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*This paper is dedicated to*
*the blessed memory of Boris Polyak,*
*the author's teacher and friend*

# A Comparison of Guaranteeing and Kalman Filters

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**Abstract**—We propose a new approach to filtering under arbitrary bounded exogenous disturbances based on reducing this problem to an optimization problem. The approach has a low computational complexity since only Lyapunov equations are solved at each iteration. At the same time, it possesses advantages essential from an engineering-practical point of view, namely, the possibilities to limit the filter matrix and to construct optimal filter matrices separately for each coordinate of the system's state vector. A gradient method for finding the filter matrix is presented. According to the examples, the proposed recurrence procedure is rather effective and yields quite satisfactory results. This paper continues the series of research works devoted to feedback control design from an optimization perspective.

*Keywords*: linear system, exogenous disturbances, filtering, Kalman filter, Luenberger observer, optimization, Lyapunov equation, gradient method, Newton's method, convergence

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## 1. INTRODUCTION

The classical formulation of the filtering problem (i.e., state estimation for a dynamic system by measurements) under random disturbances admits an almost exhaustive solution using the Kalman filter [1]; see the monographs [2] and [3] for details. However, quite often the only knowledge is that all disturbances are bounded (and arbitrary in other respects); in this case, we can construct guaranteeing (rather than probabilistic) estimates of the states. In the early 1970s, ellipsoidal filtering was developed by Schweppe [4], Kurzhansky [5], and Chernous'ko [6]. Later, the papers [7, 8] considered filtering with bounded nonrandom disturbances for time-invariant problems. Within this approach, the error of a desired state estimate must belong to the same ellipsoid for all time instants, i.e., the estimate must be uniform, and the filter is designed in the class of linear time-invariant filters. As it turned out, filtering is completely solvable in this class of problems and estimations: an optimal filter and a corresponding state estimate were constructed. In [7, 8], the technique of linear matrix inequalities (LMIs) [9] was applied, and the original problem was reduced to a parametric semidefinite programming problem. We refer to the monograph [10] for systematic exposure of this technique.

On the other hand, control problems for linear systems can be treated and solved as optimization problems. Such methods have become very popular, despite that their rigorous justification has appeared only recently; see [11–15]. In [16], such an approach was first applied to control problems with exogenous disturbances; in [17], to the output-feedback control design based on an observer; in [18], to the design of PID controllers.

This paper continues both lines of research and pursues the following objectives: simplify the algorithm for computing the guaranteeing filter and compare it numerically with the Kalman filter. We propose an optimization algorithm for solving the filtering problem under nonrandom bounded exogenous disturbances. It has a low computational complexity since only Lyapunov equations are solved at each iteration. At the same time, compared to the LMI approach, it is advantageous from an engineering-practical point of view due to the possibility to limit the filter matrix. According to the examples, the proposed recurrence procedure is rather effective and yields quite satisfactory results.

Note that, in contrast to the Kalman filter, the proposed approach allows constructing optimal filter matrices separately for each coordinate of the system's state vector.

This paper is organized as follows. Section 2 contains the problem statement. In Section 3, we discuss the proposed approach to constructing a guaranteeing filter. Section 4 presents and substantiates the algorithm for computing the optimal filter matrix. Section 5 is devoted to the continuous problem statement. In Section 6, we describe and discuss the calculation results for several examples. Possible generalizations of the results are considered in the Conclusions.

In the sequel, we adopt the following notations: $|\cdot|$ is the Euclidean norm of a vector; $\|\cdot\|$ is the spectral norm of a matrix; $\|\cdot\|_F$ is the Frobenius norm of a matrix; $^{\mathrm{T}}$ is the transpose symbol; tr is the trace of a matrix; $\langle\cdot,\cdot\rangle$ is the Frobenius scalar product of matrices; $I$ is an identity matrix of appropriate dimensions; $\lambda_i(A)$ are the eigenvalues of a matrix $A$; $\sigma_i(A)$ are the singular values of a matrix $A$; $\sigma(A) = -\max_i \mathrm{Re}\,(\lambda_i(A))$ is the degree of stability of a Hurwitz matrix $A$; $\rho(A) = \max_i |\lambda_i(A)|$ is the spectral radius of a Schur matrix $A$. All matrix inequalities are understood in the sense of the definiteness of the corresponding matrices.

## 2. PROBLEM STATEMENT

We consider a discrete-time system of the form

$$\begin{aligned} x_{k+1} &= Ax_k + B_1u_k + D_1w_k,\\ y_k &= Cx_k + B_2u_k + D_2w_k,\\ z_k &= C_1x_k, \end{aligned} \tag{1}$$

with given matrices $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times p}$, $B_2 \in \mathbb{R}^{\ell\times p}$, $C \in \mathbb{R}^{\ell\times n}$, $C_1 \in \mathbb{R}^{r\times n}$, $D_1 \in \mathbb{R}^{n\times m}$, and $D_2 \in \mathbb{R}^{\ell\times m}$, the state vector $x_k \in \mathbb{R}^n$, an initial condition $x_0$, the input $u_k \in \mathbb{R}^p$, the observed $y_k \in \mathbb{R}^\ell$ and estimated $z_k \in \mathbb{R}^r$ outputs, and a bounded exogenous disturbance (noise) $w_k \in \mathbb{R}^m$, i.e.,

$$|w_k| \leqslant 1 \quad \text{for all } k = 0, 1, \ldots.$$

The pairs $(A, D_1)$ and $(A, C)$ are supposed to be controllable and observable, respectively.

Let the state $x_k$ of the system be unmeasurable and information about the system be provided by its output $y_k$. To estimate the output $z_k$, we use a filter described by a linear difference equation with respect to the state estimate $\widehat{x}_k$:

$$\widehat{x}_{k+1} = A\widehat{x}_k + B_1u_k + L(y_k - C\widehat{x}_k - B_2u_k), \quad \widehat{x}_0 = 0, \tag{2}$$

where $L \in \mathbb{R}^{n\times\ell}$. We emphasize that the filter has a preset structure (linear time-invariant) and only the constant matrix $L$ is to be chosen. This structure is the same as in the well-known Luenberger observer [19, 20]. In fact, this filter can be considered a generalization of the Luenberger observer to problems with disturbances.

It is required to minimize the estimation error

$$z_k - \widehat{z}_k = C_1(x_k - \widehat{x}_k) = C_1 e_k,$$

where the residual $e_k = x_k - \widehat{x}_k$, due to (1) and (2), satisfies the difference equation

$$e_{k+1} = (A - LC)e_k + (D_1 - LD_2)w_k, \quad e_0 = x_0. \tag{3}$$

An *admissible* filter matrix $L$ stabilizes system (3), making $(A - LC)$ a Schur matrix. Its existence follows from the observability of the original system.

We underline that the considerations below deal with the case of bounded nonrandom disturbances. For random Gaussian noises, it is natural to apply Kalman filtering, but among other objectives, this paper draws attention to filtering with bounded noises. In addition, some examples demonstrate the operation of the Kalman filter with bounded disturbances and, vice versa, the application of the bounded noise model to random disturbances.

Note also that this paper involves a more general problem statement than [7]: the system description includes the input $u_k$.

Finally, the constraint on the exogenous disturbance of the form

$$|w_k| \leqslant \gamma \quad \text{for all} \quad k = 0, 1, \ldots$$

is taken into account obviously, by the matrix scaling: $D_1 := \gamma D_1$ and $D_2 := \gamma D_2$.

## 3. GUARANTEEING FILTER

This paper proposes a guaranteeing approach to solving the filtering problem under bounded noises. Here, direct formulas can be explicitly derived using gradient descent. The approach is based on the concept of invariant ellipsoids; for details, see [10, 21].

The assertion below expresses one of the main results of the paper.

**Theorem 1.** *Let $L^*$, $P^*$ be the solution of the optimization problem*

$$\min f(L, \alpha), \quad f(L, \alpha) = \operatorname{tr} C_1 P C_1^{\mathrm{T}} + \rho \|L\|_F^2, \tag{4}$$

*subject to the constraint*

$$\frac{1}{\alpha}(A - LC)P(A - LC)^{\mathrm{T}} - P + \frac{1}{1-\alpha}(D_1 - LD_2)(D_1 - LD_2)^{\mathrm{T}} = 0 \tag{5}$$

*for the matrix variables $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and $L \in \mathbb{R}^{n\times \ell}$ and a scalar parameter $0 < \alpha < 1$.*

*Then the output of system (1) with zero initial condition is estimated by the observer (2) with the matrix $L^*$, and the estimation error $(z_k - \widehat{z}_k)$ belongs to the minimal bounding ellipsoid with the matrix*

$$C_1 P^* C_1^{\mathrm{T}}.$$

Proceeding to the proof of this theorem, we recall the following result from [22].

**Lemma 1.** *Assume that $A$ is a Schur matrix, $\rho = \max_i |\lambda_i(A)| < 1$, the pair $(A, D)$ is controllable, and the matrix $P(\alpha) \succ 0$, $\rho^2 < \alpha < 1$, satisfies the discrete Lyapunov equation*

$$\frac{1}{\alpha} APA^{\mathrm{T}} - P + \frac{1}{1-\alpha} DD^{\mathrm{T}} = 0. \tag{6}$$

*Then:*

1) *An optimal bounding ellipsoid for the output of the system*

$$x_{k+1} = Ax_k + Dw_k,$$
$$z_k = Cx_k,$$

*with an initial condition* $x_0$ *and bounded exogenous disturbances* $|w_k| \leqslant 1$ *is found by minimizing the univariate function* $f(\alpha) = \operatorname{tr} CP(\alpha)C^{\mathrm{T}}$ *on the interval* $\rho^2 < \alpha < 1$.

2) *If* $\alpha^*$ *is the minimum point and* $x_0$ *satisfies the condition*

$$x_0^{\mathrm{T}} P^{-1}(\alpha^*) x_0 \leqslant 1,$$

*then the guaranteeing estimate is given by*

$$|z_k|^2 \leqslant f(\alpha^*), \quad k = 0, 1, \ldots.$$

We consider the value $C_1 e_k$ as the linear output of system (3). When enclosing the residual $e_k$ into the invariant ellipsoid

$$\mathcal{E} = \left\{ e \in \mathbb{R}^n: \quad e^{\mathrm{T}} P^{-1} e \leqslant 1 \right\}, \quad P \succ 0,$$

the value $C_1 e_k$ will be contained in the bounding ellipsoid

$$\mathcal{E}_z = \left\{ e_z \in \mathbb{R}^r: \quad e_z^{\mathrm{T}} (C_1 P C_1^{\mathrm{T}})^{-1} e_z \leqslant 1 \right\}. \tag{7}$$

The size of this ellipsoid has to be minimized. Thus, we estimate the asymptotic filtering accuracy. (In the case of small deviations, the accuracy is even uniform in $k$.)

