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# Control of Dynamic Systems under Input and Output Constraints

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**Abstract**—This paper extends the method originally proposed in [1] to systems with an arbitrary number of inputs and outputs. The method ensures that these signals will be in given domains. Two sequential changes of coordinates are introduced to solve the problem. The first change reduces the plant's output to a new variable of a dimension not exceeding that of the control vector (input). The second change allows passing from the control problem with constraints to the one without them. The effectiveness of this method is illustrated for two problems. The first problem is designing a state-feedback controller for linear systems with constraints imposed on the input and state variables. The second problem is designing an output-feedback controller for linear systems with constraints imposed on the output and input. In both problems, the stability of the closed loop system is verified in terms of linear matrix inequalities. The results are accompanied by simulation examples to show the effectiveness of the proposed method.

*Keywords*: dynamic system, the change of coordinates, stability, control

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## 1. INTRODUCTION

In [1, 2], control methods ensuring that the plant's output will be in a given domain were surveyed. Also, the papers [1, 2] proposed a new solution of this problem: a special change of coordinates was applied to pass from the control problem with constraints to the one without them.

However, the solution [1, 2] and most of the literature described in [1, 2] are limited to plants with the following features:

1) The dimension of the control signal (input) is not smaller than that of the controlled signal (output).

2) No constraints are imposed on the input.

Regarding the first limitation, there are many applications where the control signal has a smaller dimension than the controlled signal. For example, we mention control of partially driven systems: walking robots, numerically controlled machines, aircraft and watercraft, some pendulum systems, etc. Many methods were developed to solve such problems, in particular [3–6], but they provide a given control performance level in the steady-state mode only, not at any time.

Regarding the second limitation, the following question arises naturally: what magnitude is needed for the input to ensure that the output will be in a given domain at any time?

In this paper, we extend the approach [1, 2] to solve three problems as follows:

1) design a controller for plants in which the input has a smaller dimension than the output;

2) ensure that the output and input will be in a given domain at any time;

3) use linear matrix inequalities (LMIs) to analyze the stability of the closed loop system and design the controller's parameters.

This paper is organized as follows. Section 2 poses a general control problem with ensuring that the input and output will be in a given domain at any time. In Section 3, two changes of coordinates are proposed. The first change reduces the plant's output to a new variable of a dimension not exceeding that of the input. The second change allows passing from the control problem with constraints to the one without them. In Section 4, we apply the result of Section 3 to design a state-feedback controller for linear systems with constraints imposed on the input and state variables. In Section 5, we design an output-feedback controller for linear systems with constraints imposed on the output and input. The results are accompanied by simulation examples to show the effectiveness of the proposed method.

This paper involves the following *notations and definitions*: $\mathbb{R}^n$ is the Euclidean $n$-dimensional space with the norm $|\cdot|$; $\mathbb{R}^{n\times m}$ is the set of all real matrices of dimensions $n \times m$; $P > 0$ is a symmetric positive definite matrix; $\lambda_{\min}(P)$ is the smallest eigenvalue of a matrix $P > 0$; $p = d/dt$; finally, $*$ is a symmetric block in a symmetric matrix.

**Definition 1** [3, 7]. A continuous function $\alpha : [0, a) \to [0; \infty)$ belongs to the class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.

## 2. PROBLEM STATEMENT

We consider the dynamic system

$$\begin{aligned} \dot{x} &= F(x, u, t), \\ y &= H(x, u, t), \end{aligned} \tag{1}$$

where $t \geqslant 0$ and $x \in \mathbb{R}^n$, $u \in \mathcal{U} \subset \mathbb{R}^m$, and $y = col\{y_1, \ldots, y_l\} \in \mathcal{Y} \subset \mathbb{R}^l$ denote the state vector, control signal (input) and controlled signal (output), respectively. The functions $F$ and $H$ are defined for all $x$, $u$, and $t$; $F$ is supposed to be piecewise continuous and bounded in $t$, whereas $H$ is supposed to be continuously differentiable with respect to all arguments and bounded in $t$. Let the plant (1) be controllable and observable for any $x \in \mathbb{R}^n$.

In contrast to [1], this paper does not impose the constraint $\dim u \geqslant \dim y$: the signals $u$ and $y$ have arbitrary dimensions. This constraint is eliminated by the change of coordinates

$$\xi = G(y, u, t), \tag{2}$$

where the dimension of $\xi$ does not exceed that of the input, i.e., $\xi = col\{\xi_1, \ldots, \xi_v\}$, $v \leqslant m$, and the function $G$ is continuously differentiable with respect to all arguments and bounded in $t$.

*Remark 1.* In fact, we can choose a more general class of functions than the class of continuously differentiable functions $G$. For this purpose, we recall the following definition with some comments below.

**Definition 2** [8]. A function $u \to G(y, u, t)$ is said to be differentiable at a point $u \in \mathbb{R}^m$ in a direction $v \in \mathbb{R}^m$ if there exists the finite limit

$$G'_u(y, u, t; v) = \lim_{\alpha \to +0} \frac{G(y, u + \alpha v, t) - G(y, u, t)}{\alpha}.$$

If a function $u \to G(y, u, t)$ is differentiable at a point $u$ in any direction $v \in \mathbb{R}^m$, it is said to be differentiable in the directions at this point and the function $v \to G'_u(y, u, t; v)$ is called the derivative of $u \to G(y, u, t)$ in the directions.

Let the function $G(y, u, t)$ have the local Lipschitz property, be continuously differentiable with respect to $y$ and $t$, and be differentiable in the directions at $u$. Then, for any absolutely continuous functions $y(t)$ and $u(t)$, the function $G(y(t), u(t), t)$ is absolutely continuous as well. Furthermore, by the differentiation rule of a complex function in a direction (see [8]), we have

$$\frac{d}{dt}G(y(t), u(t), t) = \frac{\partial G}{\partial y}\dot{y} + G'_u(y(t), u(t), t; \dot{u}(t)) + \frac{\partial G}{\partial t}$$

for almost all $t \geqslant 0$. Here, $G'_u(y(t), u(t), t; \dot{u}(t))$ is the derivative of the function $u \to G(y(t), u, t)$ at a point $u(t)$ in the direction $\dot{u}(t)$.

Thus, whenever the function $G$ is not differentiable with respect to $u$ but is differentiable in the directions at $u$, the value $\frac{\partial G(y(t),u(t),t)}{\partial u}\dot{u}(t)$ in all considerations below can be replaced by $G'_u(y(t), u(t), t; \dot{u}(t))$.

The problem is to design a controller ensuring the condition

$$\underline{g}_i(t) < \xi_i(t) < \overline{g}_i(t), \ i = 1, \ldots, v \tag{3}$$

for all $t$. The function $G$ and the continuously differentiable functions $\underline{g}_i(t)$ and $\overline{g}_i(t)$ are chosen to satisfy the constraints $u \in \mathcal{U}$ and $y \in \mathcal{Y}$. Therefore, the constraints (3) together with the transformation (2) must include the input and output constraints for the plant (1). By choosing the functions $\underline{g}_i(t)$ and $\overline{g}_i(t)$, we can also specify different configurations of the domain for the transients in $\xi_i(t)$.

