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CONTROL IN SOCIAL ECONOMIC SYSTEMS

Design of Efficient Investment Portfolios with a Shortfall Probability as a Measure of Risk

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Abstract—The paper presents a constructive description of the set of all efficient (Pareto-optimal) investment portfolios in a new setting, where the risk measure named "shortfall probability" (SP) is understood as the probability of a shortfall of investor's capital below a prescribed level. Under a normality assumption, it is shown that SP has a generalized convexity property, the set efficient portfolios is constructed. Relations between the set of mean-SP and the set of mean-variance efficient portfolios as well as between mean-SP and mean-Value-at-Risk (VaR) sets of efficient portfolios are studied. It turns out that mean-SP efficient set is a proper subset of the mean-variance efficient set; interrelation with the mean-VaR efficient set is more complicated, however, mean-SP efficient set is proved to be a proper subset of mean-VaR efficient set under a sufficiently high confidence level. Besides a normal distribution, elliptic distributions are considered as an alternative for modeling the investor's total return distribution. The obtained results provides the investor with a risk measure, that is more vivid than the variance and Value-at-Risk, and with determination of the corresponding set of effective portfolios.

Keywords: risk analysis, portfolio optimization, value at risk, shortfall probability

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1. INTRODUCTION

After a pioneering work by Markowitz [1], where variance was considered as a measure of risk, another measures of risk, e.g., value at risk (VaR), became a popular extension of this risk management framework. VaR determines the maximum amount that a portfolio value could lose over a given period of time with a given probability as a result of changes in market prices or rates of returns. The concept of VaR is very appealing because it is consistent with the mean-variance paradigm (see, e.g., [2, 3]) and, from the other hand, the regulators such as The Securities and Exchange Commission require registrants to provide quantitative information about market risk with VaR being one of the disclosure alternatives. However, VaR is still criticized (see, e.g., Rockafellar et. al [4] and Szego [5] for this argument) with respect to incapability of distinguishing between "large" and "small" losses lying behind the given threshold. In [6], a class of deviation measures of risk is suggested and a relationship between deviation measures and coherent risk measures [7] is established.

The shortfall probability (SP) or, in other terms, the failure probability is used in engineering applications as a risk measure; see for example Gardoni and Murphy [8, 9]. In [10], it is pointed out that the failure probability has a lack of convexity and smoothness as a function of the design parameters in engineering optimization problems.

Pinar [11], Duffie and Pan [12] assume a joint normal/log-normal distribution of the underlying market parameters for calculating VaR. A dynamic model of investment optimization under VaR constraints and with bankruptcy is studied in [13].

Our work differs from previous results in several respects. The main idea of this paper is the following: despite the lack of convexity and smoothness in engineering optimization problems (see above [10]), we will show that within the framework of a problem of determining efficient portfolios in an asset market under normality assumption, SP has a generalized convexity property and a smooth dependence on a portfolio weight vector. First, we indicate some interesting (in the authors' opinion) properties of the SP risk measure such as an invariance of a currency unit and generalized convexity property, and investigate how it relates to variance and VaR measures. Second, we analytically characterize the mean-SP efficient set, making use of a parametrization by the total return expectation, and compare it to the mean-variance and mean-VaR efficient sets.

The rest of the paper is organized as follows. Section 2 analyzes a problem of designing the set of all mean-SP efficient portfolios in a market without a risk-free assert, and compares the result with the efficient sets of mean-variance and mean-VaR portfolios. Section 3 studies a case of elliptical multivariate distributions instead of a multivariate normally distribution of the return rates of risky assets. Section 4 concludes the paper.
The problem of determination of a "best" portfolio from the mean-SP efficient set is beyond the scope of the paper.

2. EFFECTIVE INVESTMENT PORTFOLIOS IN A MARKET WITH RISKY ASSERTS

Consider a market without a risk-free asset (see, e.g., [11, 14]) in which a stochastic vector of return rates of risky assets during one stage of investment is $R = (R_1, \ldots, R_n)$. Let $a \in R^n$ denote a portfolio weight vector of n assets. The typical budget constraint is $\sum_{i=1}^{n} a_i = 1$. In this setting, it means the self-financing of the investor and his/her opportunity of "short sales", i.e., borrowing some asserts at the current prices with the aim to invest the money into the others. We assume that the rates of return follow a normal distribution with a mean vector $m = (m_1, \ldots, m_n)$ and a covariance matrix C.