According to Lemma 1, the original problem has been reduced to the matrix optimization problem (4)–(5). In addition to the component determining the size of the bounding ellipsoid (7) by the trace criterion, the minimized function $f(L, \alpha)$ includes a filter matrix penalty. (The coefficient $\rho > 0$ regulates its importance.) Its presence ensures the coercivity of the minimized function in $L$. The form $f(L, \alpha)$ emphasizes that, given $L$ and $\alpha$, the matrix $P$ is found from the Lyapunov Equation (5); thus, the independent variables are $L$ and $\alpha$.

We stress an important feature as follows.

*Remark 1.* The guaranteeing approach under consideration allows constructing optimal filter matrices for each coordinate of the system's state vector separately. (This possibility disappears for the Kalman filter.) Indeed, let the transposed $i$th coordinate vector be the matrix $C_1$ in the problem of Theorem 1. Then we arrive at the filter matrix minimizing the residual $(x_k^{(i)} - \widehat{x}_k^{(i)})$.

## 4. CALCULATING THE OPTIMAL FILTER MATRIX

Recall that a guaranteeing approach to filtering under bounded noises was proposed in [7]. It is based on the technique of linear matrix inequalities and involves solving a parametric semidefinite programming problem. There is no need to apply this, technically rather complicated, apparatus for the optimization problem (4)–(5). (Despite that both the minimized function and the constraint are jointly nonconvex in the variables $P$, $L$, and $\alpha$.) This section introduces a regular iterative approach to its solution, with the gradient descent method applied for the variable $L$ and Newton's minimization method for the variable $\alpha$. Here is a general scheme of the algorithm.

**Algorithm 1** to minimize $f(L, \alpha)$:

1. Specify the parameters $\varepsilon > 0$, $\gamma > 0$, $0 < \tau < 1$, and the initial admissible approximation $L_0$. Calculate the value $\alpha_0 = \left(1 + \rho^2(A - L_0C)\right)/2$.

2. In iteration $j$, given the values $L_j$ and $\alpha_j$, find the gradient $H_j = \nabla_L f(L_j, \alpha_j)$. If $\|H_j\| \leqslant \varepsilon$, then take $L_j$ as an approximate solution.

3. Make a move of the gradient descent method:

$$L_{j+1} = L_j - \gamma_j H_j,$$

choosing the step $\gamma_j > 0$ by fractioning $\gamma$ until the conditions:

a. $L_{j+1}$ makes $(A - LC)/\sqrt{\alpha_j}$ a Schur matrix.

b. $f(L_{j+1}) \leqslant f(L_j) - \tau\gamma_j\|H_j\|^2$.

4. For the resulting value $L_{j+1}$, minimize $f(L_{j+1}, \alpha)$ in $\alpha$ using Newton's method:[1]

$$\alpha_{j+1} = \alpha_j - \frac{f'(\alpha_j)}{f''(\alpha_j)}.$$

Having obtained $\alpha_{j+1}$, get back to item 2.

The values $\nabla_L f(L, \alpha)$, $f'(\alpha)$, and $f''(\alpha)$ in Algorithm 1 are given by

$$\begin{aligned}
\nabla_L f(L, \alpha) &= 2\left(\rho L - \frac{1}{\alpha}Y(A - LC)PC^{\mathrm{T}} - \frac{1}{1-\alpha}Y(D_1 - LD_2)D_2^{\mathrm{T}}\right), \\
f'(\alpha) &= \operatorname{tr} Y\left(\frac{1}{(1-\alpha)^2}(D_1 - LD_2)(D_1 - LD_2)^{\mathrm{T}} - \frac{1}{\alpha^2}(A - LC)P(A - LC)^{\mathrm{T}}\right), \\
f''(\alpha) &= 2\operatorname{tr} Y\left(\frac{1}{(1-\alpha)^3}(D_1 - LD_2)(D_1 - LD_2)^{\mathrm{T}} + \frac{1}{\alpha^3}(A - LC)(P - X)(A - LC)^{\mathrm{T}}\right),
\end{aligned}$$

where the matrices $P$, $Y$, and $X$ satisfy the discrete Lyapunov Equations (5),

$$\frac{1}{\alpha}(A - LC)^{\mathrm{T}}Y(A - LC) - Y + C_1^{\mathrm{T}}C_1 = 0, \tag{8}$$

and

$$\begin{aligned}
\frac{1}{\alpha}(A - LC)X(A - LC)^{\mathrm{T}} - X + \frac{1}{(1-\alpha)^2}(D_1 - LD_2)(D_1 - LD_2)^{\mathrm{T}} \\
- \frac{1}{\alpha^2}(A - LC)P(A - LC)^{\mathrm{T}} = 0,
\end{aligned}$$

respectively.

The rationale of Algorithm 1 with the corresponding formulas is placed in Appendix A.

An important point is choosing a trial step of the gradient descent method. A very promising choice rests on the following reasoning. For some admissible $L$, let us find the solution $P$ of the discrete Lyapunov equation

$$(A - LC)P(A - LC)^{\mathrm{T}} - P = -I.$$

Considering the increment in $L$, i.e.,

$$L \to L - \gamma H, \quad H = \nabla_L f(L, \alpha),$$

[1] It actually takes 3–4 iterations to obtain a solution with high accuracy if the starting point is not too close to the boundaries of the interval $\left(\rho^2(A - L_{j+1}C), 1\right)$.

we determine under which $\gamma$ the matrix $P$ will remain a matrix of the quadratic Lyapunov function for $A-(L-\gamma H)C$, i.e.,

$$(A-(L-\gamma H)C)\,P\,(A-(L-\gamma H)C)^{\mathrm{T}}-P\prec 0.$$

Due to the Schur complement lemma, it reduces to

$$\begin{pmatrix} P & A-(L-\gamma H)C \\ (A-(L-\gamma H)C)^{\mathrm{T}} & P^{-1}\end{pmatrix}\succ 0.$$

This matrix inequality, written as

$$\begin{pmatrix} P & A-LC \\ (A-LC)^{\mathrm{T}} & P^{-1}\end{pmatrix}+\gamma\begin{pmatrix} 0 & HC \\ (HC)^{\mathrm{T}} & 0\end{pmatrix}\succ 0,$$

holds for

$$\gamma<\lambda_{\max}^{-1}\left(\begin{pmatrix} 0 & HC \\ (HC)^{\mathrm{T}} & 0\end{pmatrix},\begin{pmatrix} P & A-LC \\ (A-LC)^{\mathrm{T}} & P^{-1}\end{pmatrix}\right).$$

## 5. THE CONTINUOUS-TIME CASE

We consider a continuous-time system of the form

$$\begin{aligned}\dot{x}&=Ax+B_1u+D_1w,\quad x(0)=x_0,\\ y&=Cx+B_2u+D_2w,\\ z&=C_1x,\end{aligned}\tag{9}$$

with given matrices $A\in\mathbb{R}^{n\times n}$, $B_1\in\mathbb{R}^{n\times p}$, $B_2\in\mathbb{R}^{\ell\times p}$, $C\in\mathbb{R}^{\ell\times n}$, $C_1\in\mathbb{R}^{r\times n}$, $D_1\in\mathbb{R}^{n\times m}$, and $D_2\in\mathbb{R}^{\ell\times m}$, the state vector $x(t)\in\mathbb{R}^n$, the input $u(t)\in\mathbb{R}^p$, the observed $y(t)\in\mathbb{R}^\ell$ and estimated $z(t)\in\mathbb{R}^r$ outputs, and a bounded exogenous disturbance (noise) $w(t)\in\mathbb{R}^m$, i.e.,

$$|w(t)|\leqslant 1\quad\text{for all}\quad t\geqslant 0.$$

The pairs $(A,D_1)$ and $(A,C)$ are supposed to be controllable and observable, respectively.

To estimate the output $z$, we use a filter described by a linear differential equation with respect to the state estimate $\widehat{x}$:

$$\dot{\widehat{x}}=A\widehat{x}+B_1u+L(y-C\widehat{x}-B_2u),\quad \widehat{x}(0)=0,\tag{10}$$

where $L\in\mathbb{R}^{n\times\ell}$.

Like in the discrete-time case, it is required to minimize the estimation error

$$z-\widehat{z}=C_1(x-\widehat{x})=C_1e,$$

where the residual $e(t)=x(t)-\widehat{x}(t)$, due to (9) and (10), satisfies the differential equation

$$\dot{e}=(A-LC)e+(D_1-LD_2)w,\quad e(0)=x_0.\tag{11}$$

An *admissible* filter matrix $L$ stabilizes system (11), making $(A-LC)$ a Hurwitz matrix. Its existence follows from the observability of the original system.

The next result is a continuous-time analog of Lemma 1.

**Lemma 2** [9, 10]. *Assume that* $A$ *is a Hurwitz matrix,* $\sigma = -\max_i \operatorname{Re}(\lambda_i(A)) > 0$, *the pair* $(A, D)$ *is controllable, and the matrix* $P(\alpha) \succ 0$, $0 < \alpha < 2\sigma$, *satisfies the Lyapunov equation*

$$\left(A + \frac{\alpha}{2}I\right)P + P\left(A + \frac{\alpha}{2}I\right)^{\mathrm{T}} + \frac{1}{\alpha}DD^{\mathrm{T}} = 0.$$

*Then:*

1) *An optimal bounding ellipsoid for the output of the system*

$$\begin{aligned} \dot{x} &= Ax + Dw, \quad x(0) = x_0, \\ z &= Cx, \end{aligned}$$

*with bounded exogenous disturbances* $|w(t)| \leqslant 1$ *is found by minimizing the univariate function* $f(\alpha) = \operatorname{tr} CP(\alpha)C^{\mathrm{T}}$ *on the interval* $0 < \alpha < 2\sigma$.

2) *If* $\alpha^*$ *is the minimum point and* $x(0)$ *satisfies the condition*

$$x^{\mathrm{T}}(0)P^{-1}(\alpha^*)x(0) \leqslant 1,$$

*then the guaranteeing estimate is given by*

$$|z(t)|^2 \leqslant f(\alpha^*), \quad 0 \leqslant t < \infty.$$

Following similar considerations as in the discrete-time case, we use Lemma 2 to establish the following result.

**Theorem 2.** *Let* $L^*$, $P^*$ *be the solution of the optimization problem*

$$\min f(L, \alpha), \quad f(L, \alpha) = \operatorname{tr} C_1 P C_1^{\mathrm{T}} + \rho \|L\|_F^2,$$

*subject to the constraint*

$$\left(A - LC + \frac{\alpha}{2}I\right)P + P\left(A - LC + \frac{\alpha}{2}I\right)^{\mathrm{T}} + \frac{1}{\alpha}(D_1 - LD_2)(D_1 - LD_2)^{\mathrm{T}} = 0 \quad (12)$$

*for the matrix variables* $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ *and* $L \in \mathbb{R}^{n\times \ell}$ *and a scalar parameter* $\alpha > 0$.

*Then the output of system* (9) *with zero initial condition is estimated by the observer* (10) *with the matrix* $L^*$, *and the estimation error* $(z - \widehat{z})$ *belongs to the minimal bounding ellipsoid with the matrix*

$$C_1 P^* C_1^{\mathrm{T}}.$$

The properties of the minimized function and its derivatives (see Appendix B) allow developing a minimization method and justify its convergence.