To summarize: in this paper, we design a controller law ensuring condition (3). On the other hand, the definition of the variable $\xi$ determines the behavior of the signals $y$ and $u$. In Sections 4 and 5, the function $G$ is specified as a quadratic form; in combination with (3), this definition requires the variables $y$ and $u$ to be in some domain whose projections on the coordinate planes are rings with variable radii depending on $\underline{g}_i(t)$ and $\overline{g}_i(t)$. (See the details below.) In Remarks 5 and 7, we give examples of alternative changes (2), which lead to other shapes of the domains for $y$ and $u$.

## 3. THE SOLUTION METHOD

Following [1], we introduce the change of variable

$$\xi(t) = \Phi(\varepsilon(t), t), \tag{4}$$

where $\varepsilon(t) \in \mathbb{R}^v$ is a continuously differentiable function and $\Phi(\varepsilon, t) = col\{\Phi_1(\varepsilon, t), \ldots, \Phi_v(\varepsilon, t)\}$ satisfies several conditions:

(a) $\underline{g}_i(t) < \Phi_i(\varepsilon, t) < \overline{g}_i(t)$, $i = 1, \ldots, v$ for any $t \geqslant 0$ and any $\varepsilon \in \mathbb{R}^v$.

(b) There exists an inverse mapping $\varepsilon = \Phi^{-1}(\xi, t)$ for any $\xi$ from (3) and any $t \geqslant 0$.

(c) The function $\Phi(\varepsilon, t)$ is continuously differentiable with respect to $\varepsilon$ and $t$, and $\det\left(\frac{\partial \Phi(\varepsilon, t)}{\partial \varepsilon}\right) \neq 0$ for any $\varepsilon \in \mathbb{R}^v$ and any $t \geqslant 0$.

(d) $\left|\frac{\partial \Phi(\varepsilon, t)}{\partial t}\right| \leqslant \gamma > 0$ for any $\varepsilon \in \mathbb{R}^v$ and any $t \geqslant 0$. The value $\gamma$ is known since (4) is specified by the control designer.

Information on the dynamics of $\varepsilon(t)$ is necessary to construct the controller. We find the total time derivatives of $y(t)$ and $\xi(t)$ along the trajectories of (1), (2), (4) and equate the results for (2) and (4):

$$\begin{aligned}
&\dot{y} = \frac{\partial H}{\partial x}F + \frac{\partial H}{\partial u}\dot{u} + \frac{\partial H}{\partial t}, \\
&\frac{\partial \Phi(\varepsilon,t)}{\partial \varepsilon}\dot{\varepsilon} + \frac{\partial \Phi(\varepsilon,t)}{\partial t} = \frac{\partial G}{\partial y}\dot{y} + \frac{\partial G}{\partial u}\dot{u} + \frac{\partial G}{\partial t}.
\end{aligned} \tag{5}$$

Considering condition (c), we express $\dot{\varepsilon}$ from (5) as

$$\dot{\varepsilon} = \left(\frac{\partial \Phi(\varepsilon,t)}{\partial \varepsilon}\right)^{-1}\left[\frac{\partial G}{\partial y}\frac{\partial H}{\partial x}F + \left(\frac{\partial G}{\partial y}\frac{\partial H}{\partial u} + \frac{\partial G}{\partial u}\right)\dot{u} + \frac{\partial G}{\partial y}\frac{\partial H}{\partial t} + \frac{\partial G}{\partial t} - \frac{\partial \Phi(\varepsilon,t)}{\partial t}\right]. \tag{6}$$

*Remark 2.* If the state vector $x$ is unmeasurable, the controller can be designed using the expression

$$\dot{\varepsilon} = \left(\frac{\partial \Phi(\varepsilon,t)}{\partial \varepsilon}\right)^{-1}\left[\frac{\partial G}{\partial y}\dot{y} + \frac{\partial G}{\partial u}\dot{u} + \frac{\partial G}{\partial t} - \frac{\partial \Phi(\varepsilon,t)}{\partial t}\right] \tag{7}$$

instead of (6). It follows from the second expression in (5).

Formulas (6) and (7) are employed for controller design in the forthcoming sections. Now we provide the main result of this section.

**Theorem 1.** *Let the transformation* (4) *satisfy conditions* (a)–(d) *and* $\underline{g}_i(0) < \xi_i(0) < \overline{g}_i(0)$, $i = 1, \ldots, v$. *Assume also that there exist a stabilizing controller* $u = u(t, y, \varepsilon)$ *for system* (6) *or* (7), *with piecewise continuity in* $t$ *and the local Lipschitz property in* $y$ *and* $\varepsilon$, *and a Lyapunov function* $V(t, \varepsilon)$ *such that*

$$\begin{aligned}
&\alpha_1(|\varepsilon|) \leqslant V(t,\varepsilon) \leqslant \alpha_2(|\varepsilon|), \\
&\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \varepsilon}\left(\frac{\partial \Phi(\varepsilon,t)}{\partial \varepsilon}\right)^{-1}\left[\frac{\partial G}{\partial y}\dot{y} + \frac{\partial G}{\partial u}\dot{u} + \frac{\partial G}{\partial t} - \frac{\partial \Phi(\varepsilon,t)}{\partial t}\right] \leqslant -\alpha_3(|\varepsilon|),
\end{aligned} \tag{8}$$

*where* $\alpha_1$, $\alpha_2$, *and* $\alpha_3$ *are functions from the class* $\mathcal{K}$. *Then condition* (3) *holds.*

Due to conditions (8), the function $\varepsilon(t)$ is bounded in the limit. Then condition (a) implies condition (3).

*Remark 3.* Theorem 1 remains valid under weaker differentiability assumptions [9] for the function $G$, namely, its local Lipschitz property in all variables, continuous differentiability in $y$ and $t$, and differentiability in the directions at $u$. In this case, the second term on the right-hand side of the second equality in (5) is replaced by $G'_u(y, u, t; v)$; see Definition 2. Hence, equalities (5)–(7) hold for almost all $t$. As a result, $\frac{\partial G}{\partial u}\dot{u}$ in formulas (6) and (7) is replaced by the corresponding derivative in the directions, and all the considerations apply to almost all $t$. Therefore, in the proofs of Theorems 2 and 3 below, the corresponding (in)equalities for derivatives hold for almost all $t$.

Some particular changes (4) of coordinates were proposed in [1]. They differ and are not connected with each other. Here, we present a new change of coordinates to interconnect the transformations from [1] and obtain a series of new changes.

*Example 1.* In (4), let

$$\Phi(\varepsilon,t) = \frac{\overline{g}(t) - \underline{g}(t)}{2}T(\varepsilon) + \frac{\underline{g}(t) + \overline{g}(t)}{2}, \tag{9}$$

where $\varepsilon \in \mathbb{R}$ and $T(\varepsilon)$ is a strictly monotonic function such that $-1 < T(\varepsilon) < 1$ for any $\varepsilon$.