Throughout the manuscript, we use the following natural assumptions:

- vectors m and $\mathbf{1} = (1, \ldots, 1) \in R^n$ are linearly independent, i.e., the mean rates of return do not equal the same value,
- the covariance matrix C is positive definite.

At the end of the stage, the realized wealth (or total return) is a random variable given by

$$X_a = \sum_{i=1}^{n} a_i R_i.$$

The objective function the investor wishes to maximize is the mean value of the total return

$$\mu(a) \overset{def}{=} E\,X_a = \sum_{i=1}^{n} a_i m_i,$$

where $m_i = E\ R_i$; and the objective function to be minimized is a shortfall probability (SP)

$$SP[\alpha, X_a] \overset{def}{=} P\{X_a \leqslant \alpha\}$$

where α is a prescribed upper bound for the total return.

The introduced measure of risk $SP[\alpha, X] = P\{X \leqslant \alpha\}$ is not a "coherent" measure of risk according to Artzner et al. [7], since it does not satisfy the homogeneity property, $SP[\alpha, \lambda\rho X] \neq \lambda\rho SP[\alpha, X]$. Also, the translation invariance property is not satisfied, $SP[\alpha, X+c] \neq SP[\alpha, X]-c$. However, SP posses an *invariance of a currency unit* property, i.e., if a capital X is converted into money γX in another currency with a coefficient $\gamma > 0$ then the probability $SP[\gamma\alpha, \gamma X] = P\{\gamma X \leqslant \gamma\alpha\} = SP[\alpha, X]$ does not change.

Now, we define one of the mostly used measures of risk in modern portfolio theory (see, e.g., [14, 15]) and then compare it with the SP measure above.

Definition 1. Value at Risk (VaR) is the rate of return v such that $-v$ is the quantile of distribution function $F(x) = P\{X \leqslant x\}$ of the order $1-\beta$ with a prescribed level of confidence $\beta \in (0.5, 1)$. Roughly saying, $VaR[\beta, X]$ is the root of the equation $F(-v) = 1-\beta$.

Remark 1. Some researchers (Artzner et al. [7], Rockafellar et al. [4]) have pointed out the shortcomings of VaR as a measure of risk. Namely, VaR is not a coherent measure of risk since it fails to satisfy the subadditivity property, except the case of normality of $F(x)$. Also, VaR provides no handle on the extent of the losses that might be suffered beyond the threshold amount indicated by this measure. SP seems to be opposite to VaR in the sense that SP risk measure directly indicates the value of probability of the investor's capital fall, but not the value of maximum amount losses under a given confidence level. This makes SP an alternative to VaR in solving problems of finding efficient portfolios.

Now, we turn to the framework of normal modeling the distribution of the total return. As is shown, e.g., in [14], $VaR[\beta, X_a] = x_\beta^N \sigma(a) - \mu(a)$, where x_β^N is the β-order quantile of the standard normal distribution and the variance is $Var\ X_a = \sigma^2(a) = aCa'$, with a' being the transpose of row vector a. In this case, the risk measure $VaR[\beta, X]$ (see, e.g., [7]) is coherent and, hence, convex: $VaR[\beta, \rho X_1 + (1-\rho)X_2] \leqslant \rho VaR[\beta, X_1] + (1-\rho)VaR[\beta, X_2]$.

Consider

$$SP[\alpha, X_a] = \Phi\left(\frac{\alpha - \mu(a)}{\sigma(a)}\right),$$

where $\Phi(x)$ is the distribution function of the standard normal stochastic value. One can see that the argument of $\Phi(\cdot)$ coincides, up to the sign, with Sharpe's ratio $(\sum_1^n m_i a_i - \alpha)/\sqrt{aCa'}$, where α plays the role of the return rate of a "virtual" risk-free assert. Note that a natural assumption in the model with a risk-free assert is $\alpha < \min_{i=1,\dots,n} m_i$. In this connection, we will show in Proposition 1 that SP has a "generalized" convex property on a definite set of risks (c.f. VaR risk measure).

Definition 2. A function $f(x)$ on a convex set D is called *strongly quasi-convex* [16] if for any $x_1, x_2 \in D$ such that $x_1 \neq x_2$ the inequality $f(\rho x_1 + (1-\rho)x_2) < \max\{f(x_1), f(x_2)\}$ holds for any $\rho \in (0, 1)$.

