots,i_r\}$*, and let it contain no vertices of other basic bicomponents. Then:*

1)

$$\left|\det U\begin{pmatrix} i_1\dots i_r \\ 1\dots r\end{pmatrix}\right| = \sum_{p=1}^{m_s}\left|\det U\begin{pmatrix} K_p \\ 2\dots r\end{pmatrix}\right|, \tag{8}$$

where $K_p = (i_1\dots i_r)\setminus j_p$, $p = 1,\dots,m_s$;

2) $\det U\binom{j_p\,K_p}{1\dots r} \neq 0$ *and* $\det U\binom{K_p}{2\dots r} \neq 0$, $p = 1,\dots,m_s$, *have the same sign.*

Note that $(i_1\dots i_r)\setminus j_p$ means that element j_p has been removed from the ordered set $(i_1\dots i_r)$. And the view $j_p\,K_p$ indicates that the element j_p has been added to the left of the ordered set K_p.

Using the Propositions 2 and 3, one can prove the following theorem, which was first proved in [10] for a system without non-basic agents.

Theorem 3. *For a system with an arbitrary communication digraph, the sum of the elements of the matrix first row of* U^+ *is equal to 1:*

$$\sum_{i=1}^{n} u^+_{1i} = 1.$$

So, the method of orthogonal projection to MAS with any communication digraph and vector of initial values $x(0)$ leads to consensus, and the consensus is determined by the product

$$(u^+_{11},\dots,u^+_{1n})(x_1(0),\dots,x_n(0))^T.$$

In particular, if the communication digraph contains a spanning tree, then in this case the matrix U will be square, nonsingular, and, according to (6), we have

$$u^+_{1i} = u^{-1}_{1i} = \frac{\left|\det U\binom{(1\dots n)\setminus i}{2\dots n}\right|\det U}{(\det U)^2} = \frac{\left|\det U\binom{(1\dots n)\setminus i}{2\dots n}\right|}{\det U} = l^{\vdash}_{1i}.$$

The last equality follows from Theorem 1, according to which $|\det U\binom{(1\dots n)\setminus i}{2\dots n}|$ coincides with the algebraic complement of any element of the ith row of the Laplacian matrix of the communication digraph.

Proposition 4. *If* i_1 *and* j_1 *belong to the same base component, then* $\frac{u^+_{1i_1}}{u^+_{1j_1}} = \frac{l^{\vdash}_{1i_1}}{l^{\vdash}_{1j_1}}$.

Using the Proposition 3, expression of (6) can be represented as

$$\Sigma = \sum_{i_1=1}^{n}\sum_{i_2<\dots<i_r}\det U\begin{pmatrix} i_2\dots i_r \\ 2\dots r\end{pmatrix}\det U\begin{pmatrix} i_1\dots i_r \\ 1\dots r\end{pmatrix} = \sum_{i_1<\dots<i_r}\left(\det U\begin{pmatrix} i_1\dots i_r \\ 1\dots r\end{pmatrix}\right)^2 = \sum_{s=1}^{v}\sum_{i=1}^{q_s}\varrho^2_{si},$$

where $q_s = (m_1 m_2\cdots m_v)/m_s$, $s = 1,\dots,v$.

For each basic bicomponent s with the vertex set N_s, the number ϱ_{si} is equal to the product of the weight of the set of all trees in the bicomponent s and the weight of the maximum out-forests with fixed vertices.

Let $P_1 = \{1, \dots, m_1\}$ be the set of vertices of the first basic bicomponent. According to the expression (6),

$$W_1 = \sum_{i=1}^{m_1} u_{1i}^{+} = D^{-1} \sum_{i=1}^{m_1} \sum_{K_i} \det U \begin{pmatrix} K_i \\ 2 \dots r \end{pmatrix} \det U \begin{pmatrix} i\, K_i \\ 1 \dots r \end{pmatrix}$$

$$= D^{-1} \sum_{m_1+1<\dots<i_r} \left(\det U \begin{pmatrix} P_1\, i_{m_1+1} \dots i_r \\ 1 \dots r \end{pmatrix} \right)^2 = D^{-1} \sum_{i=1}^{q_1} \varrho_{1i}^2, \tag{9}$$

where $K_t = (1 \dots m_1 i_{m_1+1} \dots i_r) \setminus t$, $t = 1, \dots, m_1$. We recall that in (9), for each basic bicomponent s, the number ϱ_{si} is equal to the product of the weight of the set of all trees of the basic bicomponent s and the product of the weights of all trees with fixed roots from the remaining basic bicomponents, and the block definer L_R. Using (9), we can determine the ratio of the sums of weights in different bicomponents.