**Algorithm 2** to minimize $f(L, \alpha)$:

1. Specify the parameters $\varepsilon > 0$, $\gamma > 0$, and $0 < \tau < 1$ and the initial admissible approximation $L_0$. Calculate the value $\alpha_0 = \sigma(A - L_0 C)$.

2. In iteration $j$, given the values $L_j$ and $\alpha_j$, find the gradient $H_j = \nabla_L f(L_j, \alpha_j)$. If $\|H_j\| \leqslant \varepsilon$, then take $L_j$ as an approximate solution.

3. Make a move of the gradient descent method:

$$L_{j+1} = L_j - \gamma_j H_j,$$

choosing the step $\gamma_j > 0$ by fractioning $\gamma$ until the conditions:

a. $L_{j+1}$ makes $(A - LC + \frac{\alpha_j}{2} I)$ a Hurwitz matrix.

b. $f(L_{j+1}) \leqslant f(L_j) - \tau\gamma_j \|H_j\|^2$.

4. For the resulting value $L_{j+1}$, minimize $f(L_{j+1}, \alpha)$ in $\alpha$ and obtain $\alpha_{j+1}$. Get back to item 2.

The values $\nabla_L f(L,\alpha)$, $f'(\alpha)$, and $f''(\alpha)$ in Algorithm 2 are given by

$$\nabla_L f(L,\alpha) = 2\left(\rho L - YPC^{\mathrm{T}} - \frac{1}{\alpha} Y(D_1 - LD_2)D_2^{\mathrm{T}}\right),$$
$$f'(\alpha) = \operatorname{tr} Y\left(P - \frac{1}{\alpha^2}(D_1 - LD_2)(D_1 - LD_2)^{\mathrm{T}}\right),$$
$$f''(\alpha) = 2\operatorname{tr} Y\left(X + \frac{1}{\alpha^3}(D_1 - LD_2)(D_1 - LD_2)^{\mathrm{T}}\right),$$

where the matrices $P$, $Y$, and $X$ satisfy the discrete Lyapunov Equations (12),

$$\left(A - LC + \frac{\alpha}{2} I\right)^{\mathrm{T}} Y + Y\left(A - LC + \frac{\alpha}{2} I\right) + C_1^{\mathrm{T}} C_1 = 0, \tag{13}$$

and

$$\left(A - LC + \frac{\alpha}{2} I\right) X + X\left(A - LC + \frac{\alpha}{2} I\right)^{\mathrm{T}} + P - \frac{1}{\alpha^2}(D_1 - LD_2)(D_1 - LD_2)^{\mathrm{T}} = 0, \tag{14}$$

respectively.

The rationale of Algorithm 2 with the corresponding formulas is placed in Appendix B.

The trial step in the gradient descent method is chosen by the following reasoning. For some admissible $L$, let us find the solution $P$ of the Lyapunov equation

$$(A - LC)P + P(A - LC)^{\mathrm{T}} = -I.$$

Considering the increment in $L$, i.e.,

$$L \to L - \gamma H, \quad H = \nabla_L f(L,\alpha),$$

we determine under which $\gamma$ the matrix $P$ will remain a matrix of the quadratic Lyapunov function for $A - (L - \gamma H)C$, i.e.,

$$(A - (L - \gamma H)C)\,P + P\,(A - (L - \gamma H)C)^{\mathrm{T}} \prec 0.$$

Due to the original equation, we have

$$\gamma\left(HCP + P(HC)^{\mathrm{T}}\right) \prec I$$

and, consequently,

$$\gamma < \lambda_{\max}^{-1}\left(HCP + P(HC)^{\mathrm{T}}\right).$$

## 6. EXAMPLES AND DISCUSSION

We consider a discrete-time state-space model of a plant described by

$$\begin{aligned} x_{k+1} &= Ax_k + B_1 u_k + G w_k, \\ y_k &= Cx_k + B_2 u_k + v_k, \end{aligned} \tag{15}$$

with the state vector $x_k \in \mathbb{R}^n$, an initial condition $x_0$, the input $u_k \in \mathbb{R}^p$, the observed output $y_k \in \mathbb{R}^\ell$, a noise $w_k \in \mathbb{R}^m$, and a measurement error $v_k \in \mathbb{R}^\ell$. Here, $A$, $B_1$, $B_2$, $C$, and $G$ are given matrices of compatible dimensions, and the values $w_k$ and $v_k$ are supposed to be independent.

For system (15), the Kalman filter has the following form. (In this case, the values $w_k$ and $v_k$ are supposed to be random with the Gaussian distribution with zero mean and covariance matrices $Q$ and $R$, respectively.)

The extrapolation stage:

$$\widehat{x}_{k+1|k} = A\widehat{x}_{k|k} + B_1 u_k,$$
$$P_{k+1|k} = AP_{k|k}A^{\mathrm{T}} + GQG^{\mathrm{T}}.$$

The correction stage:

$$K_k = P_{k|k-1}C^{\mathrm{T}}(CP_{k|k-1}C^{\mathrm{T}} + R)^{-1},$$
$$\widehat{x}_{k|k} = \widehat{x}_{k|k-1} + K_k(y_k - C\widehat{x}_{k|k-1} - B_2 u_k),$$
$$P_{k|k} = (I - K_k C)P_{k|k-1}.$$

We study three problem statements as follows.

1. The model $\mathcal{M}_1$ with random disturbances: the noise $w_k$ and the measurement error $v_k$ are Gaussian with zero mean and covariance matrices $Q$ and $R$, respectively:

$$w_k \sim N(0, Q), \quad v_k \sim N(0, R).$$

2. The model $\mathcal{M}_2$ with bounded random disturbances: the noise $w_k$ and the measurement error $v_k$ are uniformly distributed on the cubes $[-w, w]^m$ and $[-v, v]^\ell$, respectively:

$$w_k \sim U([-w, w]^m), \quad v_k \sim U([-v, v]^\ell).$$

3. The model $\mathcal{M}_3$ with bounded nonrandom disturbances: the noise $w_k$ and the measurement error $v_k$ take arbitrary values on the cubes $[-w, w]^m$ and $[-v, v]^\ell$, respectively:

$$|w_k|_\infty \leqslant w, \quad |v_k|_\infty \leqslant v.$$

Within these models, let us compare the operation of the Kalman filter and the guaranteeing approach for several examples. (Even though the guaranteeing estimates are not valid for the model $\mathcal{M}_1$ whereas the Kalman filter for the model $\mathcal{M}_3$.)

The figures below demonstrate the true trajectory of the system, its observation (if any), and the estimates provided by the Kalman filter (QF) and the guaranteeing filter (GF). In the latter case, we also present a guaranteeing tube containing the estimate under all admissible disturbances.

*Example 1.* Consider a truck on straight, frictionless rails [23]. Initially, the truck is stationary at zero position and is affected by exogenous disturbances. The position of the truck is measured every $\Delta t$ seconds, and the measurements are imprecise. The problem is to track the position $s$ and velocity $\dot{s} = v$ of the truck.

The corresponding system can be represented in the form (15) with

$$x_k = \begin{pmatrix} s \\ \dot{s} \end{pmatrix}, \quad x_0 = 0,$$

$$A = \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix}, \quad B_1 = B_2 = 0, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} (\Delta t)^2/2 \\ \Delta t \end{pmatrix}.$$

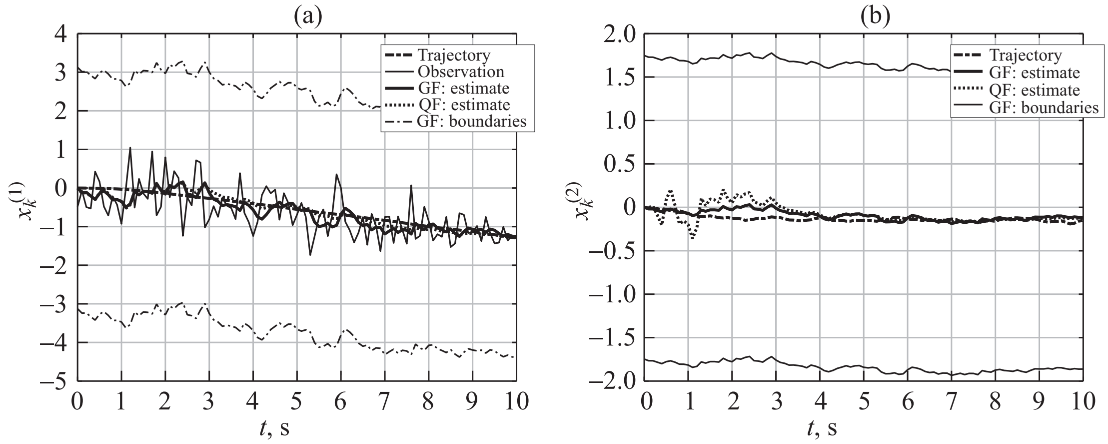


**Fig. 1.** The dynamics of coordinates and their estimates in Example 1 (model $\mathcal{M}_1$).

1. Within the model $\mathcal{M}_1$, the truck moves at time instant $k$ with a constant acceleration distributed by the Gaussian law with zero mean and a standard deviation $\sigma_x$ and the measurement error has the Gaussian distribution with zero mean and a standard deviation $\sigma_y$:

$$w_k \sim N(0, \sigma_x), \quad v_k \sim N(0, \sigma_y).$$

We construct a Kalman filter and use the guaranteeing approach by combining the disturbances $w_k$ and $v_k$ into the common disturbance vector

$$\tilde{w}_k = \begin{pmatrix} w_k \\ v_k \end{pmatrix}. \tag{16}$$

In this case, the matrices $D_1$ and $D_2$ in system (1) take the form

$$D_1 = 3\sigma_x\sqrt{2}\begin{pmatrix} G & 0 \end{pmatrix}, \quad D_2 = 3\sigma_y\sqrt{2}\begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Hence, the values $w_k$ and $v_k$ can be independently varied within the ranges

$$|w_k| \leqslant 3\sigma_x, \quad |v_k| \leqslant 3\sigma_y.$$

Numerical calculations based on the guaranteeing approach yielded the optimal filter matrices

$$L_1^* = \begin{pmatrix} 0.2359 \\ 0.1412 \end{pmatrix}$$

(for the coordinate $x^{(1)} = s$) and

$$L_2^* = \begin{pmatrix} 0.1122 \\ 0.0386 \end{pmatrix}$$

(for the coordinate $x^{(2)} = \dot{s}$).

The comparison results with the Kalman filter for

$$\Delta t = 0.1, \quad \sigma_x = 0.1, \quad \sigma_y = 0.5$$

are shown in Figs. 1a and 1b.