Compared to [1], the change (9) is advantageous: it allows separating the functions $\overline{g}(t)$, $\underline{g}(t)$, and $T(\varepsilon)$. The functions $\overline{g}(t)$ and $\underline{g}(t)$ define the desired domain for the controlled variable and are specified by the control designer. The function $T(\varepsilon)$ defines the change of coordinates. For example, $T(\varepsilon)$ can be chosen as

$$T(\varepsilon) = \frac{\varepsilon}{1+|\varepsilon|}, \quad T(\varepsilon) = \frac{e^{\varepsilon}-1}{e^{\varepsilon}+1} = \tanh(0.5\varepsilon), \quad T(\varepsilon) = \frac{2}{\pi}\arctan(\varepsilon), \ \ldots.$$

In what follows, we apply the proposed approach to the same models as in [1]. The results can be extended to the case of unknown model parameters by analogy with [2].

## 4. STATE-FEEDBACK CONTROLLER UNDER STATE AND INPUT CONSTRAINTS

We consider the plant

$$\dot{x} = Ax + Bu + Df, \tag{10}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$, $u \in \mathcal{U} \subset \mathbb{R}$, $f \in \mathbb{R}$, and $|f(t)| \leqslant \bar{f}$ for all $t$. The matrices $A$, $B$, and $D$ are known and have compatible dimensions. Let the pair $(A, B)$ be controllable. In this case, $y = x$. Assume that the sets $\mathcal{X}$ and $\mathcal{U}$ have the form

$$\mathcal{X} = \left\{x \in \mathbb{R}^n : \ x^{\mathrm{T}} P_x x \leqslant 1\right\}, \quad \mathcal{U} = \left\{u \in \mathbb{R} : \ p_u|u| \leqslant 1\right\}, \tag{11}$$

where $P_x > 0$ and $p_u > 0$ are specified by the control designer.

To compare the constraints imposed on $x$ and $u$, we introduce the changes of variables $\tilde{x} = \sqrt{\lambda_{\min}(P_x)}x$ and $\tilde{u} = p_u u$. They will serve for reducing proportionally $x^{\mathrm{T}} P_x x \leqslant 1$ to a new ellipsoid with unit semi-major axis and $p_u|u| \leqslant 1$ to a new segment of length 2. With these changes, we transform Eq. (10) and the sets $\mathcal{X}$ and $\mathcal{U}$ to

$$\begin{aligned} &\dot{\tilde{x}} = A\tilde{x} + \tilde{B}\tilde{u} + \tilde{D}f, \\ &\tilde{\mathcal{X}} = \left\{\tilde{x} \in \mathbb{R}^n : \ \frac{1}{\lambda_{\min}\{P_x\}}\tilde{x}^{\mathrm{T}} P_x \tilde{x} \leqslant 1\right\}, \quad \tilde{\mathcal{U}} = \left\{\tilde{u} \in \mathbb{R} : \ |\tilde{u}| \leqslant 1\right\}, \end{aligned} \tag{12}$$

where $\tilde{B} = \frac{\sqrt{\lambda_{\min}\{P_x\}}}{p_u}B$ and $\tilde{D} = \sqrt{\lambda_{\min}\{P_x\}}D$.

Let the new control signal (input) $\tilde{u}$ be the sum

$$\tilde{u} = \tilde{u}_1 + \tilde{u}_2, \tag{13}$$

where $\tilde{u}_1$ is intended to stabilize (12) and $\tilde{u}_2$ to ensure the existing constraints on $\tilde{x}$ and $\tilde{u}$.

According to Young's inequality, $2a^{\mathrm{T}}b \leqslant \mu a^{\mathrm{T}}a + \mu^{-1}b^{\mathrm{T}}b$ for any $a, b \in \mathbb{R}^n$ and $\mu > 0$. Therefore, we study the following upper bound: $\tilde{u}^2 = \tilde{u}_1^2 + 2\tilde{u}_1\tilde{u}_2 + \tilde{u}_2^2 \leqslant (1+r)\tilde{u}_1^2 + (1+r^{-1})\tilde{u}_2^2$, $r > 0$. Since $y = x$ in (10), for (2) we define the variable

$$\xi = \frac{1}{\lambda_{\min}\{P_x\}}\tilde{x}^{\mathrm{T}} P_x \tilde{x} + (1+r)\tilde{u}_1^2 + (1+r^{-1})(|\tilde{u}_2| + \delta)^2, \tag{14}$$

where $\delta > 0$ is specified by the control designer. This value is needed to implement the control law $\tilde{u}_2$ considering $|\tilde{u}_2| + \delta \neq 0$. Denoting $P_1 = \frac{1}{\lambda_{\min}\{P_x\}}P_x$, $p_2 = 1 + r$, and $p_3 = 1 + r^{-1}$, we write (14) as

$$\xi = \tilde{x}^{\mathrm{T}} P_1 \tilde{x} + p_2\tilde{u}_1^2 + p_3(|\tilde{u}_2| + \delta)^2. \tag{15}$$

Because $\xi$ is a scalar value, inequality (3) can be represented as

$$\underline{g}(t) < \xi(t) < \overline{g}(t). \tag{16}$$

In view of

$$\tilde{x}^{\mathrm{T}} P_1 \tilde{x} + \tilde{u}^2 \leqslant \tilde{x}^{\mathrm{T}} P_1 \tilde{x} + p_2 \tilde{u}_1^2 + p_3 \tilde{u}_2^2 \leqslant \tilde{x}^{\mathrm{T}} P_1 \tilde{x} + p_2 \tilde{u}_1^2 + p_3 (|\tilde{u}_2| + \delta)^2,$$

choosing $\overline{g}(t) \leqslant 1$ satisfies the constraints (12) in (16).

It is required to design a controller ensuring (16). Then the state variables and input will lie in the domains $\mathcal{X}$ and $\mathcal{U}$ with the additional condition (16).

Note that applying the method [1] to the plant (10) yields only one component of the vector $x$ in the given domain without input constraints.

Now we provide the main result of Section 4.

**Theorem 2.** *Let the transformation* (2) *satisfy conditions* (a)–(d), $\frac{\partial \Phi(\varepsilon,t)}{\partial \varepsilon} > 0$ *for any* $\varepsilon$ *and any* $t$, *and* $\underline{g}(0) < \xi(0) < \overline{g}(0)$. *Assume also that for given* $\delta > 0$, $\mu > 0$, $P_1 > 0$, $p_2 > 0$, $p_3 > 0$, $\beta > 0$, $c > 0$, *and* $K \in \mathbb{R}^{1\times n}$, *there exist* $H > 0$, $\alpha > 0$, *and* $\tau_i > 0$, $i = 1, \dots, 6$, *satisfying the linear inequalities*

$$\begin{gathered}
\begin{bmatrix} -\alpha + 0.5\tau_1 & 0.5 v \mu^{-1} \tilde{D}^{\mathrm{T}} \tilde{D} & -0.5 \\ * & -\tau_2 & 0 \\ * & * & -\tau_3 \end{bmatrix} \leqslant 0, \quad v = \pm \bar{f}, \\
c\tau_1 \geqslant \bar{f}^2 \tau_2 + \gamma^2 \tau_3,
\end{gathered} \tag{17}$$

$$\begin{gathered}
\begin{bmatrix} \bar{A}^{\mathrm{T}} H + H\bar{A} + \beta H & H\tilde{B} & H\tilde{D} \\ * & -\tau_4 & 0 \\ * & * & -\tau_5 \end{bmatrix} \leqslant 0, \\
H \geqslant \bar{P}_1, \\
K^{\mathrm{T}} K \leqslant \tau_6 H, \\
\frac{\inf\{\bar{g}(t)\}}{\lambda_{\min}\{\bar{P}_1\}} \beta \geqslant \frac{\inf\{\bar{g}(t)\}}{p_3} \tau_4 + \bar{f}^2 \tau_5, \\
\frac{\inf\{\bar{g}(t)\}}{\lambda_{\min}\{\bar{P}_1\}} \tau_6 \leqslant \frac{1}{1+r}.
\end{gathered} \tag{18}$$