For example, the function $-\phi(x)$, where $\phi(x)$ denotes the density of standard normal distribution, is strongly quasi-convex on $(-\infty, \infty)$ but not convex; the function $\max\{-\phi(x), -1/(2\sqrt{\pi})\}$ is neither strongly quasi-convex, nor convex.

Define a set of normally distributed random values $\mathbf{D}_\alpha = \{X: \ EX > \alpha\}$, in which the relation $X_1 \neq X_2$ is understood as $P\{X_1 \neq X_2\} > 0$.

Proposition 1. *$SP[\alpha, X]$ is strongly quasi-convex on $\mathbf{D}_\alpha$.*

Proof. Let risks X_1 and X_2 belong to $\mathbf{D}_\alpha$ and $X_1 \neq X_2$. Denote $\mu_\rho = \rho EX_1 + (1-\rho)EX_2$ and $\sigma_\rho^2 = Var(\rho X_1 + (1-\rho)X_2)$. Calculate the derivative

$$\frac{d}{d\rho}\Phi((\alpha-\mu_\rho)/\sigma_\rho) = \frac{\phi((\alpha-\mu_\rho)/\sigma_\rho)}{\sigma_\rho^3}[(EX_2 - EX_1)\sigma^2\rho$$
$$-(\alpha-\mu_\rho)(\rho Var\ X_1 - (1-\rho)Var\ X_2 + (1-2\rho)cov(X_1, X_2)],$$

where $\phi(x)$ denotes the density of standard normal distribution. The term in the square brackets is an increasing function in ρ as its derivative $(\mu_\rho - \alpha)Var(X_1 - X_2) > 0$. The following cases are possible: First, $SP[\alpha, \rho X_1 + (1-\rho)X_2]$ either decreases or increases in ρ on its whole domain $(0, 1)$. Then, evidently, Definition 2 is met. Second, the function in the square brackets changes the sign from minus to plus at some point $\rho_0 \in (0, 1)$. In this case, $SP[\alpha, \rho X_1 + (1-\rho)X_2]$ decreases on the interval lying left to ρ_0 and increases on the interval right to ρ_0. By Definition 2.2, $SP[\alpha, X]$ is strongly quasi-convex. □

Remark 2. Here we compare the investor's preferences induced by VaR, SP, and variance risk measure. Since $VaR[\beta, X_a] = x_\beta^N \sigma(a) - \mu(a)$, inequality $VaR(\beta, X_{a^1}) > VaR(\beta, X_{a^2})$ is equivalent to

$$x_\beta^N \sigma(a^1) - \mu(a^1) > x_\beta^N \sigma(a^2) - \mu(a^2). \tag{1}$$

For the risk measure $SP[\alpha, X_a] = \Phi((\alpha - \mu(a))/\sigma(a))$, the corresponding inequality, $\Phi((\alpha - \mu(a^1))/\sigma(a^1)) > \Phi((\alpha - \mu(a^2))/\sigma(a^2))$, is equivalent to

$$(\alpha - \mu(a^1))\sigma(a^2) > (\alpha - \mu(a^2))\sigma(a^1). \tag{2}$$

One can see that relation (2) is quite different from (1), which means that the investor's preferences with respect to VaR differ from that of SP. Note that the investor's preferences of another measure of risk [17], $V[X_a] = \sigma^2(a)$,

$$\sigma^2(a^1) > \sigma^2(a^2), \tag{3}$$

also differ from (2).

Return to the bi-objective optimization problem of maximizing the mean portfolio value and minimizing SP measure of risk. In the normality framework, it is

$$\begin{cases} \mu(a) \equiv \sum_{i=1}^n a_i m_i \to \max, \\ SP[\alpha, X_a] \equiv \Phi(\frac{\alpha - \mu(a)}{\sigma(a)}) \to \min, \\ \text{s.t. } a \in A = \{a \in R^n : \sum_{i=1}^n a_i = 1\}. \end{cases} \tag{4}$$

Definition 3. A portfolio a^0 is said to be an efficient portfolio in (4) if there is no portfolio a^1 such that $\mu(a^1) \geqslant \mu(a^0)$, $SP[\alpha, X_{a^1}] \leqslant SP[\alpha, X_{a^0}]$, and at least one inequality holds strict.