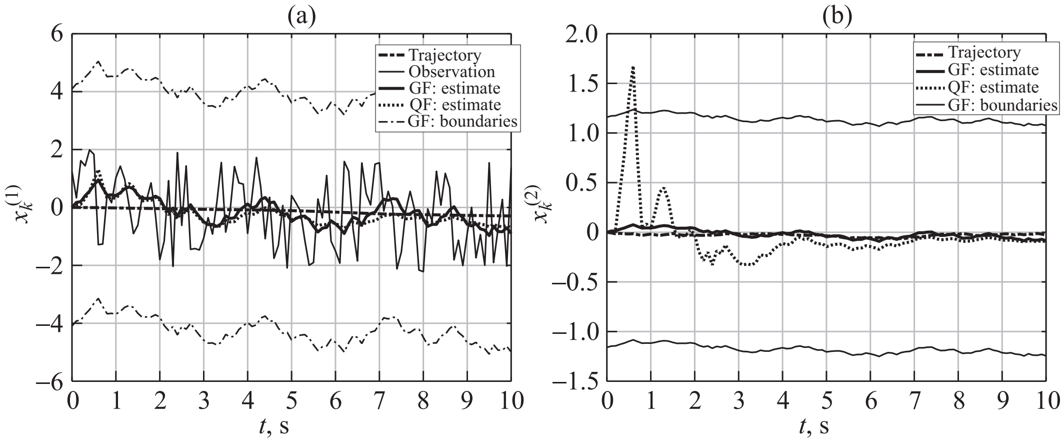


**Fig. 2.** The dynamics of coordinates and their estimates in Example 1 (model $\mathcal{M}_2$).

2. Within the model $\mathcal{M}_2$, the truck moves at time instant $k$ with a constant acceleration uniformly distributed on an interval $[-a, a]$ and the measurement error has the uniform distribution on an interval $[-v, v]$:

$$w_k \sim U(-a, a), \quad v_k \sim U(-v, v).$$

We construct the Kalman filter with the parameters $\sigma_x = a/3$ and $\sigma_y = v/3$ and use the guaranteeing approach for the disturbance (16) and the matrices

$$D_1 = a\sqrt{2}\begin{pmatrix} G & 0 \end{pmatrix}, \quad D_2 = v\sqrt{2}\begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Numerical calculations based on the guaranteeing approach yielded the optimal filter matrices

$$L_1^* = \begin{pmatrix} 0.1393 \\ 0.0489 \end{pmatrix}$$

(for the coordinate $x^{(1)} = s$) and

$$L_2^* = \begin{pmatrix} 0.0574 \\ 0.0101 \end{pmatrix}$$

(for the coordinate $x^{(2)} = \dot{s}$).

The comparison results with the Kalman filter for

$$\Delta t = 0.1, \quad a = 0.1, \quad v = 2$$

are shown in Figs. 2a and 2b.

3. Within the model $\mathcal{M}_3$, the acceleration $w_k$ and the measurement error $v_k$ take arbitrary values on the intervals $[-a, a]$ and $[-v, v]$, respectively:

$$|w_k| \leqslant a, \quad |v_k| \leqslant v.$$

We construct the Kalman filter with the parameters $\sigma_x = a$ and $\sigma_y = v$ and use the guaranteeing approach for the disturbance (16) and the matrices

$$D_1 = a\sqrt{2}\begin{pmatrix} G & 0 \end{pmatrix}, \quad D_2 = v\sqrt{2}\begin{pmatrix} 0 & 1 \end{pmatrix}.$$

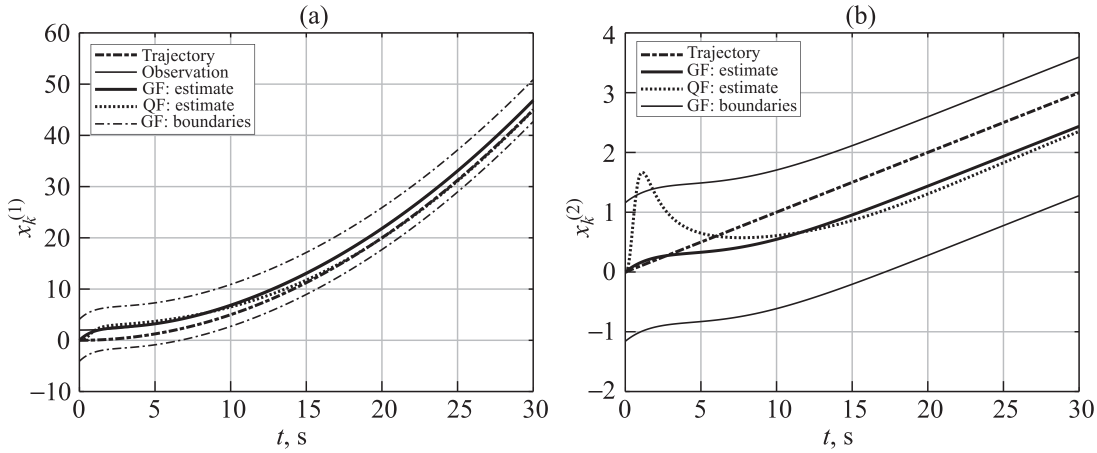


**Fig. 3.** The dynamics of coordinates and their estimates in Example 1 (model $\mathcal{M}_3$).

Numerical calculations based on the guaranteeing approach yielded the optimal filter matrices

$$L_1^* = \begin{pmatrix} 0.1397 \\ 0.0492 \end{pmatrix}$$

(for the coordinate $x^{(1)} = s$) and

$$L_2^* = \begin{pmatrix} 0.0574 \\ 0.0101 \end{pmatrix}$$

(for the coordinate $x^{(2)} = \dot{s}$).

The comparison results with the Kalman filter for

$$\Delta t = 0.1, \quad a = 0.1, \quad v = 2$$

are shown in Figs. 3a and 3b.

*Example 2.* This example [24] consists in estimating the projectile motion along a ballistic trajectory under exogenous disturbances and observable (noisy) coordinates. The corresponding system has the form (15), where

$$x = \begin{pmatrix} s_x \\ s_y \\ v_x \\ v_y \end{pmatrix}$$

is the state vector of the system (the projections of the coordinate and velocity of the projectile on the horizontal and vertical axes) and

$$A = \begin{pmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1-b & 0 \\ 0 & 0 & 0 & 1-b \end{pmatrix}, \quad u = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -g\Delta t \end{pmatrix},$$

$$B_1 = G = I, \quad B_2 = 0, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

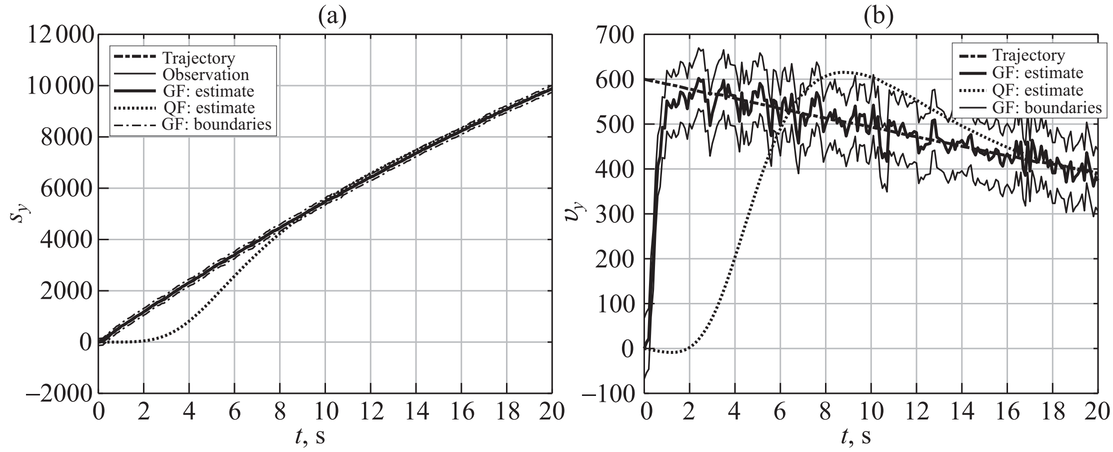


**Fig. 4.** The dynamics of coordinates and their estimates in Example 2 (model $\mathcal{M}_1$).

Here, $\Delta t$ is the interval between measurements, $0 < b \ll 1$ is the air resistance coefficient, and $g$ is the gravitational constant. The disturbance $w_k$ and measurement noise $v_k$ are Gaussian with zero mean and covariance matrices $Q_k \succ 0$ and $R_k \succ 0$, respectively. The parameters have the values

$$\Delta t = 0.1, \quad b = 10^{-4}, \quad g = 9.8, \quad Q = 0.1I, \quad R = 500I.$$

1. Within the "standard" model $\mathcal{M}_1$,

$$w_k \sim N(0, \sigma_x I), \quad \sigma_x^2 = 0.1,$$
$$v_k \sim N(0, \sigma_y I), \quad \sigma_y^2 = 500.$$

As in the previous example, to use the guaranteeing approach, we combine the disturbances $w_k$ and $v_k$ into the common disturbance vector. The matrices $D_1$ and $D_2$ in system (1) take the form

$$D_1 = 3\sigma_x\sqrt{2}\begin{pmatrix} G & 0 \end{pmatrix}, \quad D_2 = 3\sigma_y\sqrt{2}\begin{pmatrix} 0 & I \end{pmatrix}.$$

Hence, the values $w_k$ and $v_k$ can be independently varied within the ranges

$$|w_k| \leqslant 3\sigma_x, \quad |v_k| \leqslant 3\sigma_y.$$

Numerical calculations based on the guaranteeing approach yielded the following optimal filter matrices for each of the four coordinates:

$$L_1^* = \begin{pmatrix} 0.5946 & 0 \\ 0 & 0.6822 \\ 0.8467 & 0 \\ 0 & 1.0590 \end{pmatrix}, \quad L_2^* = \begin{pmatrix} 0.7277 & 0 \\ 0 & 0.6340 \\ 1.2376 & 0 \\ 0 & 0.9879 \end{pmatrix},$$
$$L_3^* = \begin{pmatrix} 0.0971 & 0 \\ 0 & 0.1389 \\ 0.0284 & 0 \\ 0 & 0.0456 \end{pmatrix}, \quad L_4^* = \begin{pmatrix} 0.1393 & 0 \\ 0 & 0.0975 \\ 0.0459 & 0 \\ 0 & 0.0285 \end{pmatrix}.$$

(Hereinafter, the entries of the filter matrices not exceeding $10^{-6}$ in absolute value are zeroed out.)

The comparison results for the coordinates $s_y$ and $v_y$ are is shown in Figs. 4a and 4b.