*Then the controller*

$$\begin{aligned}
u &= \frac{1}{p_u}(\tilde{u}_1 + \tilde{u}_2), \\
\tilde{u}_1 &= K\tilde{x}, \\
\dot{\tilde{u}}_2 &= -\frac{1}{2p_3(|\tilde{u}_2| + \delta)} \mathrm{sgn}(u_2) \left[ \alpha\varepsilon + 2\tilde{x}^{\mathrm{T}} \bar{P}_1 \bar{A} \tilde{x} + 2\tilde{x}^{\mathrm{T}} \bar{P}_1 \tilde{B} \tilde{u}_2 + \mu \mathrm{sgn}(\varepsilon) \tilde{x}^{\mathrm{T}} \bar{P}_1^2 \tilde{x} \right],
\end{aligned} \tag{19}$$

*ensures* (16), *where* $\bar{A} = A + \tilde{B}K$ *and* $\bar{P}_1 = P_1 + p_2 K^{\mathrm{T}} K$.

*Remark 4.* For the resolvability of (18), the value $K$ should be chosen based on the Hurwitz property of the matrix $\bar{A}$.

**Proof.** Note that in this paper, the solutions of equations with discontinuous right-hand sides are considered in the Filippov sense. Therefore, in the proofs of Theorems 2 and 3, the corresponding (in)equalities for the derivatives hold for almost all $t$. In view of (13) and (19), we transform (12) and (15) to

$$\begin{aligned} \dot{\tilde{x}} &= \bar{A}\tilde{x} + \tilde{B}\tilde{u}_2 + \tilde{D}f, \\ \xi &= \tilde{x}^{\mathrm{T}}\bar{P}_1\tilde{x} + p_3(|\tilde{u}_2| + \delta)^2. \end{aligned} \tag{20}$$

Due to (20), the expression (6) can be written as

$$\dot{\varepsilon} = \left(\frac{\partial\Phi(\varepsilon,t)}{\partial\varepsilon}\right)^{-1}\left[2\tilde{x}^{\mathrm{T}}\bar{P}_1\bar{A}\tilde{x} + 2\tilde{x}^{\mathrm{T}}\bar{P}_1\tilde{B}\tilde{u}_2 + 2\tilde{x}^{\mathrm{T}}\bar{P}_1\tilde{D}f + 2p_3(|\tilde{u}_2| + \delta)\mathrm{sgn}(\tilde{u}_2)\dot{\tilde{u}}_2 - \frac{\partial\Phi(\varepsilon,t)}{\partial t}\right]. \tag{21}$$

To analyze the stability of the solutions of (21), we consider the Lyapunov function

$$V_1 = 0.5\varepsilon^2. \tag{22}$$

Calculating the total time-derivative of (22) along the trajectories of (21) gives

$$\dot{V}_1 = \left(\frac{\partial\Phi(\varepsilon,t)}{\partial\varepsilon}\right)^{-1}\varepsilon\left[2\tilde{x}^{\mathrm{T}}\bar{P}_1\bar{A}\tilde{x} + 2x^{\mathrm{T}}\bar{P}_1\tilde{B}\tilde{u}_2 + 2\tilde{x}^{\mathrm{T}}\bar{P}_1\tilde{D}f + 2p_3(|\tilde{u}_2| + \delta)\mathrm{sgn}(\tilde{u}_2)\dot{\tilde{u}}_2 - \frac{\partial\Phi(\varepsilon,t)}{\partial t}\right]. \tag{23}$$

Using Young's inequality for

$$2\varepsilon\tilde{x}^{\mathrm{T}}\bar{P}_1\tilde{D}f \leqslant \mu|\varepsilon|\tilde{x}^{\mathrm{T}}\bar{P}_1^2\tilde{x} + \mu^{-1}|\varepsilon|\tilde{D}^{\mathrm{T}}\tilde{D}f^2$$

and considering the third expression in (19), we estimate (23) as

$$\dot{V}_1 \leqslant \left(\frac{\partial\Phi(\varepsilon,t)}{\partial\varepsilon}\right)^{-1}\left(-\alpha\varepsilon^2 + \mu^{-1}|\varepsilon|\tilde{D}^{\mathrm{T}}\tilde{D}f^2 - \varepsilon\frac{\partial\Phi(\varepsilon,t)}{\partial t}\right). \tag{24}$$

For $V_1 \geqslant c$, let us require $\dot{V}_1 \leqslant 0$ under the constraints $f^2 \leqslant \bar{f}^2$ and $\left(\frac{\partial\Phi(\varepsilon,t)}{\partial t}\right)^2 \leqslant \gamma^2$ (see the problem statement and condition (d)). Since $\frac{\partial\Phi(\varepsilon,t)}{\partial\varepsilon} > 0$ does not affect the sign of (24), these conditions can be written as

$$\begin{gathered} -\alpha\varepsilon^2 + \mu^{-1}\varepsilon\mathrm{sgn}(\varepsilon)\tilde{D}^{\mathrm{T}}\tilde{D}f^2 - \varepsilon\frac{\partial\Phi(\varepsilon,t)}{\partial t} \leqslant 0 \ \ \forall\left(\varepsilon, f, \frac{\partial\Phi(\varepsilon,t)}{\partial t}\right): \\ 0.5\varepsilon^2 \geqslant c, \quad f^2 \leqslant \bar{f}^2, \quad \left(\frac{\partial\Phi(\varepsilon,t)}{\partial t}\right)^2 \leqslant \gamma^2. \end{gathered} \tag{25}$$

Denoting $z = col\left\{\varepsilon, f, \frac{\partial\Phi(\varepsilon,t)}{\partial t}\right\}$, we represent (25) in the matrix form:

$$\begin{gathered} z^{\mathrm{T}}\begin{bmatrix} -\alpha & 0.5\mu^{-1}\mathrm{sgn}(\varepsilon)f\tilde{D}^{\mathrm{T}}\tilde{D} & -0.5 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} z \leqslant 0, \\ z^{\mathrm{T}}\begin{bmatrix} -0.5 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} z \leqslant -c, \quad z^{\mathrm{T}}\begin{bmatrix} 0 & 0 & 0 \\ * & 1 & 0 \\ * & * & 0 \end{bmatrix} z \leqslant \bar{f}^2, \quad z^{\mathrm{T}}\begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 1 \end{bmatrix} z \leqslant \gamma^2. \end{gathered} \tag{26}$$

According to the S-procedure [10–12], inequalities (26) hold if

$$\begin{gathered} \begin{bmatrix} -\alpha + 0.5\tau_1 & 0.5\mathrm{sgn}(\varepsilon)\mu^{-1}\tilde{D}^{\mathrm{T}}\tilde{D} & -0.5 \\ * & -\tau_2 & 0 \\ * & * & -\tau_3 \end{bmatrix} \leqslant 0, \\ c\tau_1 \geqslant \bar{f}^2\tau_2 + \gamma^2\tau_3. \end{gathered} \tag{27}$$

Because $\mathrm{sgn}(\varepsilon)f \in [-\bar{f}, \bar{f}]$, we have a polytopic uncertainty in (27) with the two polytope vertices $v = -\bar{f}$ and $v = \bar{f}$. If there exist solutions $\alpha$ and $\tau_i$, $i = 1, 2, 3$, of the first LMI in (17) at the vertices $v = -\bar{f}$ and $v = \bar{f}$, then the first LMI in (27) is solvable; for details, see [13, 14]. Hence, system (21) is stable by the input state. Due to the change (4) of coordinates and property (a), the variable $\xi$ in (20) is bounded. As a result, the signals $\tilde{x}$, $x$, and $\tilde{u}_2$ are bounded as well. The third expression in (19) implies the boundedness of $\dot{\tilde{u}}_2$.