Now, we investigate the problem of designing a set A^{SP} of all efficient portfolios in (4). The next theorem provides a description of A^{SP} as a set of portfolios parametrized by M, a parameter having sense of a fixed value of the mean of total return, $M = \mu(a)$. Further we need some notation: Let $\langle x, y\rangle$ denote the scalar product $\sum_{i=1}^n x_i y_i$ of two row vectors x and y, $\langle x, y\rangle_C = \langle x, yC^{-1}\rangle$, $||x||_C^2 = \langle x, xC^{-1}\rangle$, and $\triangle = ||\mathbf{1}||_C^2 ||m||_C^2 - \langle \mathbf{1}, m\rangle_C^2$.

Theorem 1. *The set A^{SP} is not empty if and only if*

$$\alpha < \frac{\langle \mathbf{1}, m\rangle_C}{||\mathbf{1}||_C^2}. \tag{5}$$

If (5) is met then

$$A^{SP} = \{a(M) = \frac{1}{\triangle}\left[\mathbf{1}||m||_C^2 - m\langle \mathbf{1}, m\rangle_C + M(m||\mathbf{1}||_C^2 - \mathbf{1}\langle \mathbf{1}, m\rangle_C)\right] C^{-1}\}, \tag{6}$$

where M runs over a semi-infinite interval

$$M \in \left[M^{SP}, \infty\right), \text{ with } M^{SP} = \frac{||m||_C^2 - \alpha\langle \mathbf{1}, m\rangle_C}{\langle \mathbf{1}, m\rangle_C - \alpha||\mathbf{1}||_C^2}. \tag{7}$$

The proof of Theorem 1 is given in Appendix.

Below we compare this theorem with known descriptions of the set A^V of mean-variance efficient portfolios and the set A^{VaR} of mean-VaR efficient portfolios. The next two statements are just variations of results in [17] and [14], the expressions for efficient portfolios are now parametrized by $M = \mu(a)$.

Statement 1. *The set A^V is not empty. It is defined by (6), where M runs over a semi-infinite interval*

$$M \in \left[M^V, \infty\right), \text{ where } M^V = \frac{\langle \mathbf{1}, m\rangle_C}{||\mathbf{1}||_C^2}. \tag{8}$$

Statement 2. *The set A^{VaR} is not empty if and only if*

$$\beta > \Phi\left(\sqrt{D/||\mathbf{1}||_C^2}\right), \quad \text{where } D = ||m||_C^2||\mathbf{1}||_C^2 - \langle \mathbf{1}, m\rangle_C^2 > 0. \tag{9}$$

If (9) is met, A^{VaR} is defined by (6), with M running over a semi-infinite interval

$$M \in [M^{VaR}, \infty), \; M^{VaR} = \frac{\langle \mathbf{1}, m\rangle_C}{||\mathbf{1}||_C^2} + \sqrt{\frac{D}{||\mathbf{1}||_C^2}\left(\frac{(x_\beta^N)^2}{||\mathbf{1}||_C^2(x_\beta^N)^2 - D} - \frac{1}{||\mathbf{1}||_C^2}\right)}. \tag{10}$$

The fact that the mean-SP, mean-variance, and mean-VaR efficient portfolios are defined by the same formula (6) (with, however, the different ranges of M) has the following explanation: If a portfolio a^* is efficient in the sense of any of the three settings above, then a^* necessarily solves the problem

$$\begin{cases} \min \sigma^2(a), \\ \mu(a) = M, \\ a \in A = \{a \in R^n : \; \sum_{i=1}^n a_i = 1\}, \end{cases} \tag{11}$$

where $M = \mu(a^*)$. The necessity evidently follows from the definitions of $VaR[\beta, X_a] = x_\beta^N \sigma(a) - \mu(a)$ and $V[X_a] = \sigma^2(a)$. In the case of $SP[\alpha, X_a] = \Phi((\alpha - \mu(a))/\sigma(a))$, we have $M = \mu(a) \geqslant M^{SP} > \alpha$ since

$$\frac{\langle \mathbf{1}, m\rangle_C}{||\mathbf{1}||_C^2} < \frac{||m||_C^2 - \alpha\langle \mathbf{1}, m\rangle_C}{\langle \mathbf{1}, m\rangle_C - \alpha||\mathbf{1}||_C^2}. \tag{12}$$

Indeed, the denominator $\langle \mathbf{1}, m\rangle_C - \alpha||\mathbf{1}||_C^2 > 0$ by virtue of (5). By Cauchy-Schwartz-Bunyakovskii inequality, we have $||m||_C^2||\mathbf{1}||_C^2 - \alpha\langle \mathbf{1}, m\rangle_C||\mathbf{1}||_C^2 > \langle \mathbf{1}, m\rangle_C^2 - \alpha\langle \mathbf{1}, m\rangle_C||\mathbf{1}||_C^2$. Therefore, the mean-SP efficient portfolio must solve (11). Problem (11) is studied, e.g. in [14], the optimal portfolios $a^* = a(M)$ are given in (6).