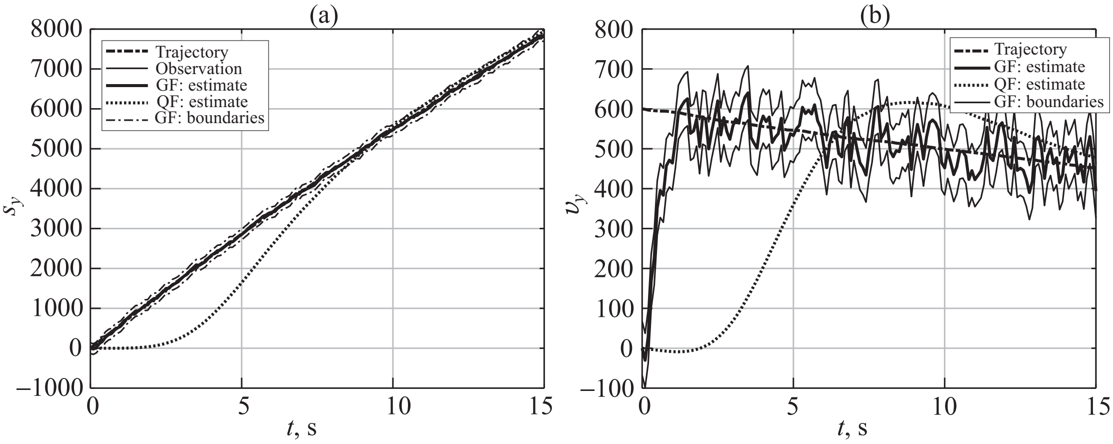


**Fig. 5.** The dynamics of coordinates and their estimates in Example 2 (model $\mathcal{M}_2$).

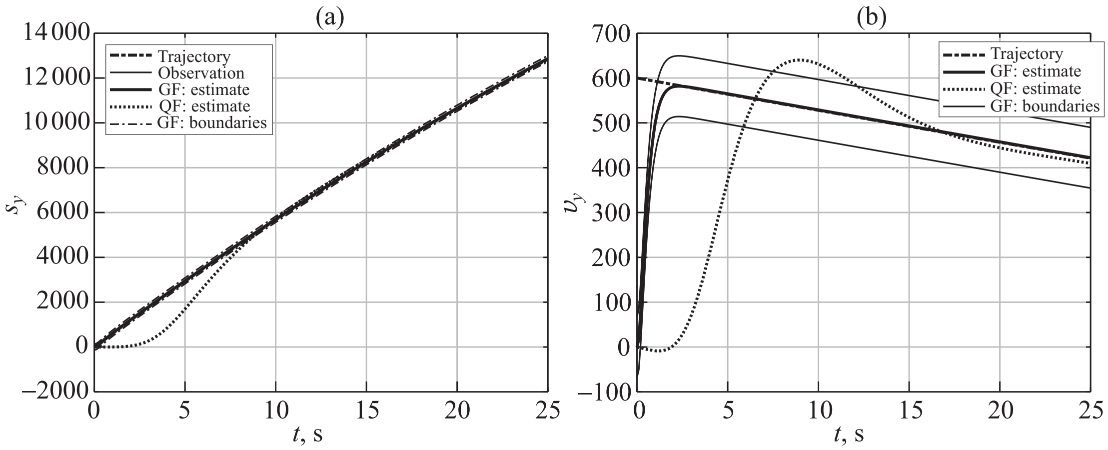


**Fig. 6.** The dynamics of coordinates and their estimates in Example 2 (model $\mathcal{M}_3$).

Note that the actual starting point

$$x_0 = \begin{pmatrix} 0 \\ 0 \\ 300 \\ 600 \end{pmatrix}$$

is outside the minimal invariant ellipsoid for the residual, which explains the behavior of the estimation trajectories in the initial section. However, due to the attraction property of the invariant ellipsoid, after a few steps, the guaranteeing estimates cover the true trajectory.

2. Within the model $\mathcal{M}_2$,

$$w_k \sim U(-w, w), \quad v_k \sim U(-v, v),$$

where $w = 3\sigma_x$ and $v = 3\sigma_y$.

In this case,

$$D_1 = w\sqrt{2}\begin{pmatrix} G & 0 \end{pmatrix}, \quad D_2 = v\sqrt{2}\begin{pmatrix} 0 & I \end{pmatrix},$$

and numerical calculations based on the guaranteeing approach yielded the same filter matrices as in the model $\mathcal{M}_2$. The comparison results with the Kalman filter for the coordinates $s_y$ and $v_y$ are shown in Figs. 5a and 5b.

3. Within the model $\mathcal{M}_3$,

$$|w_k| \leqslant \sigma_x, \quad |v_k| \leqslant \sigma_y.$$

Numerical calculations based on the guaranteeing approach with

$$D_1 = \sigma_x\sqrt{2}\begin{pmatrix} G & 0 \end{pmatrix}, \quad D_2 = \sigma_y\sqrt{2}\begin{pmatrix} 0 & I \end{pmatrix}$$

yielded the optimal filter matrices

$$L_1^* = \begin{pmatrix} 0.6362 & 0 \\ 0 & 0.7256 \\ 0.9779 & 0 \\ 0 & 1.2126 \end{pmatrix}, \quad L_2^* = \begin{pmatrix} 0.6697 & 0 \\ 0 & 0.5747 \\ 1.0517 & 0 \\ 0 & 0.8206 \end{pmatrix},$$

$$L_3^* = \begin{pmatrix} 0.0971 & 0 \\ 0 & 0.1389 \\ 0.0284 & 0 \\ 0 & 0.0456 \end{pmatrix}, \quad L_4^* = \begin{pmatrix} 0.1393 & 0 \\ 0 & 0.0975 \\ 0.0459 & 0 \\ 0 & 0.0285 \end{pmatrix}.$$

The comparison results with the Kalman filter for the coordinates $s_y$ and $v_y$ are shown in Figs. 6a and 6b.

According to the examples, the Kalman and guaranteeing filters, on the one hand, do not differ too much in their results; on the other hand, they are quite operable in all three models. As expected, for Gaussian disturbances, the Kalman filter gives slightly (but not dramatically) better estimates than the guaranteeing filter; for bounded nonrandom disturbances, the latter filter has the advantage over the former.

Note also that the guaranteeing tube containing the corresponding estimate is strongly overvalued; see the figures. This is typical behavior for guaranteed estimation methods, intended to counteract the "worst-case" realization of uncertainty.

Finally, the guaranteeing filter allows obtaining a uniform estimate of the filtering accuracy. In this context, we pay attention to the behavior of the guaranteeing and Kalman estimates on the initial segment of the trajectory. (The latter estimate also has a pronounced peak.)

## 7. CONCLUSIONS

This paper has proposed a new approach to guaranteeing filtering based on reducing this problem to a matrix optimization problem with the filter matrix as the variable. The resulting problem has been solved by the gradient descent method. Its convergence has been theoretically proved for a series of important special cases.

The examples presented above show the operability and effectiveness of the proposed algorithm. Also, it has been compared with the Kalman filter on three different problem statements.

This paper has considered time-invariant filter design; of obvious interest is to generalize this approach to dynamic filtering in the spirit of [25, 26] using the tools [27, 28].

## FUNDING

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*APPENDIX A*

**Lemma A.1.** *Let $X$ and $Y$ be the solutions of the dual discrete Lyapunov equations with a Schur matrix $A$:*

$$A^{\mathrm{T}}XA - X + W = 0 \quad \text{and} \quad AYA^{\mathrm{T}} - Y + V = 0.$$

*Then* $\operatorname{tr}(XV) = \operatorname{tr}(YW)$.

**Proof of Lemma A.1.** Indeed, direct calculations give

$$\begin{aligned}\operatorname{tr}(XV) &= \operatorname{tr}\left(X(Y - AYA^{\mathrm{T}})\right) = \operatorname{tr}(XY) - \operatorname{tr}(XAYA^{\mathrm{T}}) \\ &= \operatorname{tr}(YX) - \operatorname{tr}(YA^{\mathrm{T}}XA) = \operatorname{tr}\left(Y(X - A^{\mathrm{T}}XA)\right) = \operatorname{tr}(YW).\end{aligned}$$

The proof of Lemma A.1 is complete.

**Lemma A.2.** *The solution $P$ of the discrete Lyapunov equation*

$$APA^{\mathrm{T}} - P + Q = 0$$

*with a Schur matrix $A$ and $Q \succ 0$ satisfies the lower bounds*

$$\lambda_{\max}(P) \geqslant \frac{\lambda_{\min}(Q)}{1-\rho^2}, \quad \lambda_{\min}(P) \geqslant \frac{\lambda_{\min}(Q)}{1-\sigma_{\min}^2(A)}, \tag{A.1}$$

*where $\rho = \max\limits_i |\lambda_i(A)|$ and $\sigma_{\min}(A)$ is the smallest singular value of the matrix $A$.*

*If $Q = DD^{\mathrm{T}}$ and the pair $(A, D)$ is controllable, then*

$$\lambda_{\max}(P) \geqslant \frac{\|u^*D\|^2}{1-\rho^2} > 0, \tag{A.2}$$

*where*

$$u^*A = \lambda u^*, \quad |\lambda| = \rho, \quad \|u\| = 1,$$

*i.e., $u$ is the left eigenvector of the matrix $A$ corresponding to the eigenvalue $\lambda$ of the matrix $A$ with the greatest magnitude. The vector $u$ and the value $\lambda$ can be complex; here, $u^*$ denotes the conjugate transpose of $u$.*

**Proof of Lemma A.2.** The lower bounds (A.1) are well known; for example, see [29]. Let us prove (A.2). The explicit solution of the discrete Lyapunov equation with a Schur matrix $A$ has the form

$$P = \sum_{k=0}^{\infty} A^k DD^{\mathrm{T}}(A^{\mathrm{T}})^k.$$

Multiplying this equality by $u$ on the right and by $u^*$ on the left, due to $u^*A^k = \lambda^k u^*$ and $(A^{\mathrm{T}})^k u = (\lambda^*)^k u$, we obtain

$$\lambda_{\max}(P) \geqslant u^*Pu = \sum_{k=0}^{\infty} u^*A^kDD^{\mathrm{T}}(A^{\mathrm{T}})^k u = \sum_{k=0}^{\infty} (\lambda\lambda^*)^k u^*DD^{\mathrm{T}}u = \frac{\|u^*D\|^2}{1-\rho^2},$$

where $\|u^*D\| > 0$ by the controllability of the pair $(A, D)$; for example, see [10, Theorem D.1.5]. The proof of Lemma A.2 is complete.

Now, we optimize the function $f(\alpha)$ and consider the problem

$$\min f(\alpha), \quad f(\alpha) = \operatorname{tr} CPC^{\mathrm{T}}$$

subject to the constraint

$$\frac{1}{\alpha}APA^{\mathrm{T}} - P + \frac{1}{1-\alpha}DD^{\mathrm{T}} = 0$$

for the matrix variables $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and a scalar parameter $0 < \alpha < 1$.

Here we impose more stringent requirements for the problem statement: the matrix $C$ of the system output is supposed to be square and nonsingular. This assumption could be relaxed, but the current goal is to establish the simplest and most obvious results.