Let us establish additional conditions imposing constraints on $\tilde{u}_1$ in (19). We consider the Lyapunov function

$$V_2 = \tilde{x}^{\mathrm{T}} H \tilde{x} \tag{28}$$

and require $\dot{V} \leqslant 0$ and $H \geqslant \bar{P}_1$ for $V_2 \geqslant \frac{\inf\{\bar{g}(t)\}}{\lambda_{\min}\{\bar{P}_1\}}$ provided that $\tilde{u}_2^2 \leqslant \frac{\inf\{\bar{g}(t)\}}{p_3}$ and $f^2 \leqslant \bar{f}^2$. (The inequality $H \geqslant \bar{P}_1$ means that the ellipsoid $\tilde{x}^{\mathrm{T}} H \tilde{x} = \inf\{\bar{g}(t)\}$ is contained in the ellipsoid $\tilde{x}^{\mathrm{T}} \bar{P}_1 \tilde{x} = \inf\{\bar{g}(t)\}$.) In other words, the controller $\tilde{u}_1$ ensures that the state trajectories will be in a smaller domain compared to (16). These conditions can be written as

$$\begin{gathered}\dot{V}_2 = \tilde{x}^{\mathrm{T}}(\bar{A}^{\mathrm{T}} H + H\bar{A})\tilde{x} + 2\tilde{x}^{\mathrm{T}} H\tilde{B}\tilde{u}_2 + 2\tilde{x}^{\mathrm{T}} H\tilde{D} f \leqslant 0 \ \forall(\tilde{x}, \tilde{u}_2, f): \\ \tilde{x}^{\mathrm{T}} H\tilde{x} \geqslant \frac{\inf\{\bar{g}(t)\}}{\lambda_{\min}\{\bar{P}_1\}}, \quad \tilde{u}_2^2 \leqslant \frac{\inf\{\bar{g}(t)\}}{p_3}, \quad f^2 \leqslant \bar{f}^2.\end{gathered} \tag{29}$$

Denoting $s = col\{\tilde{x}, \tilde{u}_2, f\}$, we transform (29) to

$$\begin{gathered} s^{\mathrm{T}} \begin{bmatrix} \bar{A}^{\mathrm{T}} H + H\bar{A} & H\tilde{B} & H\tilde{D} \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} s \leqslant 0, \\ -s^{\mathrm{T}} \begin{bmatrix} H & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} s \geqslant -\frac{\inf \bar{g}(t)}{\lambda_{\min}\{\bar{P}_1\}}, \\ s^{\mathrm{T}} \begin{bmatrix} 0 & 0 & 0 \\ * & -1 & 0 \\ * & * & 0 \end{bmatrix} s \leqslant \frac{\inf\{\bar{g}(t)\}}{p_3}, \quad s^{\mathrm{T}} \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & -1 \end{bmatrix} s \leqslant \bar{f}. \end{gathered} \tag{30}$$

In view of $\tilde{u}^2 \leqslant 1$ from (12) and $\tilde{u}^2 \leqslant (1 + r)\tilde{u}_1^2 + (1 + r^{-1})\tilde{u}_2^2$ from Young's inequality, we require $(1 + r)\tilde{u}_1^2 + (1 + r^{-1})\tilde{u}_2^2 \leqslant 1$. Hence, $\tilde{u}_1^2 \leqslant \frac{1}{1+r}$. Based on the S-procedure [13] and the second expression in (19), inequalities (30) and $\tilde{x}^{\mathrm{T}} K^{\mathrm{T}} K \tilde{x} \leqslant \frac{1}{1+r}$ will hold simultaneously under $\tilde{x}^{\mathrm{T}} H \tilde{x} \leqslant \frac{\inf\{\bar{g}(t)\}}{\lambda_{\min}\{\bar{P}_1\}}$ if conditions (18) are true. The proof of Theorem 2 is complete.

*Remark 5.* This paper considers quadratic state constraints. Similar constraints were also studied in [15]. If the constraints are specified as a parallelepiped, we can introduce new constraints in the form of an ellipsoid inscribed into this parallelepiped and solve the problem with them; such an approach was described, e.g., in [16, 17].

On the other hand, it is not necessary to consider the quadratic variable $\xi$. For example, if the constraints are a priori given by the quadratic form (11), we can also use the quadratic change (15) or, alternatively, $\xi = \tilde{x}^{\mathrm{T}} P_1 \tilde{x} + (|\tilde{u}| + \delta)^2$. If all constraints are some intervals for the state vector coordinates and input, we can adopt the changes $\xi = |\tilde{x}| + |\tilde{u}_1| + |\tilde{u}_2|$ or $\xi = \sum_{i=1}^{n} p_{1i}|\tilde{x}_i| + |\tilde{u}_1| + |\tilde{u}_2|$, $p_{1i} > 0$, $i = 1, \ldots, n, \ldots$, relaxing the assumptions about the differentiability of the function $G$ with respect to $x$ and $u$; see Remarks 3 and [9]. Obviously, each change will yield other controllers and other conditions of their workability, differing from those proposed in this paper.