Proposition 2. *The set A^{SP} of mean-SP efficient portfolios is a proper subset of the set of mean-variance efficient portfolios, i.e. $A^{SP} \subset A^V$.*

Proof. Comparing the expressions for the left boundaries in (7) and (8), we have

$$\frac{||m||_C^2 - \alpha\langle \mathbf{1}, m\rangle_C}{\langle \mathbf{1}, m\rangle_C - \alpha||\mathbf{1}||_C^2} > \frac{\langle \mathbf{1}, m\rangle_C}{||\mathbf{1}||_C^2}$$

(see (12)). Thus, $M^{SP} > M^V$ and, hence, $A^{SP} \subset A^V$. $\square$

Remark 3. As is known [14], $A^{VaR} \subset A^V$. The relation between the sets A^{SP} and A^{VaR} is not straightforward since the fulfillment of inequality for the left boundaries in (7) and (10),

$$M^{SP} > (<) \; M^{VaR},$$

depends on the values of α and β. However, if β is not close to 1 then A^{SP} may be substantially narrower then A^{VaR} in the sense that $M^{SP} > M^{VaR}$ (see (7), (10) and numerical examples in Section 3). Also, M^{SP} is essentially large then M^V if α is "not small" since, as is easy to verify, M^{SP} is an increasing function of α.

Remark 4. Consider the limiting cases: $\beta \to 1$ and $\alpha \to -\infty$. From [14, p. 1169, Cor. 4] it follows that the set of mean-VaR efficient portfolios converges to the set of mean-variance efficient portfolios when $\beta \to 1$, i.e. the left boundary M^{VaR} in (10) converges to the left boundary in (8), $M^V = \langle \mathbf{1}, m\rangle_C/||\mathbf{1}||_C^2$. Thus, $A^{SP} \subset A^{VaR}$ for a sufficiently large confidence level $\beta < 1$. Note also that

$$M^{SP} = \frac{||m||_C^2 - \alpha\langle \mathbf{1}, m\rangle_C}{\langle \mathbf{1}, m\rangle_C - \alpha||\mathbf{1}||_C^2} \to M^V = \frac{\langle \mathbf{1}, m\rangle_C}{||\mathbf{1}||_C^2} \text{ as } \alpha \to -\infty.$$

So, in this limiting case, the efficient sets A^{SP} and A^V coincide.

Define a mean-SP efficient frontier as a set $\{(\mu(a), SP[\alpha, X_a]),\ a \in A^{SP}\} \subset R^2$ on (mean,SP) space.[1]

The next proposition shows that the function $SP(M) = \Phi((\alpha - M)/\sigma(a(M)))$, which corresponds to the mean-SP efficient frontier on the interval (7), has a more complicate form than the convex functions (see [14, 15]) $V(M) = \sigma^2(a(M))$ and $VaR(M) = x_\beta^N\sigma(a(M)) - M$ of mean-variance and mean-VaR cases (see Figs. 1–3 below).

Proposition 3. *Let* (5) *be met. Then, function* $SP(M)$ *is strongly quasi-convex on* $(-\infty, \infty)$.

Proof. Noting that $\sigma^2(a(M)) = (M^2||\mathbf{1}||_C^2 - 2M\langle \mathbf{1}, m\rangle_C + ||m||_C^2)/\triangle^2$, calculate the derivative

$$\frac{d}{dM}\Phi((\alpha - M)/\sigma(a(M))) = \frac{\phi((\alpha - M)/\sigma(a(M)))}{\triangle^2\sigma^3(a(M))}[-\triangle^2\sigma^2(a(M)) \\ -(\alpha - M)(M||\mathbf{1}||_C^2 - \langle \mathbf{1}, m\rangle_C)],$$

where $\phi(x)$ is the density of standard normal distribution. The term $\psi(M) = M(\langle \mathbf{1}, m\rangle_C - \alpha||\mathbf{1}||_C^2) - ||m||_C^2 + \alpha\langle \mathbf{1}, m\rangle_C$ in the square brackets is an increasing function as $\langle \mathbf{1}, m\rangle_C - \alpha||\mathbf{1}||_C^2 > 0$ due to (5). Thus, the derivative of $SP(M)$ is represented as $\gamma(M)\psi(M)$ with $\gamma(M) > 0$ and the increasing function $\psi(M)$ which changes the sign from minus to plus at the point $M^{SP} = (||m||_C^2 - \alpha\langle \mathbf{1}, m\rangle_C)/(\langle \mathbf{1}, m\rangle_C - \alpha||\mathbf{1}||_C^2)$. Then, $SP(M)$ decreases up to the point $M = M^{SP}$ and increases on the interval right to M^{SP}. By above-given Definition 2, $SP(M)$ is strongly quasi-convex. □