**Lemma A.3.** *Assume that $A$ is a Schur matrix, $\rho$ is the spectral radius of $A$, $\rho^2 < \alpha < 1$, the pair $(A, D)$ is controllable, and the matrix $C$ is such that $C^{\mathrm{T}}C \succ 0$. Then the function $f(\alpha) = \operatorname{tr} CP(\alpha)C^{\mathrm{T}}$ possesses the following properties:*

*a) The function $f(\alpha)$ is well-defined, positive, and strongly convex on the interval $\rho^2 < \alpha < 1$ and its values tend to infinity at the interval endpoints. Moreover, there exists a constant $c > 0$ such that*

$$f(\alpha) \geqslant \frac{\alpha}{(1-\alpha)(\alpha-\rho^2)}c, \quad \rho^2 < \alpha < 1; \tag{A.3}$$

*b) The function $f(\alpha)$ has the derivative*

$$f'(\alpha) = \operatorname{tr} Y\left(\frac{1}{(1-\alpha)^2}DD^{\mathrm{T}} - \frac{1}{\alpha^2}APA^{\mathrm{T}}\right),$$

*where $P$ and $Y$ are the solutions of the discrete Lyapunov equations*

$$\frac{1}{\alpha}APA^{\mathrm{T}} - P + \frac{1}{1-\alpha}DD^{\mathrm{T}} = 0 \tag{A.4}$$

*and*

$$\frac{1}{\alpha}A^{\mathrm{T}}YA - Y + C^{\mathrm{T}}C = 0, \tag{A.5}$$

*respectively.*

*c) The second derivative of the function $f(\alpha)$ is given by*

$$f''(\alpha) = 2\operatorname{tr} Y\left(\frac{1}{(1-\alpha)^3}DD^{\mathrm{T}} + \frac{1}{\alpha^3}A(P-X)A^{\mathrm{T}}\right),$$

*where $P$, $Y$, and $X$ satisfy the discrete Lyapunov Equations* (A.4), (A.5), *and*

$$\frac{1}{\alpha}AXA^{\mathrm{T}} - X + \frac{1}{(1-\alpha)^2}DD^{\mathrm{T}} - \frac{1}{\alpha^2}APA^{\mathrm{T}} = 0,$$

*respectively. Moreover, $f''(\alpha^*) > 0$ and $f''(\alpha)$ is monotonically increasing on the left and right of $\alpha^*$.*

**Proof of Lemma A.3.** a. Equation (6) can be written as

$$\left(\frac{1}{\sqrt{\alpha}}A\right)P\left(\frac{1}{\sqrt{\alpha}}A\right)^{\mathrm{T}} - P = -\frac{1}{1-\alpha}DD^{\mathrm{T}};$$

according to [10, Lemma 1.2.6], there exists a unique solution if and only if $\frac{1}{\sqrt{\alpha}}A$ is a Schur matrix: $|\lambda_i(\frac{1}{\sqrt{\alpha}}A)| < 1$, i.e., under the condition $\rho^2 < \alpha < 1$.

We estimate the value $f(\alpha) = \operatorname{tr} CP(\alpha)C^{\mathrm{T}}$ using Lemma A.2 with obvious changes:

$$f(\alpha) = \operatorname{tr} CP(\alpha)C^{\mathrm{T}} \geqslant \lambda_{\min}(C^{\mathrm{T}}C)\lambda_{\max}(P(\alpha))$$

$$\geqslant \frac{\|u^*D\|^2\lambda_{\min}(C^{\mathrm{T}}C)}{(1-\alpha)(1-\rho^2(A/\sqrt{\alpha}))} = \frac{\alpha}{(1-\alpha)(\alpha-\rho^2)}\|u^*D\|^2\lambda_{\min}(C^{\mathrm{T}}C).$$

Here, $u$ has the same meaning as in Lemma A.2 and the value $\|u^*D\|^2$ is positive by the controllability of the pair $(A/\sqrt{\alpha}, D)$. (It follows from the controllability of $(A, D)$.)

Now we show that the function $f(\alpha) = \operatorname{tr} CP(\alpha)C^{\mathrm{T}}$ is strictly convex on the interval $(\rho^2, 1)$. According to [10, Lemma 1.2.6], the solution of Eq. (A.4) can be explicitly represented as

$$P(\alpha) = \sum_{k=0}^{\infty}\left(\frac{1}{\sqrt{\alpha}}A\right)^k \frac{1}{1-\alpha}DD^{\mathrm{T}}\left(\frac{1}{\sqrt{\alpha}}A^{\mathrm{T}}\right)^k = \sum_{k=0}^{\infty} \underbrace{\frac{1}{(1-\alpha)\alpha^k}}_{g(\alpha,k)} \underbrace{A^kDD^{\mathrm{T}}(A^{\mathrm{T}})^k}_{H_k}.$$

But $H_k \succ 0$ and $g(\alpha, k) > 0$ for $0 < \alpha < 1$; therefore, on the interval $(\rho^2, 1)$ we have

$$P(\alpha) = \sum_{k=0}^{\infty} g(\alpha, k)H_k \succ 0$$

and

$$f(\alpha) = \operatorname{tr} P(\alpha)C^{\mathrm{T}}C > 0.$$

Direct calculations give

$$g'(\alpha, k) = \left(\frac{1}{1-\alpha} - \frac{k}{\alpha}\right) g(\alpha, k),$$

$$g''(\alpha, k) = \left(\left(\frac{1}{1-\alpha} - \frac{k}{\alpha}\right)^2 + \frac{1}{(1-\alpha)^2} + \frac{k}{\alpha^2}\right)g(\alpha, k) \geqslant \frac{1}{(1-\alpha)^2}g(\alpha, k).$$

(Here, differentiation is performed with respect to $\alpha$.) As a result,

$$f''(\alpha) = \sum_{k=0}^{\infty} g''(\alpha, k)\operatorname{tr} CH_kC^{\mathrm{T}} \geqslant \frac{1}{(1-\alpha)^2}f(\alpha) \geqslant \frac{1}{(1-\rho^2)^2}f(\alpha^*) > 0.$$

Thus, the second derivative of the function $f(\alpha)$ is positive and tends to infinity at the endpoints of the interval $(\rho^2, 1)$.

Next, with direct calculations of the fourth derivative, we obtain

$$g^{(IV)}(\alpha, k) = \sum_{k=0}^{\infty} \frac{k(k+1)(k+2)(k+3)}{\alpha^{k+4}} + \frac{24}{(1-\alpha)^4} \geqslant \frac{24}{(1-\alpha)^4},$$

so

$$f^{(IV)}(\alpha) = \sum_{k=0}^{\infty} g^{(IV)}(\alpha, k)\operatorname{tr} CH_kC^{\mathrm{T}}$$

$$\geqslant \frac{24}{(1-\alpha)^4}\sum_{k=0}^{\infty}\operatorname{tr} CH_kC^{\mathrm{T}} > \frac{24}{(1-\rho^2)^4}\sum_{k=0}^{\infty}\operatorname{tr} CH_kC^{\mathrm{T}} > 0,$$

i.e., the second derivative $f''(\alpha)$ is convex and grows at the interval endpoints.

b. Let us derive the formula for the derivative of $f(\alpha)$. In Eq. (A.4), the solution $P$ is a function of $\alpha$. We differentiate this equation, interpreting $P'$ as the derivative with respect to $\alpha$:

$$\frac{1}{\alpha}AP'A^{\mathrm{T}} - P' + \frac{1}{(1-\alpha)^2}DD^{\mathrm{T}} - \frac{1}{\alpha^2}APA^{\mathrm{T}} = 0. \tag{A.6}$$

Applying Lemma A.1 to the dual Eqs. (A.6) and (A.5) finally yields

$$f'(\alpha) = \operatorname{tr} CP'C^{\mathrm{T}} = \operatorname{tr} P'C^{\mathrm{T}}C = \operatorname{tr} Y\left(\frac{1}{(1-\alpha)^2}DD^{\mathrm{T}} - \frac{1}{\alpha^2}APA^{\mathrm{T}}\right).$$

c. The desired expression for the second derivative of $f(\alpha)$ can be established by analogy. Differentiating Eq. (A.6) with respect to $\alpha$, we have

$$\frac{1}{\alpha}AP''A^{\mathrm{T}} - P'' + \frac{2}{(1-\alpha)^3}DD^{\mathrm{T}} + \frac{2}{\alpha^3}APA^{\mathrm{T}} - \frac{2}{\alpha^3}AP'A^{\mathrm{T}} = 0.$$

Applying Lemma A.1 to this equation and Eq. (A.5) with $X = P'$, we arrive at

$$f''(\alpha) = \operatorname{tr} CP''C^{\mathrm{T}} = \operatorname{tr} P''C^{\mathrm{T}}C = 2\operatorname{tr} Y\left(\frac{1}{(1-\alpha)^3}DD^{\mathrm{T}} + \frac{1}{\alpha^3}A(P-X)A^{\mathrm{T}}\right).$$

The proof of Lemma A.3 is complete.

Note that the function $f(\alpha)$ and its two derivatives are calculated by solving three discrete Lyapunov equations.

Due to the above properties, this function can be minimized using Newton's method. We specify an initial approximation $\rho^2(A) < \alpha_0 < 1$, e.g., $\alpha_0 = \left(1 + \rho^2(A)\right)/2$, and apply the iterative process

$$\alpha_{j+1} = \alpha_j - \frac{f'(\alpha_j)}{f''(\alpha_j)}. \tag{A.7}$$

The next theorem ensures the global convergence of the algorithm; it can be proved by analogy with a similar result in [16].

**Theorem A.1** [16]. *In the method* (A.7), *we have the upper bounds*

$$|\alpha_j - \alpha^*| \leqslant \frac{f''(\alpha_0)}{2^j f''(\alpha^*)}|\alpha_0 - \alpha^*|, \qquad |\alpha_{j+1} - \alpha^*| \leqslant c|\alpha_j - \alpha^*|^2,$$

*where $c > 0$ is some constant* (*possibly, in explicit form*).

The first bound ensures the global convergence of the method (faster than a geometric progression with a coefficient of 1/2); the second bound, the quadratic convergence in the neighborhood of the solution. In practice, it takes 3–4 iterations to obtain a solution with a high accuracy (unless the starting point is too close to the interval endpoints).

Returning to the optimization problem (4)–(5), we minimize the function

$$f(L) = \min_{\alpha} f(L, \alpha)$$

after a preliminary study of its properties.

**Lemma A.4.** *The function $f(L)$ is well-defined and positive on the set $\mathcal{S}$ of admissible filter matrices.*

Indeed, if $(A - LC)$ is a Schur matrix, then $\rho(A - LC) < 1$ and for $\rho^2(A - LC) < \alpha < 1$, there exists a solution $P \succcurlyeq 0$ of the discrete Lyapunov Equation (5). Thus, a strictly positive function $f(L, \alpha)$ is well-defined and $f(L) > 0$ due to (A.3). As in the continuous-time case, its domain $\mathcal{S}$ may be nonconvex and disconnected and its boundaries nonsmooth.

**Lemma A.5.** *On the set $\mathcal{S}$ the function $f(L)$ is coercive (i.e., it tends to infinity on the boundary of the domain). Moreover, the following lower bounds are valid:*

$$f(L) \geqslant \frac{1}{1 - \rho^2(A - LC)} \frac{\lambda_{\min}(C_1 C_1^{\mathrm{T}})}{1 - \sigma^2_{\min}(A - LC)} \|D_1 - LD_2\|_F^2, \tag{A.8}$$

$$f(L) \geqslant \rho\|L\|^2.$$

**Proof of Lemma A.5.** We consider a sequence $\{L_j\} \subseteq \mathcal{S}$ of admissible matrices such that $L_j \to L \in \partial\mathcal{S}$, i.e., $\rho(A - LC) = 1$. In other words, for any $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$|\rho(A - L_j C) - \rho(A - LC)| = 1 - \rho(A - L_j C) < \varepsilon$$

for all $j \geqslant N(\varepsilon)$.