If a digraph consists of one basic bicomponent with a set of vertices $m_1 = n$, then from (9), by virtue of $\sum_{i=1}^{n} \det U \binom{N\setminus i}{2\dots n} = \det U \binom{1\dots n}{1\dots n} = \det U$ follows

$$W_1 = D^{-1} \sum_{i=1}^{n} \det U \begin{pmatrix} N \setminus i \\ 2 \dots n \end{pmatrix} \det U = \frac{\det U \det U}{\left(\det U\right)^2} = 1.$$

4. AN EXPLICIT EXPRESSION FOR THE EIGENPROJECTION OF THE PRODUCT OF A POSITIVE DIAGONAL MATRIX AND THE LAPLACIAN

As noted in the introduction, the asymptotic behavior of many MAS models is determined by the properties of the eigenprojection of the Laplacian matrix of the communication digraph. The kth column characterizes the “importance” of the kth agent in the final consensus. For a strongly connected communication digraph, the larger the value of $\ell_{1k}^{\vdash}$, the stronger the influence of the kth agent on the final value. We consider the following problem: if the influences of other agents on the kth agent change proportionally, how will the eigenprojection change and can the eigenprojection of the resulting matrix be expressed in terms of the eigenprojection of the original matrix? Moreover, if the influences on the kth agent change τ_k times, then the Laplacian matrix M of the digraph with new weights will be equal to TL, where L is the matrix of the digraph before changing its weights, $T = \operatorname{diag}(\tau_1, \dots, \tau_n)$.

In this section, we prove that the eigenprojection $M^{\vdash}$ can be represented as $L^{\vdash} D$, where D is diagonal matrix. According to the Theorem 1, the matrices $L^{\vdash}$ and $M^{\vdash}$ are of the same type.

If $\operatorname{rank} L^{\vdash} = 1$, then there always exists a positive diagonal matrix D such that $M^{\vdash} = L^{\vdash} D$. However, if $\operatorname{rank} L^{\vdash} > 1$, then the existence of a diagonal matrix is not obvious. Moreover, in the first case, if $M^{\vdash}$ is not known, then finding D is not a trivial problem.

Theorem 4. *If $M = TL$, where $T = \operatorname{diag}(\tau_1, \dots, \tau_n)$ is a positive diagonal matrix and L is an arbitrary Laplacian matrix, then there exists a nonnegative diagonal matrix D for which:*

$$M^{\vdash} = L^{\vdash} D. \tag{10}$$

Corollary 1 (from the theorem 4). *If the communication digraph is strongly connected and $T = \operatorname{diag}(l_{11}^{\vdash}, \dots, l_{1n}^{\vdash})$, then:*

1) *for the eigenprojection of the matrix $M = TL$ we have $M^{\vdash} = \frac{1}{n}\mathbf{1}\mathbf{1}^{\mathbf{T}}$;*

2) *matrix TL is balanced.*

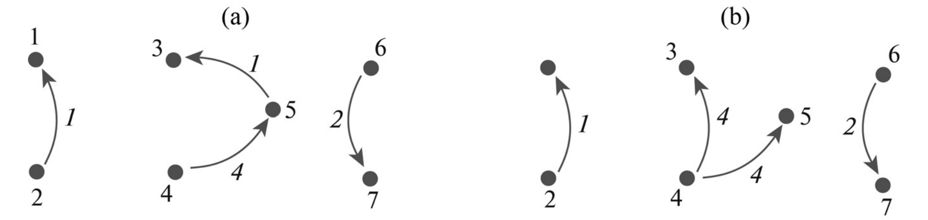

Fig. 2. Two forests (a) and (b), consisting of three outgoing trees with roots 2, 4 and 6.