3. EXAMPLES

Now, we illustrate Proposition 3 by a numerical example. Let the number of asserts be $n = 2$, the vector of mean rates of returns be $m = (1.1, 1.2)$, and the covariance matrix of the rates of returns be

$$C = \begin{pmatrix} 0.4 & 0.2 \\ 0.2 & 0.5 \end{pmatrix}.$$

Let the confidence level $\beta = 0.9$ and the value of upper bound for the total return $\alpha = 0.8$.

The efficient sets A^{SP}, A^V, and A^{VaR} are not empty since (see (5), (9)) $\alpha = 0.8 < \langle \mathbf{1}, m\rangle_C/||\mathbf{1}||_C^2 = 1.1400$ and $\beta = 0.9 > \Phi\left(\sqrt{D/||\mathbf{1}||_C^2}\right) = 0.5562$. The left boundaries of the intervals in (7), (8), and (10) are, correspondingly, $M^{SP} = 1.1978, M^V = 1.1400$, and $M^{VaR} = 1.1429$. One can see that function $SP(M)$ (see Fig. 1) is strongly quasi-convex but non-convex in distinction to convex functions $V(M)$ and $VaR(M)$ depicted on Figs. 2–3. Moreover, $A^{SP} \subset A^{VaR} \subset A^V$ as follows from relations among M^{SP}, M^V, and M^{VaR}. Note also that the value $M^{SP} = 1.1978$ differs from $M^V = 1.1400$ and $M^{VaR} = 1.1429$ approximately by significant %5.

[1] In distinction to a widely used notation, here μ (or $M = \mu$) and SP refer, accordingly, to horizontal and vertical axes. The reason is that this location of coordinate axes allows for a more vivid representation of the frontier curve.

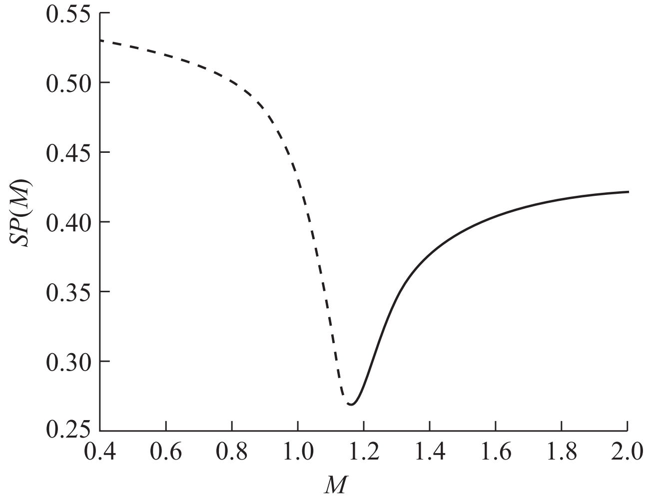

Fig. 1. The mean-SP efficient frontier – solid line.

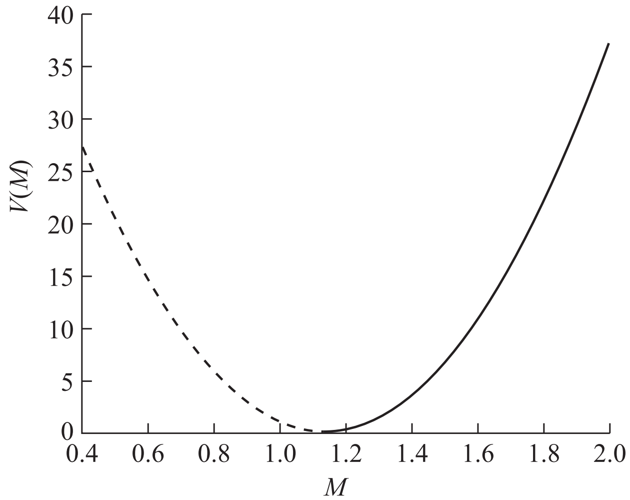

Fig. 2. The mean-variance efficient frontier – solid line.