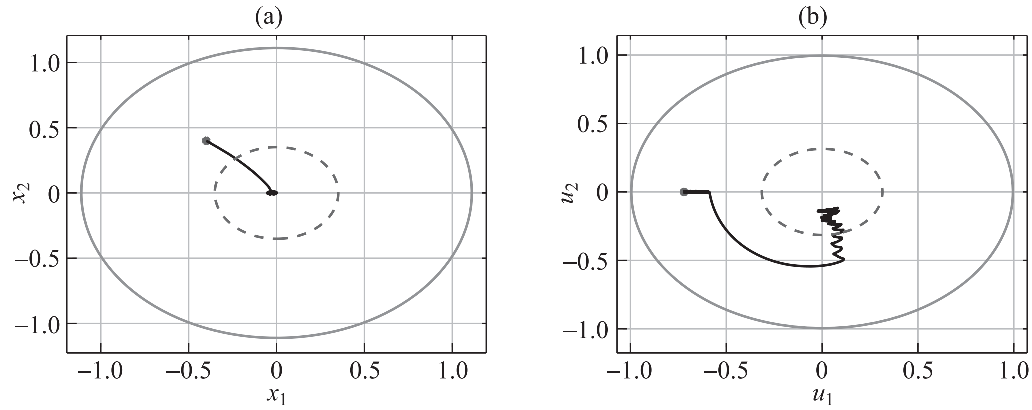


**Fig. 1.** Phase trajectories in the closed loop system: (a) $(x_1, x_2)$ and (b) $(u_1, u_2)$.

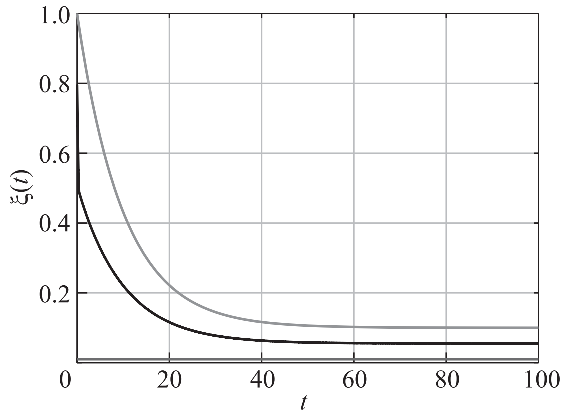


**Fig. 2.** Transient for $\xi(t)$ in the closed loop system.

*Example 2.* We consider the instable plant (10) with the following parameters:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix},$$
$$x(0) = \begin{bmatrix} -0.4 \\ 0.4 \end{bmatrix}, \quad f(t) = 0.01[\operatorname{sgn}(\sin(1.7t)) + \sin(0.3t) + \operatorname{sat}\{d(t)\}], \tag{31}$$

where $\operatorname{sat}\{\cdot\}$ is the saturation function and $d(t)$ is a band-limited white noise. This noise is generated in Matlab Simulink using the Band-Limited White Noise block with noise power 0.3 and sampling time 0.2. Under these parameters, $\bar{f} = 0.03$.

In (11), let $P_x = 0.81 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $p_u = 1$.

We choose $\delta = 0.01$ in (15), $K = [-2 \;\; -4]$ and $\mu = 0.01$ in (19), $T(\varepsilon) = \frac{e^{\varepsilon}-1}{e^{\varepsilon}+1}$, $\overline{g} = 0.89e^{-0.1t} + 0.1$, and $\underline{g} = 0.01$ in (16), and $r = 0.01$ in (14). We calculate $P_1 = I$, $p_2 = 1.01$, $p_3 = 101$, and $\gamma = \frac{3\overline{g} - \underline{g}}{2} = 1.475$. For $c = 1$ and $\beta = 0.1$, inequalities (17) and (18) have solutions, e.g., if $\alpha = 37.9$.

Figure 1 shows the phase portraits for $(x_1, x_2)$ and $(u_1, u_2)$, whereas Fig. 2 presents the transient for $\xi(t)$. The large ellipses in Fig. 1 correspond to the expressions $x^{\mathrm{T}} P_x x = \overline{g}(0)$ and $(1+r)p_u u_1^2 + (1+r^{-1})p_u u_2^2 = \overline{g}(0)$ and the small ellipses to $x^{\mathrm{T}} P_x x = \inf\{\overline{g}(t)$ and $(1+r)p_u u_1^2 + (1+r^{-1})p_u u_2^2 =$

$\inf\{\overline{g}(t)\}$. According to Fig. 1, the phase trajectories evolve from the large ellipse, reach the small ellipse after about 25 s (see Fig. 2), and stay inside it.

## 5. OUTPUT-FEEDBACK CONTROLLER

We consider the plant

$$\begin{aligned} \dot{x} &= Ax + Bu + Df, \\ y &= Lx, \end{aligned} \tag{32}$$

where $x \in \mathbb{R}^n$, $u \in \mathcal{U} \subset \mathbb{R}$, $y \in \mathcal{Y} \subset \mathbb{R}$, $f \in \mathbb{R}$, and $|f(t)| \leqslant \bar{f}$ for all $t$. The matrices $A$, $B$, $D$, and $L$ have compatible dimensions. Let the pair $(A, B)$ be controllable and the pair $(L, A)$ be observable. Assume that the plant (32) is strictly minimum-phase [3] and the domains $\mathcal{Y}$ and $\mathcal{U}$ have the form

$$\mathcal{Y} = \{y \in \mathbb{R} :\ p_y|y| \leqslant 1\}, \quad \mathcal{U} = \{u \in \mathbb{R} :\ p_u|u| \leqslant 1\}, \tag{33}$$

where $p_y > 0$ and $p_u > 0$ are specified by the control designer.

To compare the constraints imposed on $y$ and $u$, we introduce the changes of variables $\tilde{y} = p_y y$ and $\tilde{u} = p_u u$. They will serve for reducing the constraints (33) to segments of the same length. With these changes, we transform the domains $\mathcal{Y}$ and $\mathcal{U}$ to

$$\tilde{\mathcal{Y}} = \{\tilde{y} \in \mathbb{R} :\ |\tilde{y}| \leqslant 1\}, \quad \tilde{\mathcal{U}} = \{\tilde{u} \in \mathbb{R} :\ |\tilde{u}| \leqslant 1\}. \tag{34}$$

Let $\tilde{u}$ be the sum (13) and $\xi$ in (2) be

$$\xi = \tilde{y}^2 + p_2\tilde{u}_1^2 + p_3(|\tilde{u}_2| + \delta)^2, \tag{35}$$

where $\delta$, $p_2$, and $p_3$ have been determined in Section 4. With $\overline{g}(t) \leqslant 1$, (35) also includes the constraints (33). It is required to design a controller ensuring condition (16) considering (35).

Applying the method [1] to the plant (32) ensures that only the output will be in the given domain. In contrast to [1], with the method proposed above, the output and input will both be in the given domain.

We transform (32) to

$$Q(p)\tilde{y}(t) = R(p)\tilde{u}(t) + \phi(t). \tag{36}$$

Here,

$$Q(p) = \det(pI - A), \quad R(p) = \frac{p_y}{p_u} L(pI - A)^* B,$$

$(pI - A)^*$ is the adjoint matrix, and

$$\phi(t) = p_y L(pI - A)^* \left[x(0) + Df(t) + Bu(0) + Bf(0)\right].$$

In the sequel, for the sake of simplicity, the input-output representations (transfer functions) will be used simultaneously with the state-space representations; for example, see [3, 18, 19] and the references therein.

Now we provide the main result of Section 5.