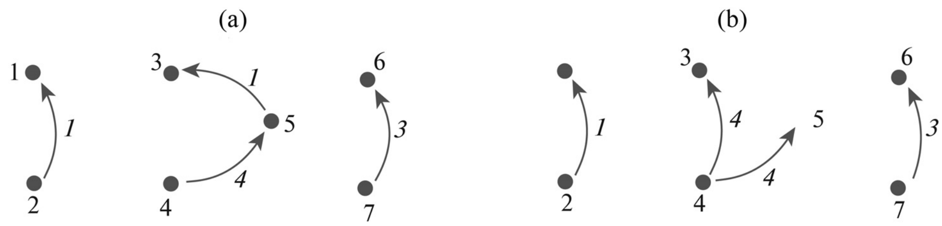

Fig. 3. Two forests (a) and (b), consisting of three outgoing trees with roots 2, 4, and 7.

Example 1. Consider a multi-agent system with a communication digraph shown in Fig. 1. We also consider the matrices L_R^0 and U, the first of which corresponds to a digraph with basic bicomponents "glued" to the vertex **0**:

$$U = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & -3 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 3 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}; \quad L_R^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ -1 & 0 & 3 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ -4 & 0 & 0 & 0 & 4 & 0 \\ -4 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Let us find out what quantities make up the element u_{16}^+ of the matrix U^+. The sum in the numerator from the expression (6) for u_{16}^+ contains six nonzero terms, each of which is the product of two minors. One of the terms is

$$\det U \begin{pmatrix} 1\,3\,5\,7\,\ldots\,12 \\ 2\,3\,\ldots\,10 \end{pmatrix} \det U \begin{pmatrix} 6\,1\,3\,5\,7\,8\,\ldots\,12 \\ 1\,2\,\ldots\,10 \end{pmatrix}.$$

According to item 3 of the Proposition 2, the absolute value of the non-zero minor $\det U\binom{1\,3\,5\,7\,8\,\ldots\,12}{2\,3\,\ldots\,10}$ equals the product of the weight of the set of all outgoing forests by $\mathcal{K}$ with roots $\{2,4,6\} = \{1,\ldots,12\} \backslash \{1,3,5,7,8,\ldots,12\}$ on $\det L_R = 80$.

Figures 2a and 2b show both forests on the set of vertices $\mathcal{K} = \{1,\ldots,7\}$ coming from the roots $2,4,6$. The weight of the first forest is 8, the second is 32, i.e. the sum of the weights of these forests is 40. If this number is multiplied by the weight of the tree shown in Fig. 1b, i.e. by 80, we get $\det U\binom{1\,3\,5\,7\,8\,\ldots\,12}{2\,3\,\ldots\,10} = 40 \times 80 = 3200$.

The minor has a similar graph interpretation $\det U\binom{1\,3\,5\,6\,8\ldots 12}{2\,3\ldots 10} = -60\times 80 = -4800$, whose absolute value is equal to the product of the weight of the set of all outgoing forests by $\mathcal{K}$ with roots $\{1\,\ldots\,12\}\setminus\{1\,3\,5\,6\,8\,\ldots\,12\} = \{2\,4\,7\}$ on $\det L_R = 80$.

Figure 3 shows both forests coming from the roots $\{2,4,7\}$: the weight of the first forest is 12, and the second is 48, i.e. the sum of the weights of the two forests is 60. So, $\det U\binom{1\,3\,5\,7\,8\ldots 12}{2\,3\ldots 10} = -60\times 80 = -4800$.

On the other hand, if we apply the Proposition 3 to the basic bicomponent $\{6,7\}$, then we get:

$$\left|\det U\begin{pmatrix}6\,1\,3\,5\,7\,8\,\ldots\,12\\ 1\,2\,\ldots\,10\end{pmatrix}\right| = \left|\det U\begin{pmatrix}7\,1\,3\,5\,6\,8\,\ldots\,12\\ 1\,2\,\ldots\,10\end{pmatrix}\right|$$

$$= \left|\det U\begin{pmatrix}1\,3\,5\,7\,8\,\ldots\,12\\ 2\,3\,\ldots\,10\end{pmatrix}\right| + \left|\det U\begin{pmatrix}1\,3\,5\,6\,8\,\ldots\,12\\ 2\,3\,\ldots\,10\end{pmatrix}\right| = 3200 + 4800 = 8000.$$