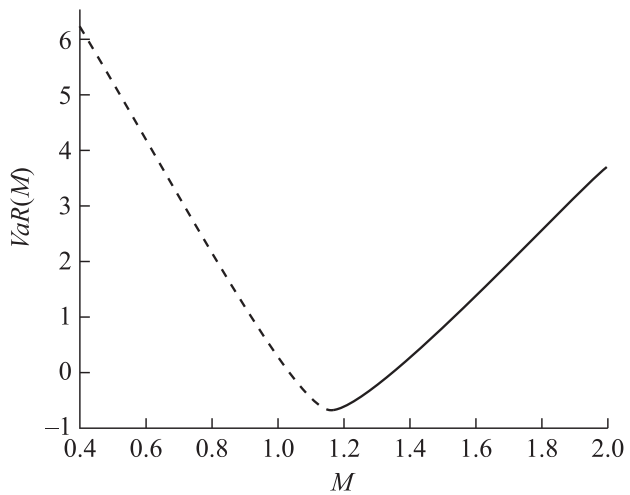

Fig. 3. The mean-VaR efficient frontier – solid line.

4. A CASE OF NON-NORMAL TOTAL RETURN

Despite that papers [12, 18] demonstrate that normality assumption is a good approximation for the total return $X_a = \sum_{i=1}^n a_i R_i$, in order to model a distribution of the total return with a "heavy" tail different from that in the normal case, we extend our results to the case of so-called multivariate elliptical distributions [19]. An attractive property of this class of distributions is that any linear function of elliptically distributed stochastic values has a distribution of the same kind. Within this framework, consider an arbitrary elliptical distribution $F(x) = P(X \leqslant x)$ (normal distribution, or Laplace, or Bessel, or Exponential Power, or Stable Laws [19]) with the mean μ and variance $\sigma^2 < \infty$. Following [15], define the standard elliptical distribution as $F_0(x) = F(\frac{x-\mu}{\sigma})$ – an elliptical distribution with zero mean and unit variance. Now, $SP[\alpha, X_a] = F_0((\alpha - \mu(a))/\sigma(a))$ and $VaR[\beta, X_a] = z_\beta \sigma(a) - \mu(a)$, where z_β is the β-order quantile of $F_0(x)$. Therefore, the theory developed in Section 2 is still valid when the rates of return have a multivariate elliptical distribution, except with z_β instead of the quantile x_β^N and the standard elliptical distribution $F_0(x)$ instead of the standard normal distribution $\Phi(x)$.

For example, Laplace distribution [19] has two parameters: μ and $\lambda > 0$. The mean and variance are, correspondingly, μ and $2\lambda^2$. Denote by y_β the β-order quantile of the Laplace distribution with parameters $\mu = 0$ and $\lambda = 1$. As this distribution has zero mean and variance 2, we get $z_\beta = y_\beta/\sqrt{2}$.

5. CONCLUSIONS

The present paper investigates a problem of constructing the set of all efficient portfolios in the mean-SP setting, where the SP measure of risk is the probability that the total return falls below a prescribed level. Using the parametrization by the mean of total return, we have found a constructive characterization of the set of efficient mean-SP portfolios, and show that it is always a subset of the set of efficient mean-variance portfolios, and it is also a subset of the set of efficient mean-VaR portfolios when the confidence level is "not very close" to the unit. Finally, we study a non-normality situation and show that the main results developed in this paper are still valid when the rates of returns follow a multivariate elliptical distribution. Further extensions of the paper may include: the study of a market with a risk-free assert under the SP measure of risk, a comparison of the mean-SP efficient portfolios with that in mean-conditional value-at-risk (mean-CVaR) setting [20], and the investigation of a model in which a background risk is correlated with the rates of returns [15].

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APPENDIX

The proof of Theorem 1. Since $\Phi(x)$ is an increasing function, a necessary condition of efficiency of a fixed portfolio $a^* \in A^{SP}$ is that it must solve the problem

$$\begin{cases} \min\,(\alpha - \mu(a))/\sigma(a), \\ \mu(a) = M, \\ a \in A = \{a \in R^n : \ \sum_{i=1}^n a_i = 1\}, \end{cases} \tag{A.1}$$

where $M = \mu(a^*)$.