**Theorem 3.** *Let the transformation* (2) *satisfy conditions* (a)–(d), $\frac{\partial \Phi(\varepsilon,t)}{\partial \varepsilon} > 0$ *for any* $\varepsilon$ *and any* $t$, *and* $\underline{g}(0) < \xi(0) < \overline{g}(0)$. *Assume also that for given* $\delta > 0$, $\mu > 0$, $p_1 > 0$, $p_2 > 0$, $p_3 > 0$,

*$\beta > 0$, $c > 0$, and $k \in \mathbb{R}$, there exist $H > 0$, $\alpha > 0$, and $\tau_i > 0$, $i = 1, \ldots, 6$, satisfying the linear inequalities*

$$\begin{bmatrix} -\alpha + 0.5\tau_1 & 0.5v\mu^{-1} & -0.5 \\ * & -\tau_2 & 0 \\ * & * & -\tau_3 \end{bmatrix} \leqslant 0, \quad v = \pm\bar{f}, \qquad (37)$$

$$c\tau_1 \geqslant \hat{\phi}^2\tau_2 + \gamma^2\tau_3,$$

$$\begin{bmatrix} \bar{A}^{\mathrm{T}}H + Q\bar{A} + \beta H & \frac{1}{p_u}HB & HD \\ * & -\tau_4 & 0 \\ * & * & -\tau_5 \end{bmatrix} \leqslant 0,$$

$$H \geqslant p_y^2\bar{p}_1 L^{\mathrm{T}}L,$$

$$L^{\mathrm{T}}K^{\mathrm{T}}KL \leqslant \tau_6 H, \qquad (38)$$

$$\frac{\inf\{\bar{g}(t)\}}{\bar{p}_1}\beta \geqslant \frac{\inf\{\bar{g}(t)\}}{p_3}\tau_4 + \hat{\phi}^2\tau_5,$$

$$\frac{\inf\{\bar{g}(t)\}}{\bar{p}_1}\tau_6 \leqslant \frac{1}{1+r}.$$

*Then the controller*

$$\begin{aligned} u &= \frac{1}{p_u}(\tilde{u}_1 + \tilde{u}_2), \\ \tilde{u}_1 &= k\tilde{y}, \\ \dot{\tilde{u}}_2 &= -\frac{1}{2p_3(|\tilde{u}_2| + \delta)}\operatorname{sgn}(\tilde{u}_2)\left[\alpha\varepsilon + 2\bar{p}_1\tilde{y}\frac{pR(p)}{\bar{Q}(p)}\tilde{u}_2 + \mu\bar{p}_1^2\operatorname{sgn}(\varepsilon)\tilde{y}^2\right], \end{aligned} \qquad (39)$$

*ensures* (16)*, where $\bar{A} = A + \frac{k}{p_u}BL$ and $\bar{p}_1 = 1 + k^2 p_2$.*

*Remark 6.* For the resolvability of (38), the value $k$ should be chosen based on the Hurwitz property of the matrix $\bar{A}$.

**Proof.** Due to $\tilde{u}_1$ in (39), the expressions (35) and (36) can be written as

$$\begin{aligned} \bar{Q}(p)\tilde{y}(t) &= R(p)\tilde{u}_2(t) + \phi(t), \\ \xi &= \bar{p}_1\tilde{y}^2 + p_3(|\tilde{u}_2| + \delta)^2. \end{aligned} \qquad (40)$$

Here, $\bar{Q}(p) = Q(p) - kR(p)$. Substituting (36) into (6) yields

$$\dot{\varepsilon} = \left(\frac{\partial\Phi(\varepsilon,t)}{\partial\varepsilon}\right)^{-1}\left[2\bar{p}_1\tilde{y}\frac{pR(p)}{\bar{Q}(p)}\tilde{u}_2 + 2\bar{p}_1\tilde{y}\bar{\phi}(t) + 2p_3(|\tilde{u}_2| + \delta)\operatorname{sgn}(\tilde{u}_2)\dot{\tilde{u}}_2 - \frac{\partial\Phi(\varepsilon,t)}{\partial t}\right], \qquad (41)$$

where $\bar{\phi}(t) = \frac{p}{\bar{Q}(p)}\phi(t)$ is a bounded function. Let us denote $\hat{\phi} = \sup\{\bar{\phi}(t)\}$.

To analyze the stability of (41), we consider the Lyapunov function (22). Calculating the derivative of (22) along the trajectories of (41) gives

$$\dot{V}_1 = \left(\frac{\partial\Phi(\varepsilon,t)}{\partial\varepsilon}\right)^{-1}\varepsilon\left[2\bar{p}_1\tilde{y}\frac{pR(p)}{\bar{Q}(p)}\tilde{u}_2 + 2\bar{p}_1\tilde{y}\bar{\phi} + 2p_3(|\tilde{u}_2| + \delta)\operatorname{sgn}(\tilde{u}_2)\dot{\tilde{u}}_2 - \frac{\partial\Phi(\varepsilon,t)}{\partial t}\right]. \qquad (42)$$

Using Young's inequality for $2\varepsilon\bar{p}_1\tilde{y}\bar{\phi} \leqslant |\varepsilon|\mu\bar{p}_1^2\tilde{y}^2 + \bar{\phi} + |\varepsilon|\mu^{-1}\bar{\phi}^2$ and considering the third expression in (39), we obtain

$$\dot{V}_1 = \left(\frac{\partial\Phi(\varepsilon,t)}{\partial\varepsilon}\right)^{-1}\left[-\alpha\varepsilon^2 + \mu^{-1}|\varepsilon|\varepsilon\bar{\phi}^2 - \varepsilon\frac{\partial\Phi(\varepsilon,t)}{\partial t}\right]. \qquad (43)$$

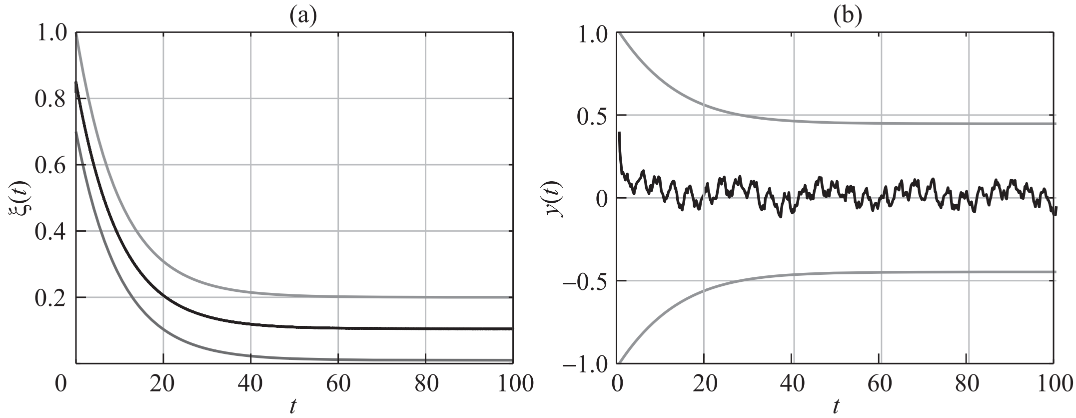


**Fig. 3.** Transients for (a) $\xi(t)$ and (b) $y(t)$ under constraints (16) given by exponential functions.

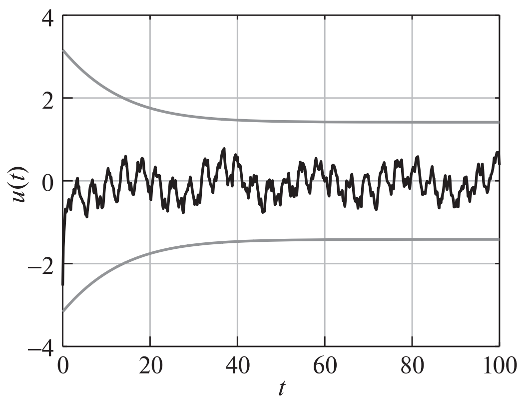


**Fig. 4.** Transients for $u(t)$ under constraints (16) given by exponential functions.

The remaining manipulations to derive conditions (37) are similar to the proof of Theorem 2.