5. CONCLUSION

In this paper, we obtain a new representation of the orthogonal projection method using the U^+ matrix, previously presented only for a narrow class of communication digraphs. It is shown that the projection method is a natural generalization of the consensus protocol and is represented by elements of the pseudoinverse matrix U^+. It is established that the eigenprojection of the matrix TL, where T is a positive diagonal matrix, can be represented as $(TL)^{\vdash} = L^{\vdash}D$, where D is a positive diagonal matrix. It is proved that if the digraph is strongly connected, then all the diagonal elements of the diagonal matrix D are equal. From the main results, as a corollary, a simple method for regularizing an arbitrary digraph is obtained.

APPENDIX

Proof of Proposition 2. 1) The matrix U has the following form:

$$U = \begin{pmatrix} 1 & l_{12} & \cdots & l_{1k'} & \mathbf{0}_{1,n-k}\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 1 & l_{k2} & \cdots & l_{kk'} & \mathbf{0}_{1,n-k}\\ \mathbf{1}_{n-k,1} & * & * & * & L_R \end{pmatrix} = \begin{pmatrix} 1 & U_1 & & \\ 1 & \ddots & & \\ \vdots & & U_v & \\ \mathbf{1}_{n-k,1} & * & * & L_R \end{pmatrix},$$

where $k' = k - v + 1$, $k = \sum_{i=1}^{v} m_i$ (v is a number of basic bicomponents), and the matrix L_R is square and nonsingular. Assume that the mentioned minor was obtained by deleting rows among which there is at least one row with a number from the set $\{k+1,\ldots,n\}$. Then the submatrix $U\binom{i_1\ldots i_r}{1\ldots r}$ will have a block-triangular form, and its right lower square block $L_{R'}$ contains a zero row. Therefore $\det L_{R'} = 0$ and $\det U\binom{i_1\ldots i_r}{1\ldots r} = 0$.

2) Let the set $\{i_2,\ldots,i_r\}$ contain all vertices $\{j_1,\ldots,j_{m_s}\}$ of one sth basic bicomponent. Then the submatrix with rows $\{j_1,\ldots,j_{m_s}\}$ contains $m_s - 1$ nonzero columns. The minor $\det U\binom{i_2\ldots i_r}{2\ldots r}$ consists of terms, each of which is the product of $r-1$ submatrix elements taken from different rows and columns. Therefore, each term contains a zero factor and the minor $\det U\binom{i_2\ldots i_r}{2\ldots r}$ is equal to zero.

3) If the minor $\det U\binom{i_2\ldots i_r}{2\ldots r}$ is different from zero, then according to item 2, the set $\{i_2,\ldots,i_r\}$ consists of $m_s - 1$ rows ($s = 1,\ldots,v$) from each basic bicomponent and rows corresponding to

vertices from $\bar{\mathcal{K}}$. Obviously, the minor is equal to the determinant of the block-diagonal matrix, i.e.

$$\det U\begin{pmatrix} i_2 \dots i_r \\ 2 \dots r \end{pmatrix} = \det \begin{pmatrix} U_1' & & & \\ & \ddots & & 0 \\ & & U_v' & \\ & * & & L_R \end{pmatrix},$$

where the matrix U_i' is obtained from U_i by deleting one i_kth row. According to the matrix tree theorem, the determinant of the matrix U_i' is equal to the minor of any element of the i_kth row of U_i and its absolute value is equal to the sum of the weights of all outgoing trees from the i_kth vertex of the ith basic bicomponent. It is true for any block U_t'.

Thus, the absolute value of the non-zero minor $\det U\binom{i_2 \dots i_r}{2 \dots r}$ is equal to the product of $\det L_R$ and the weight of the set of all outgoing forests on $\mathcal{K}$ with roots $\{1, \dots, n\} \backslash \{i_2, \dots, i_r\}$.