Suppose, at first, that $\alpha \geqslant M$. If $\alpha = M$ then a^* is not efficient since any portfolio a^1: $\mu(a^1) > M$ dominates a^* in the sense that $(\alpha - \mu(a^1))/\sigma(a^1) < (\alpha - \mu(a^*))/\sigma(a^*) = 0$ and $\mu(a^1) > \mu(a^*)$. If $\alpha > M$ then problem (A.1) reduces to maximizing $\sigma(a)$. It is easy to construct a portfolio sequence $\{a^m\}$ such that $\mu(a^m) = M$ and $\sigma(a^m) \to \infty$ as $m \to \infty$. Then, for sufficiently large m, $\sigma(a^m) > \sigma(a^*)$ and, hence, a^m dominates a^*. We have shown that a condition $\alpha < M$ is necessary for efficiency of a^*. Under this condition, problem (A.1) reduces to

$$\min \sigma^2(a), \quad \text{s.t. } \mu(a) = M, \ \sum_{i=1}^{n} a_i = 1. \tag{A.2}$$

Problem (A.2) is already solved by a standard method of Lagrange multipliers (see, e.g., [14, 17]). It is shown that (A.2) has a unique optimal point

$$a^*(= a(M)) = \frac{1}{\triangle}\left[\mathbf{1}||m||_C^2 - m\langle\mathbf{1}, m\rangle_C + M(m||\mathbf{1}||_C^2 - \mathbf{1}\langle\mathbf{1}, m\rangle_C)\right] C^{-1}. \tag{A.3}$$

Now, we will investigate the intervals of monotonicity of the function $SP[\alpha, X_{a(M)}] = \Phi((\alpha - M)/\sigma(a(M))$. Taking into account that

$$\sigma^2(a(M)) = \langle a(M), a(M)C\rangle = (M^2||\mathbf{1}||_C^2 - 2M\langle\mathbf{1}, m\rangle_C + ||m||_C^2)/\triangle^2,$$

the derivative

$$\frac{d}{dM}\Phi((\alpha - M)/\sigma(a(M))) = \frac{\phi((\alpha - M)/\sigma(a(M))}{\triangle^2\sigma^3(a(M))}[-\triangle^2\sigma^2(a(M)) \\ -(\alpha - M)(M||\mathbf{1}||_C^2 - \langle\mathbf{1}, m\rangle_C)],$$

where $\phi(x) > 0$ denotes the density of standard normal distribution. Consider the function in the square brackets,

$$r(M) = M(\langle\mathbf{1}, m\rangle_C - \alpha||\mathbf{1}||_C^2) - ||m||_C^2 + \alpha\langle\mathbf{1}, m\rangle_C. \tag{A.4}$$

1) Let $\alpha \geqslant \alpha_0 = \langle\mathbf{1}, m\rangle_C/||\mathbf{1}||_C^2$. If $\alpha > \alpha_0$ then, as $M > \alpha > 0$, $r(M) < r(\alpha) = -\alpha^2||\mathbf{1}||_C^2 - ||m||_C^2 + 2\alpha\langle\mathbf{1}, m\rangle_C$. By Cauchy-Schwartz-Bunyakovskii inequality, we have $r(\alpha) < -(\alpha||\mathbf{1}||_C - ||m||_C)^2 \leqslant 0$. If $\alpha = \alpha_0$ then $r(M) \equiv -||m||_C^2 + \alpha_0\langle\mathbf{1}, m\rangle_C < 0$.

2) Let $\alpha < \alpha_0$. It follows from (A.4) that $r(M) > 0$ $(= 0)$ if and only if

$$M > (=) M^{SP} = \frac{||m||_C^2 - \alpha\langle\mathbf{1}, m\rangle_C}{\langle\mathbf{1}, m\rangle_C - \alpha||\mathbf{1}||_C^2}, \tag{A.5}$$

i.e., the function $SP[\alpha, X_{a(M)}]$ increases only on the interval $[M^{SP}, \infty)$. To sum up, (i) the condition $\alpha < \langle\mathbf{1}, m\rangle_C/||\mathbf{1}||_C^2$ is necessary and sufficient for existence of an efficient mean-SP portfolio, (ii) the set of efficient mean-SP portfolios is defined as $A^{SP} = \{a(M),\ M \in [M^{SP}, \infty)\}$, the expressions for $a(M)$ and M^{SP} are given by (A.3) and (A.5) correspondingly. □

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