Let $P_j$ be the solution of Eq. (5) associated with the filter matrix $L_j$:

$$\frac{1}{\alpha_j}(A - L_j C) P_j (A - L_j C)^{\mathrm{T}} - P_j + \frac{1}{1 - \alpha_j}(D_1 - L_j D_2)(D_1 - L_j D_2)^{\mathrm{T}} = 0;$$

let $Y_j$ be the solution of its dual discrete Lyapunov equation

$$\frac{1}{\alpha_j}(A - L_j C)^{\mathrm{T}} Y_j (A - L_j C) - Y_j + C_1 C_1^{\mathrm{T}} = 0.$$

In view of Lemma A.2, we have

$$\begin{aligned}
f(L_j) &= \mathrm{tr}\,(C_1 P_j C_1^{\mathrm{T}}) + \rho\|L_j\|_F^2 \geqslant \mathrm{tr}\,\left(P_j C_1 C_1^{\mathrm{T}}\right) \\
&= \mathrm{tr}\left(Y_j \frac{1}{1 - \alpha_j}(D_1 - L_j D_2)(D_1 - L_j D_2)^{\mathrm{T}}\right) \\
&\geqslant \frac{1}{1 - \alpha_j}\lambda_{\min}(Y_j)\|D_1 - L_j D_2\|_F^2 \geqslant \frac{1}{1 - \alpha_j}\frac{\lambda_{\min}(C_1 C_1^{\mathrm{T}})}{1 - \sigma^2_{\min}(A - L_j C)}\|D_1 - L_j D_2\|_F^2 \\
&\geqslant \frac{1}{1 - \rho^2(A - L_j C)} \frac{\lambda_{\min}(C_1 C_1^{\mathrm{T}})}{1 - \sigma^2_{\min}(A - L_j C)}\|D_1 - L_j D_2\|_F^2 \\
&\geqslant \frac{1}{\varepsilon}\frac{\lambda_{\min}(C_1 C_1^{\mathrm{T}})}{1 - \sigma^2_{\min}(A - L_j C)}\|D_1 - L_j D_2\|_F^2 \xrightarrow[\varepsilon \to 0]{} +\infty
\end{aligned}$$

since $\rho^2(A - L_j C) < \alpha_j < 1$.

On the other hand,

$$f(L_j) = \mathrm{tr}\,(C_1 P_j C_1^{\mathrm{T}}) + \rho\|L_j\|_F^2 \geqslant \rho\|L_j\|_F^2 \geqslant \rho\|L_j\|^2 \xrightarrow[\|L_j\| \to +\infty]{} +\infty.$$

The proof of Lemma A.5 is complete.

We introduce the level set

$$\mathcal{S}_0 = \{L \in \mathcal{S}\colon \quad f(L) \leqslant f(L_0)\}.$$

Obviously, Lemma A.5 implies the following result.

**Corollary A.1.** *For any $L_0 \in \mathcal{S}$, the set $\mathcal{S}_0$ is bounded.*

On the other hand, the function $f(L)$ achieves minimum on the set $\mathcal{S}_0$. (This function is continuous by the properties of the solution of the discrete Lyapunov equation and is considered on a compact set.) However, the set $\mathcal{S}_0$ has no common points with the boundary of $\mathcal{S}$ due to (A.8). The function $f(L)$ is differentiable on $\mathcal{S}_0$; see below. Consequently, we arrive at the following result.

**Corollary A.2.** *There exists a minimum point $L_*$ on the set $\mathcal{S}$, and the gradient of the function $f(L)$ vanishes at this point.*

Let us analyze the properties of the gradient of the function $f(L,\alpha)$.

**Lemma A.6.** *The function $f(L,\alpha)$ is defined on the set of stabilizing $L$ for $\rho^2(A-LC)<\alpha<1$. On this admissible set, the function is differentiable, and its gradient is given by*

$$\nabla_\alpha f(L,\alpha) = \operatorname{tr} Y\left(\frac{1}{(1-\alpha)^2}(D_1-LD_2)(D_1-LD_2)^{\mathrm{T}} - \frac{1}{\alpha^2}(A-LC)P(A-LC)^{\mathrm{T}}\right), \tag{A.9}$$

$$\nabla_L f(L,\alpha) = 2\left(\rho L - \frac{1}{\alpha}Y(A-LC)PC^{\mathrm{T}} - \frac{1}{1-\alpha}Y(D_1-LD_2)D_2^{\mathrm{T}}\right), \tag{A.10}$$

*where the matrices $P$ and $Y$ are the solutions of the discrete Lyapunov Equations (5) and (8), respectively.*

*The minimum of $f(L,\alpha)$ is achieved at an inner point of the admissible set and is determined by the conditions*

$$\nabla_L f(L,\alpha) = 0, \quad \nabla_\alpha f(L,\alpha) = 0.$$

*In addition, $f(L,\alpha)$ as a function of $\alpha$ is strictly convex on $\rho^2(A-LC)<\alpha<1$ and achieves minimum at an inner point of this interval.*

**Proof of Lemma A.6.** We have the constrained optimization problem

$$\min f(L,\alpha), \quad f(L,\alpha) = \operatorname{tr} C_1 P C_1^{\mathrm{T}} + \rho\|L\|_F^2$$

subject to the discrete Lyapunov Equation (5) for the matrix $P$ of the invariant ellipsoid.

Following Lemma A.3, differentiation with respect to $\alpha$ is performed using the relations (A.9), (5), and (8). To differentiate with respect to $L$, we add an increment $\Delta L$ and denote by $\Delta P$ the corresponding increment of $P$. As a result, the relation (5) takes the form

$$\begin{aligned}\frac{1}{\alpha}\left(A-(L+\Delta L)C\right)(P+\Delta P)\left(A-(L+\Delta L)C\right)^{\mathrm{T}} - (P+\Delta P)\\ +\frac{1}{1-\alpha}\left(D_1-(L+\Delta L)D_2\right)\left(D_1-(L+\Delta L)D_2\right)^{\mathrm{T}} = 0.\end{aligned}$$

Leaving the notation $\Delta P$ for the principal terms of the increment, we obtain

$$\begin{aligned}&\frac{1}{\alpha}\left((A-LC)P(A-LC)^{\mathrm{T}} - \Delta LCP(A-LC)^{\mathrm{T}} - (A-LC)P(\Delta LC)^{\mathrm{T}} + (A-LC)\Delta P(A-LC)^{\mathrm{T}}\right)\\ &-(P+\Delta P) + \frac{1}{1-\alpha}\left((D_1-LD_2)(D_1-LD_2)^{\mathrm{T}} - \Delta LD_2(D_1-LD_2)^{\mathrm{T}} - (D_1-LD_2)(\Delta LD_2)^{\mathrm{T}}\right) = 0.\end{aligned}$$

Subtracting Eq. (12) from this equation yields

$$\begin{aligned}&\frac{1}{\alpha}(A-LC)\Delta P(A-LC)^{\mathrm{T}}\\ &-\Delta P - \frac{1}{\alpha}\left(\Delta LCP(A-LC)^{\mathrm{T}} + (A-LC)P(\Delta LC)^{\mathrm{T}}\right)\\ &-\frac{1}{1-\alpha}\left(\Delta LD_2(D_1-LD_2)^{\mathrm{T}} + (D_1-LD_2)(\Delta LD_2)^{\mathrm{T}}\right) = 0.\end{aligned} \tag{A.11}$$

We calculate the increment of the functional $f(L)$ by linearizing the corresponding values:

$$\Delta f(L) = \operatorname{tr} C_1 \Delta P C_1^{\mathrm{T}} + \rho \operatorname{tr} L^{\mathrm{T}} \Delta L + \rho \operatorname{tr} (\Delta L)^{\mathrm{T}} L = \operatorname{tr} \Delta P C_1^{\mathrm{T}} C_1 + 2\rho \operatorname{tr} L^{\mathrm{T}} \Delta L.$$

By Lemma B.1, from the dual Eqs. (A.11) and (8) we have

$$\begin{aligned}\Delta f(L) &= -2\operatorname{tr} Y \left( \frac{1}{\alpha} \Delta L C P (A - LC)^{\mathrm{T}} + \frac{1}{1-\alpha} \Delta L D_2 (D_1 - L D_2)^{\mathrm{T}} \right) + 2\rho \operatorname{tr} L^{\mathrm{T}} \Delta L \\ &= 2\operatorname{tr} \left( \rho L^{\mathrm{T}} \Delta L - \frac{1}{\alpha} C P (A - LC)^{\mathrm{T}} Y - \frac{1}{1-\alpha} D_2 (D_1 - L D_2)^{\mathrm{T}} Y \right) \Delta L \\ &= \left\langle 2 \left( \rho L - \frac{1}{\alpha} Y (A - LC) P C^{\mathrm{T}} - \frac{1}{1-\alpha} Y (D_1 - L D_2) D_2^{\mathrm{T}} \right), \Delta L \right\rangle.\end{aligned}$$

Thus, the relation (A.10) is derived and the proof of Lemma A.6 is complete.

The gradient of the function $f(L)$ is not Lipschitz on the set $\mathcal{S}$. But it can be shown to possess this property on the subset $\mathcal{S}_0$ (similar to [16]).

These properties of the minimized function and its derivatives justify the minimization method implemented as Algorithm 1.

*APPENDIX B*

**Lemma B.1** [16]. *Let $X$ and $Y$ be the solutions of the dual Lyapunov equations with a Hurwitz matrix $A$:*

$$A^{\mathrm{T}} X + X A + W = 0 \quad \text{and} \quad A Y + Y A^{\mathrm{T}} + V = 0.$$

*Then* $\operatorname{tr}(XV) = \operatorname{tr}(YW)$.

The properties of the function $f(\alpha)$ established in [16] fully apply to the case under consideration. In particular, the function $f(\alpha)$ is well-defined, positive, and strongly convex on the interval $0 < \alpha < 2\sigma(A - LC)$ and its values tend to infinity at the interval endpoints. Moreover, there exists a constant $c > 0$ such that

$$f(\alpha) \geqslant \frac{c}{\alpha(2\sigma - \alpha)}, \quad 0 < \alpha < 2\sigma(A - LC). \tag{B.1}$$

The function $f(\alpha)$ can be effectively minimized using Newton's method. We specify an initial approximation $0 < \alpha_0 < 2\sigma(A - LC)$, e.g., $\alpha_0 = \sigma(A - LC)$, and apply the iterative process

$$\alpha_{j+1} = \alpha_j - \frac{f'(\alpha_j)}{f''(\alpha_j)},$$

where, according to [16],

$$\begin{aligned} f'(\alpha) &= \operatorname{tr} Y \left( P - \frac{1}{\alpha^2} (D_1 - L D_2)(D_1 - L D_2)^{\mathrm{T}} \right), \\ f''(\alpha) &= 2\operatorname{tr} Y \left( X + \frac{1}{\alpha^3} (D_1 - L D_2)(D_1 - L D_2)^{\mathrm{T}} \right), \end{aligned} \tag{B.2}$$

and $P$, $Y$, and $X$ are the solutions of the Lyapunov Equations (12), (13), and (14), respectively. Theorem A.1 remains valid as well.