Proof of Proposition 3. 1) Without loss of generality, we assume that the vertices are numbered as $j_p = p, p = 1, \dots, m_s$. Let $\{i_1, \dots, i_r\}$ be a subset $\{i_1, \dots, i_{r'}\}$, where $r' = r - |\bar{\mathcal{K}}|$, is a subset of the base vertex set. Consider the determinant $\det U\binom{1 \dots m_s \dots i_r}{1 \dots r}$ and represent it in block form as

$$\det \begin{pmatrix} 1 & l_{12} & \dots & l_{1m_s} & 0_{1,r'-m_s} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & l_{m_s 2} & \cdots & l_{m_s m_s} & 0_{1,r'-m_s} & 0 \\ \mathbf{1}_{r'-p,1} & 0_{r'-p,1} & \cdots & 0_{r'-p,1} & Q_{r'-m_s} & 0 \\ * & * & * & * & * & L_R \end{pmatrix} = \det \begin{pmatrix} Q_{m_s} & \mathbf{0} & \mathbf{0} \\ * & Q_{r'-m_s} & \mathbf{0} \\ * & * & L_R \end{pmatrix}.$$

According to the matrix tree Theorem 2, the algebraic complement of the first element of any kth row of the block Q_{m_s} is equal to the sum of the weights of the trees outgoing from the vertex k. Therefore, the determinant of the matrix Q_{m_s} is equal to the sum of the weights of all outgoing trees of the basic bicomponent with the vertex set $\{1, \dots, m_s\}$ and $\left|\det U\binom{1 \dots m_s \dots i_r}{1 \dots r}\right|$ the product of the sum of weights of all outgoing trees of the basic bicomponent with vertices $\{1, \dots, m_s\}$ and $|\det Q_{r'-m_s}|$, which is equal to the sum of the weights of the maximum outgoing forests on $\mathcal{K} \setminus \{1 \dots m_s\}$ with vertices: $\mathcal{K} \backslash \{i_1, \dots, i_{r'}\}$, and $\det L_R$. Thus, the equality (8) is satisfied.

2) We show that the signs of minors $\det U\binom{j_p\, K_p}{1 \dots r} \neq 0$ and $\det U\binom{K_p}{2 \dots r} \neq 0$, $p = 1, \dots, m_s$, is same as.

Note that $\det U\binom{i_1 \dots i_r}{1 \dots r} = \sum_{k=1}^{m_s} (-1)^{1+k+l} \det U_{k1}^s \xi = (-1)^l \zeta \xi$, where l is a number of rows in $U\binom{i_1 \dots i_r}{1 \dots r}$ up to sth basic bicomponent, $U^s = (\mathbf{1}_{m_s} U_s)$, $\zeta > 0$ is a weight of the set of all outgoing trees in sth basic bicomponent, ξ is a product of determinants of other blocks. Since $U\binom{j_p\, K_p}{1 \dots r}$ differs from $U\binom{i_1 \dots i_r}{1 \dots r}$ by a permutation of one row, their determinants can only differ in sign, and we have

$$\det U\begin{pmatrix} j_p\, K_p \\ 1 \dots r \end{pmatrix} = \det U\begin{pmatrix} i_1 \dots i_r \\ 1 \dots r \end{pmatrix}(-1)^{l+p-1} = (-1)^{2l+p-1}\zeta\xi,$$

where p is the row number in the sth block.

So, the sign of $\det U\binom{j_p\, K_p}{1 \dots r}$ is equal to $(-1)^{p-1}\xi$. It can be easily established that $\det U\binom{K_p}{2 \dots r}$ has the sign $(-1)^{p+1}\xi$.

Proof of Theorem 3. Due to (6), the sum of the elements of the first row U^+ can be written as

$$\sum_{i_1 \in N} u_{1i_1}^+ = \frac{\Sigma_{\mathcal{K}} + \Sigma_{\bar{\mathcal{K}}}}{D}, \tag{A.1}$$

where $\Sigma_{\mathcal{K}}$ corresponds to elements from $\mathcal{K}$, $\Sigma_{\bar{\mathcal{K}}}$ to vertices from $\bar{\mathcal{K}} = N \setminus \mathcal{K}$ and

$$D = \sum_{i_1<\ldots<i_r} \left(\det U \begin{pmatrix} i_1 \ldots i_r \\ 1 \ldots r \end{pmatrix}\right)^2.$$