Let us establish additional conditions imposing constraints on $\tilde{u}_1$ in (39). We consider the Lyapunov function

$$V_2 = x^{\mathrm{T}} H x \tag{44}$$

and write (40) as

$$\dot{x} = \bar{A}x + \frac{1}{p_u} B\tilde{u}_2 + D\phi, \quad \tilde{y} = p_y L x. \tag{45}$$

We require $\dot{V}_2 \leqslant 0$ and $H \geqslant p_y^2 \bar{p}_1 L^{\mathrm{T}} L$ for $V_2 \geqslant \frac{\inf\{\bar{g}(t)\}}{\bar{p}_1}$ provided that $\tilde{u}_2^2 \leqslant \frac{\inf\{\bar{g}(t)\}}{p_3}$ and $\phi^2 \leqslant \hat{\phi}^2$. (The inequality $H \geqslant p_y^2 \bar{p}_1 L^{\mathrm{T}} L$ means that the ellipsoid $x^{\mathrm{T}} H x = \inf\{\bar{g}(t)\}$ is contained in the cylinder $x^{\mathrm{T}} p_y^2 \bar{p}_1 L^{\mathrm{T}} L x = \inf\{\bar{g}(t)\}$.) These conditions can be written as

$$\begin{gathered} \dot{V}_2 = x^{\mathrm{T}}(\bar{A}^{\mathrm{T}} H + H\bar{A})x + 2x^{\mathrm{T}} \frac{1}{p_u} H B \tilde{u}_2 + 2x^{\mathrm{T}} H D \phi \leqslant 0 \ \forall (x, \tilde{u}_2, \phi) : \\ x^{\mathrm{T}} H x \geqslant \frac{\inf\{\bar{g}(t)\}}{\bar{p}_1}, \quad \tilde{u}_2^2 \leqslant \frac{\inf\{\bar{g}(t)\}}{p_3}, \quad \phi^2 \leqslant \hat{\phi}^2. \end{gathered} \tag{46}$$

In view of $(1 + r)\tilde{u}_1^2 \leqslant 1$, we specify $x^{\mathrm{T}} p_y^2 L^{\mathrm{T}} K^{\mathrm{T}} K L x \leqslant \frac{1}{1+r}$ for $x^{\mathrm{T}} H x \leqslant \frac{\inf\{\bar{g}(t)\}}{\bar{p}_1}$. The remaining manipulations to derive conditions (38) are similar to the proof of Theorem 2. The proof of Theorem 3 is complete.

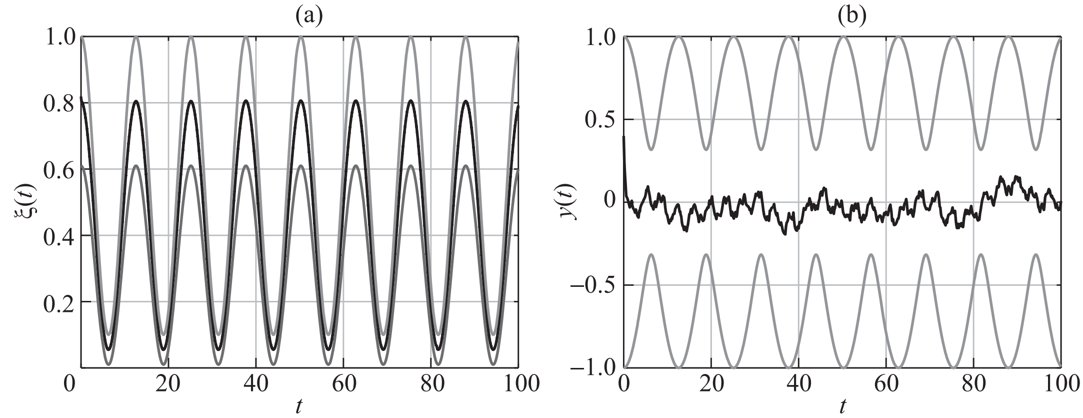


**Fig. 5.** Transients for (a) $\xi(t)$ and (b) $y(t)$ under constraints (16) given by sinusoidal functions.

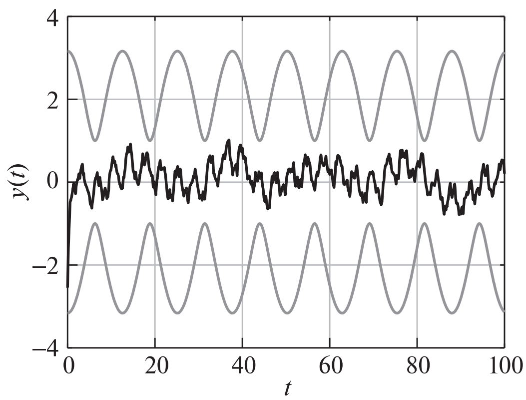


**Fig. 6.** Transients for $u(t)$ under constraints (16) given by sinusoidal functions.

*Remark 7.* In Section 5, other controllers can be obtained by defining the transformation (2) differently, e.g., as $\xi = \tilde{y}^2 + \tilde{u}^2$, $\xi = |\tilde{y}| + |\tilde{u}_1| + |\tilde{u}_2|, \ldots$; see Remark 5. In Section 5, it is particularly given by (35).

*Example 3.* We consider the instable plant (32) with the following parameters:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0.1 \\ 0.2 \\ 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}, \quad x(0) = 0.1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

with the same signal $f(t)$ as in Example 2. Then $R(p) = (p+1)^2$, $Q(p) = (p+1)^3$, and $\hat{\phi} = 0.22$.

In (33), let $p_y = 1$ and $p_u = 0.3$.

We choose $\delta = 0.01$, $k = -2$, and $\mu = 0.01$ in (39), $r = 0.01$ in (35), and $T(\varepsilon)$ from Example 2. For $\beta = 0.1$ and $c = 1$, inequalities (37) have solutions, e.g., if $\alpha = 374.3$.

Figures 3 and 4 show trajectories for $\xi(t)$, $y(t)$, and $u(t)$ under $\overline{g} = 0.79e^{-0.1t} + 0.2$ and $\underline{g} = 0.69e^{-0.1t} + 0.01$ in (3); Figs. 5 and 6, the same trajectories under $\overline{g} = 0.45\cos(0.5t) + 0.54$ and $\underline{g} = 0.3\cos(0.5t) + 3.01$ in (3). According to Figs. 3–6, the signals $\xi$, $y$, and $u$ never violate the given constraints (33) (i.e., stay inside the corresponding domains) as well as the additional constraints (16) specified by the control designer. They can be defined as exponential (Figs. 3 and 4) or sinusoidal (Figs. 5 and 6) functions.

## 6. CONCLUSIONS

This paper has extended the method [1] to dynamic systems with an arbitrary number of inputs and outputs. The method ensures that these signals will be in given domains. The developed method has been applied to design state- and output-feedback controllers for linear systems with input and output constraints provided that the dimension of the output exceeds that of the input. Also, in contrast to [1], the stability of the closed loop system has been analyzed and the controller parameters have been designed using linear matrix inequalities. The simulation results have confirmed the theoretical conclusions and have demonstrated the effectiveness of this method.

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