The following lemma is a continuous-time analog of Lemma A.4.

**Lemma B.2.** *The function $f(L)$ is well-defined and positive on the set $\mathcal{S}$ of admissible filter matrices.*

Indeed, if $(A - LC)$ is a Hurwitz matrix, then $\sigma(A - LC) > 0$ and for $0 < \alpha < 2\sigma(A - LC)$, there exists a solution $P \succcurlyeq 0$ of the Lyapunov Equation (12). Thus, a strictly positive function $f(L, \alpha)$ is well-defined and $f(L) > 0$ due to (B.1). Its domain $\mathcal{S}$ may be nonconvex and disconnected and its boundaries nonsmooth; see [16].

**Lemma B.3.** *On the set $\mathcal{S}$ of admissible matrices the function $f(L)$ is coercive (i.e., it tends to infinity on the boundary of the domain). Moreover, the following lower bounds are valid:*

$$f(L) \geqslant \frac{\lambda_{\min}(C_1C_1^{\mathrm{T}})\|D_1 - LD_2\|_F^2}{4\sigma(A - LC)\left(\|A - LC\| + \sigma(A - LC)\right)}, \tag{B.3}$$
$$f(L) \geqslant \rho\|L\|^2.$$

**Proof of Lemma B.3.** We consider a sequence $\{L_j\} \subseteq \mathcal{S}$ of admissible matrices such that $L_j \to L \in \partial\mathcal{S}$, i.e., $\sigma(A - LC) = 0$. In other words, for any $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$|\sigma(A - L_jC) - \sigma(A - LC)| = \sigma(A - L_jC) < \varepsilon$$

for all $j \geqslant N(\varepsilon)$.

Let $P_j$ be the solution of Eq. (12) associated with the filter matrix $L_j$:

$$\left(A - L_jC + \frac{\alpha_j}{2}I\right)P_j + P_j\left(A - L_jC + \frac{\alpha_j}{2}I\right)^{\mathrm{T}} + \frac{1}{\alpha_j}(D_1 - L_jD_2)(D_1 - L_jD_2)^{\mathrm{T}} = 0;$$

let $Y_j$ be the solution of its dual Lyapunov equation

$$\left(A - L_jC + \frac{\alpha_j}{2}I\right)^{\mathrm{T}}Y_j + Y_j\left(A - L_jC + \frac{\alpha_j}{2}I\right) + C_1C_1^{\mathrm{T}} = 0.$$

In view of [16, Lemma A.3], we have

$$\begin{aligned} f(L_j) &= \operatorname{tr}(C_1P_jC_1^{\mathrm{T}}) + \rho\|L_j\|_F^2 \\ &\geqslant \operatorname{tr}(P_jC_1C_1^{\mathrm{T}}) = \operatorname{tr}\left(Y_j\frac{1}{\alpha_j}(D_1 - L_jD_2)(D_1 - L_jD_2)^{\mathrm{T}}\right) \\ &\geqslant \frac{1}{\alpha_j}\lambda_{\min}(Y_j)\|D_1 - L_jD_2\|_F^2 \geqslant \frac{1}{\alpha_j}\frac{\lambda_{\min}(C_1C_1^{\mathrm{T}})}{2\|A - L_jC + \frac{\alpha_j}{2}I\|}\|D_1 - L_jD_2\|_F^2 \\ &\geqslant \frac{1}{4\sigma(A - L_jC)}\frac{\lambda_{\min}(C_1C_1^{\mathrm{T}})}{\|A - L_jC + \frac{\alpha_j}{2}I\|}\|D_1 - L_jD_2\|_F^2 \\ &\geqslant \frac{\lambda_{\min}(C_1C_1^{\mathrm{T}})}{4\varepsilon(\|A - L_jC\| + \varepsilon)}\|D_1 - L_jD_2\|_F^2 \xrightarrow[\varepsilon \to 0]{} +\infty, \end{aligned}$$

since $0 < \alpha_j < 2\sigma(A - L_jC)$ and

$$\|A - L_jC + \frac{\alpha_j}{2}I\| \leqslant \|A - L_jC\| + \frac{\alpha_j}{2}.$$

On the other hand,

$$f(L_j) = \operatorname{tr}(C_1P_jC_1^{\mathrm{T}}) + \rho\|L_j\|_F^2 \geqslant \rho\|L_j\|_F^2 \geqslant \rho\|L_j\|^2 \xrightarrow[\|L_j\| \to +\infty]{} +\infty.$$

The proof of Lemma B.3 is complete.

We introduce the level set

$$\mathcal{S}_0 = \{L \in \mathcal{S}: \quad f(L) \leqslant f(L_0)\}.$$

Obviously, Lemma B.3 implies the following result.

**Corollary B.3.** *For any $L_0 \in \mathcal{S}$, the set $\mathcal{S}_0$ is bounded.*

On the other hand, the function $f(L)$ achieves minimum on the set $\mathcal{S}_0$. (This function is continuous by the properties of the solution of the Lyapunov equation and is considered on a compact set.) However, the set $\mathcal{S}_0$ has no common points with the boundary of $\mathcal{S}$ due to (B.3). The function $f(L)$ is differentiable on $\mathcal{S}_0$; see below. Consequently, we arrive at the following result.

**Corollary B.4.** *There exists a minimum point $L_*$ on the set $\mathcal{S}$, and the gradient of the function $f(L)$ vanishes at this point.*

Let us analyze the properties of the gradient of the function $f(L,\alpha)$.

**Lemma B.4.** *The function $f(L,\alpha)$ is defined on the set of stabilizing $L$ for $0 < \alpha < 2\sigma(A - LC)$. On this admissible set, the function is differentiable, and its gradient is given by*

$$\begin{aligned}
\nabla_\alpha f(L,\alpha) &= \operatorname{tr} Y \left(P - \frac{1}{\alpha^2}(D_1 - LD_2)(D_1 - LD_2)^{\mathrm{T}}\right), \\
\nabla_L f(L,\alpha) &= 2\left(\rho L - YPC^{\mathrm{T}} - \frac{1}{\alpha} Y (D_1 - LD_2) D_2^{\mathrm{T}}\right),
\end{aligned} \tag{B.4}$$

*where the matrices $P$ and $Y$ are the solutions of the Lyapunov Equations* (12) *and* (13), *respectively.*

*The minimum of $f(L,\alpha)$ is achieved at an inner point of the admissible set and is determined by the conditions*

$$\nabla_L f(L,\alpha) = 0, \quad \nabla_\alpha f(L,\alpha) = 0.$$

*In addition, $f(L,\alpha)$ as a function of $\alpha$ is strictly convex on $0 < \alpha < 2\sigma(A - LC)$ and achieves minimum at an inner point of this interval.*

**Proof of Lemma B.4.** We have the constrained optimization problem

$$\min f(L,\alpha), \quad f(L,\alpha) = \operatorname{tr} C_1 P C_1^{\mathrm{T}} + \rho \|L\|_F^2$$

subject to the Lyapunov Equation (12) for the matrix $P$ of the invariant ellipsoid.

Differentiation with respect to $\alpha$ is performed using the relations (B.2), (12), and (13). To differentiate with respect to $L$, we add an increment $\Delta L$ and denote by $\Delta P$ the corresponding increment of $P$. As a result, the relation (12) takes the form

$$\begin{aligned}
\left(A - (L + \Delta L)C + \frac{\alpha}{2} I\right)(P + \Delta P) + (P + \Delta P)\left(A - (L + \Delta L)C + \frac{\alpha}{2} I\right)^{\mathrm{T}} \\
+ \frac{1}{\alpha}\left(D_1 - (L + \Delta L)D_2\right)\left(D_1 - (L + \Delta L)D_2\right)^{\mathrm{T}} = 0.
\end{aligned}$$

Leaving the notation $\Delta P$ for the principal terms of the increment, we obtain

$$\begin{aligned}
\left(A - (L + \Delta L)C + \frac{\alpha}{2} I\right) P + P\left(A - (L + \Delta L)C + \frac{\alpha}{2} I\right)^{\mathrm{T}} \\
+ \left(A - LC + \frac{\alpha}{2} I\right)\Delta P + \Delta P\left(A - LC + \frac{\alpha}{2} I\right)^{\mathrm{T}} \\
+ \frac{1}{\alpha}\left((D_1 - LD_2)(D_1 - LD_2)^{\mathrm{T}} - \Delta L D_2 (D_1 - LD_2)^{\mathrm{T}} - (D_1 - LD_2)(\Delta L D_2)^{\mathrm{T}}\right) = 0.
\end{aligned}$$

Subtracting Eq. (12) from this equation yields

$$\left(A - LC + \frac{\alpha}{2}I\right)\Delta P + \Delta P\left(A - LC + \frac{\alpha}{2}I\right)^{\mathrm{T}} - \Delta LCP - P(\Delta LC)^{\mathrm{T}} \tag{B.5}$$

$$-\frac{1}{\alpha}\left(\Delta LD_2(D_1 - LD_2)^{\mathrm{T}} + (D_1 - LD_2)(\Delta LD_2)^{\mathrm{T}}\right) = 0. \tag{B.6}$$

We calculate the increment of the functional $f(L)$ by linearizing the corresponding values:

$$\Delta f(L) = \mathrm{tr}\, C_1 \Delta P C_1^{\mathrm{T}} + \rho \mathrm{tr}\, L^{\mathrm{T}}\Delta L + \rho \mathrm{tr}\,(\Delta L)^{\mathrm{T}} L = \mathrm{tr}\,\Delta P C_1^{\mathrm{T}} C_1 + 2\rho \mathrm{tr}\, L^{\mathrm{T}}\Delta L.$$

By Lemma B.1, from the dual Eqs. (B.6) and (13) we have

$$\begin{aligned}\Delta f(L) &= -\mathrm{tr}\, 2Y\left(\Delta LCP + \frac{1}{\alpha}\Delta L D_2(D_1 - LD_2)^{\mathrm{T}}\right) + 2\rho \mathrm{tr}\, L^{\mathrm{T}}\Delta L \\ &= 2\mathrm{tr}\left(\rho L^{\mathrm{T}}\Delta L - CPY\Delta L - \frac{1}{\alpha}D_2(D_1 - LD_2)^{\mathrm{T}} Y\Delta L\right) \\ &= \left\langle 2\left(\rho L - YPC^{\mathrm{T}} - \frac{1}{\alpha}Y(D_1 - LD_2)D_2^{\mathrm{T}}\right), \Delta L\right\rangle.\end{aligned}$$

Thus, the relation (B.4) is derived and the proof of Lemma B.4 is complete.

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