According to item 1 of the Proposition 2, if the set $\{i_1, \ldots, i_r\}$ does not contain all vertices from $\bar{\mathcal{K}}$, then the corresponding term $\det U\binom{i_1\ldots i_r}{1\ldots r}$ in D representation is equal to zero. Also, by virtue of item 1 of the Proposition 2, we have

$$\Sigma_{\bar{\mathcal{K}}} = \sum_{i_1\in\bar{\mathcal{K}}} \sum_{i_2<\ldots<i_r} \det U \begin{pmatrix} i_2 \ldots i_r \\ 2 \ldots r \end{pmatrix} \det U \begin{pmatrix} i_1 \ldots i_r \\ 1 \ldots r \end{pmatrix} = 0.$$

Note that $\Sigma_{\mathcal{K}}$ and D include the factor $\det L_R$. The proof that $\frac{\Sigma_{\mathcal{K}}}{D}$ is equal to 1 is given in [11].

Proof of Proposition 4. Let us prove the assertion using the representation $u^+_{1k_1}$. Indeed, all corresponding factors $\det U\binom{i_1\ldots i_r}{1\ldots r}$, $\det U\binom{j_1\ldots j_r}{1\ldots r}$ in the expression (6) for $u^+_{1i_1}$ and $u^+_{1j_1}$ differ only in sign, while $\det U\binom{i_2\ldots i_r}{2\ldots r}$ and $\det U\binom{j_2\ldots j_r}{2\ldots r}$ are weights of outgoing trees from i_1 and j_1.

Proof of Theorem 4. Let us prove the theorem constructively, i.e. we construct a diagonal matrix D. Without loss of generality, we assume that the Laplacian matrix L has a block-triangular form, and for L we also construct an auxiliary matrix B^L (A.2), which is obtained from L by replacing the first column in each block L_s corresponding to the basic bicomponent s per column of ones:

$$B^L = \begin{pmatrix} B^L_1 & 0 & \cdots & 0 & 0 \\ 0 & B^L_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & B^L_v & 0 \\ * & * & \cdots & * & L_R \end{pmatrix}, \tag{A.2}$$

the diagonal blocks $B^L_s, s = 1, \ldots, v$, for the matrix B^L are defined as

$$B^L_s = \begin{pmatrix} 1 & l^s_{12} & \ldots & l^s_{1m_s} \\ 1 & l^s_{22} & \ldots & l^s_{2m_s} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & l^s_{m_s2} & \ldots & l^s_{m_sm_s} \end{pmatrix}. \tag{A.3}$$

Similarly to B^L and B^L_s, we define matrices B^M and B^M_s for M.

Note that the ith row M is obtained by multiplying the analogous row L by τ_i. Let us define an eigenprojection $M^\vdash$ according to item 1 of the Theorem 1:

$$m^\vdash_{ij} = \frac{\varepsilon(\mathcal{F}^{j\to i})}{\varepsilon(\mathcal{F})}. \tag{A.4}$$

According to the Theorem 2, for each sth basic bicomponent, the sum of the weights of all outgoing trees (on the set of all vertices from the given basic bicomponent) is equal to $\det(B_s)$. Therefore $\varepsilon(\mathcal{F}) = \det(B^M)$.

Let the vertex i be not reachable from j in any maximal outgoing forest. Since the graphs corresponding to the Laplacian matrices L and $M = TL$ have the same structure, then $m^\vdash_{ij} = l^\vdash_{ij} = 0$.

We consider the case when i is reachable from j in at least one maximal out-forest and j is a vertex from the sth basic bicomponent. Denote by $\{\Gamma_k(V, E_k)\}$ the set of all spanning subgraphs, which is obtained from the set of all maximal outgoing spanning digraphs in which i is reachable from the root vertex j, with the addition of all missing arcs from basic bicomponents. Let B^{L^k} and B^{M^k} be the corresponding matrices of the resulting digraphs constructed by analogy with B^L and B^M. The diagonal blocks B^{L^k} and B^{M^k}, which correspond to the basic bicomponents, match with the similar blocks of the matrices B^L and B^M, respectively. Obviously, the number $\varepsilon(\mathcal{F}^{j\to i})$ for the digraph corresponding to the matrix M is equal to the sum of the algebraic complements of the elements (j, i') of the B^{M^k} matrices, i.e. $\varepsilon(\mathcal{F}^{j\to i}) = \sum_k B^{M^k}_{ji'}$, where $i' \in \{1, \dots, n\}$ is the column number B^{M^k}, which corresponds to the number of a column of ones in the submatrix corresponding to the basic bicomponent s.

Let $j' \in \{1, \dots, m_s\}$ be the row number of the block with the number s that matches the row j.

Unlike i', the number j' points to the row of the block s. Then due to $B^M_q = B^{M_k}_q$ for all q and k we get

$$m^{\vdash}_{ij} = \frac{\varepsilon(\mathcal{F}^{j\to i})}{\varepsilon(\mathcal{F})} = \frac{\sum\limits_k \det B^{M^k}_{ji'}}{\det B^M} = \frac{\prod\limits_{q=1,q\neq s}^{v} \det B^M_q \left|\det B^M_s \begin{pmatrix} (1\dots m_s)\setminus j' \\ 2\dots m_s \end{pmatrix}\right| \sum\limits_k \det M^k_R}{\det M_R \prod\limits_{q=1}^{v} \det B^M_q}$$

$$= \frac{\left|\det B^M_s \binom{(1\dots m_s)\setminus j'}{2\dots m_s}\right| \sum\limits_k \det M^k_R}{\det M_R \det B^M_s}.$$

Note that L^k_R and M^k_R are blocks corresponding to non-basic vertices in the matrices B^{L^k} and B^{M^k} respectively. Next, we represent the resulting expression through the matrix L:

$$m^{\vdash}_{ij} = \frac{\prod\limits_{i=1,i\neq j'}^{m_s} \tau_i \left|\det L^M_s \begin{pmatrix} (1\dots m_s)\setminus j' \\ 2\dots m_s \end{pmatrix}\right| \sum\limits_k \det L^k_R}{\prod\limits_{i=1}^{m_s} \tau_i \sum\limits_{p=1}^{m_s} \frac{1}{\tau_p} \det B^L_s \begin{pmatrix} (1\dots .m_s)\setminus p \\ 2\dots m_s \end{pmatrix} \det L_R} = \frac{\left|\det L^M_s \binom{(1\dots m_s)\setminus j'}{2\dots m_s}\right| \sum\limits_k \det L^k_R}{\tau_j \sum\limits_{p=1}^{m_s} \frac{1}{\tau_p} \det B^L_s \begin{pmatrix} (1\dots m_s)\setminus p \\ 2\dots m_s \end{pmatrix} \det L_R}.$$

Multiply the denominator and numerator by $\prod_{i=1}^{v} \det(B^L_i)$. According to the matrix tree theorem, this number is nonzero and is equal to the weight of the set of all outgoing spanning forests in a digraph that consists of only basic bicomponents. Then:

$$m^{\vdash}_{ij} = \frac{\left|\det L^M_s \binom{(1\dots m_s)\setminus j'}{2\dots m_s}\right| \sum\limits_k \det L^k_R \quad \det B^L_s \prod\limits_{q\neq s} \det B^L_q}{\tau_j \sum\limits_{p=1}^{m_s} \frac{1}{\tau_p} \det B^L_s \begin{pmatrix} (1\dots .m_s)\setminus p \\ 2\dots m_s \end{pmatrix} \det L_R \prod\limits_{q=1}^{v} \det B^L_q}.$$

Note that in the last fraction

$$l^{\vdash}_{ij} = \frac{\left|\det L^M_s \binom{(1\dots m_s)\setminus j'}{2\dots m_s}\right| \sum\limits_k \det L^k_R \prod\limits_{q\neq s} \det B^L_q}{\det L_R \prod\limits_{q=1}^{v} \det B^L_q}.$$

Then

$$m_{ij}^{\vdash} = l_{ij}^{\vdash} \frac{\det B_s^L}{\tau_j \det C_s}, \tag{A.5}$$

where the matrices C_s, $s = 1, \ldots, v$, are defined as follows:

$$C_s = \begin{pmatrix} \frac{1}{\tau_1^s} & l_{12}^s & \ldots & l_{1m_s}^s \\ \frac{1}{\tau_2^s} & l_{22}^s & \ldots & l_{2m_s}^s \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\tau_{m_s}^s} & l_{m_s 2}^s & \ldots & l_{m_s m_s}^s \end{pmatrix}. \tag{A.6}$$

We construct two diagonal matrices $F = \operatorname{diag}(f_1, \ldots, f_n)$ and $H = \operatorname{diag}(h_1, \ldots, h_n)$ as follows: $f_t = \det B_s^L$ and $h_t = \det C_s$ if the vertex t belongs to the sth basic bicomponent. For all other diagonal elements of the matrices F and H we set $f_t = h_t = 1$.

Then

$$M^{\vdash} = L^{\vdash} T^{-1} F H^{-1} = L^{\vdash} D. \tag{A.7}$$

Note that in (A.5) no requirements are imposed on the vertex i. In particular, it may belong to some basic bicomponent.

REFERENCES

1. Olfati-Saber, R., Fax, J.A., and Murray, R.M., Consensus and Cooperation in Networked Multi-Agent Systems, *Proceedings of the IEEE*, 2007, vol. 95, no. 1, pp. 215–233.
2. Jadbabaie, A., Lin, J., and Morse, A.S., Coordination of Groups of Mobile Autonomous Agents Using Nearest Neighbor Rules, *IEEE Transactions on Automatic Control*, 2003, vol. 48, no. 6, pp. 988–1001.
3. Olfati-Saber, R.M. and Murray, R.M., Consensus Problems in Networks of Agents with Switching Topology and Time-Delays, *IEEE Trans. Automat. Control*, 2004, vol. 49, no. 9, pp. 1520–1533.
4. Ren, W., Beard, R.W., and Atkins, E.M., Information Consensus in Multivehicle Cooperative Control, *IEEE Control Systems Magazine*, 2007, vol. 27, no. 2, pp. 71–82.
5. Mesbahi, M. and Egerstedt, M., Graph Theoretic Methods in Multiagent Networks, in *Graph Theoretic Methods in Multiagent Networks*, Princeton University Press, 2010.
6. Chebotarev, P. and Agaev, R., The Forest Consensus Theorem, *IEEE Trans. Automat. Control*, 2014, vol. 59, no. 9, pp. 2475–2479.
7. Agaev, R. and Chebotarev, P., Models of Latent Consensus, *Autom. Remote Control*, 2017, vol. 78, no. 1, pp 88–99. https://doi.org/10.1134/S0005117917010076
8. Agaev, R.P., On the Role of the Eigenprojection of the Laplacian Matrix for Reaching Consensus in Multiagent Second-Order Systems, *Autom. Remote Control*, 2019, vol. 80, no. 11, pp. 2033–2042.
9. Gantmakher, F., Matrix Theory. Moscow: Nauka, 1967.
10. Agaev, R. and Chebotarev, P., The Projection Method for Reaching Consensus and the Regularized Power Limit of a Stochastic Matrix, *Autom. Remote Control*, 2011, vol. 72, no.12, pp. 2458–2476. https://doi.org/10.1134/S0005117911120034
11. Agaev, R. and Khomutov, D., Graph Interpretation of the Method of Orthogonal Projection for Regularization in Multiagent Systems, *14th International Conference "Management of Large-scale System Development" (MLSD). IEEE*, 2021, pp. 1–4.

12. Rothblum, G., Computation of the Eigenprojection of a Nonnegative Matrix at Its Spectral Radius, in *Stochastic Systems: Modeling, Identification and Optimization, II*, Berlin, Heidelberg: Springer, 1976, pp. 188–201.

13. Horn, R. and Johnson, C., Matrix analysis. M.: Mir, 1989.

14. Ben-Israel, A. and Greville, T.N.E., Generalized Inverses: Theory and Applications, 2nd Ed., Springer, 2003.

15. Fiedler, M. and Sedláček, J.O., W-Basich Orientovanych Grafu, *Časopis pro pěstování matematiky*, 1958, vol. 83, no. 2, pp. 214–225.

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