ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2023, Vol. 84, No. 4, pp. 389–396.
Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, 2023.

Studying the Dynamic Properties of a Distributed Thermomechanical Controlled Plant with Intrinsic Feedback. II

E. M. Solnechnyi

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
e-mail: solnechn@ipu.ru

Received May 11, 2021
Revised October 30, 2022
Accepted November 30, 2022

Abstract—The dynamic properties of the response of a one-dimensional elastic mechanical system to an external mechanical action are examined. Transfer functions are calculated in two channels: from the force action at one of the system boundaries to the displacement of the medium sections and to the temperature. The asymptotic behavior of the transfer function is analyzed for each channel in the neighborhood of the origin on the complex plane. The case of no heat exchange between the system and the environment is considered separately.

Keywords: distributed thermomechanical plant, dynamic properties, transfer functions, asymptotics

DOI: 10.25728/arcRAS.2023.80.26.001

1. INTRODUCTION

Thermomechanical systems with mechanical vibrations and heat transfer processes are widely used in modern engineering. Therefore, it is necessary to study the dynamic properties of such systems in mathematical terms and develop control methods for them.

The literature on the thermoelasticity phenomenon is quite extensive. The early works [1–3] were followed by [4], where thermoelasticity was investigated as part of general elasticity effects. In the recent literature, we mention the publications [5–7] devoted to various properties of thermoelastic media. The book [8] developed a modern theory of thermomechanics of elastoplastic deformation. A coupled dynamic thermoelasticity problem for a one-dimensional medium was stated in [9].

In this paper, we analyze the dynamic properties of a one-dimensional distributed elastic thermomechanical system. The mathematical model of processes in such a system is based on the classical work [4]. In contrast to [10], the system is subjected to a mechanical (force) action at one of its boundaries instead of a thermal action. The system dynamics equations have the form

$$\begin{cases} \dfrac{\partial^2\varphi}{\partial t^2} = c^2\dfrac{\partial^2\varphi}{\partial x^2} - \beta\dfrac{\partial\theta}{\partial x}, \\ \beta_{\text{therm}}\dfrac{\partial^2\varphi}{\partial t\partial x} + \dfrac{\partial\theta}{\partial t} = a\dfrac{\partial^2\theta}{\partial x^2}, \end{cases} \tag{1.1}$$

where $t \geq 0$, $0 \leq x \leq l$, and a, c, β, and β_{therm} are positive constants. (For details, we refer, e.g., to the monograph [4].)

In these equations, $\varphi(x)(t)$ denotes the displacement of the section located at a distance $l - x$ from the point of application of the force action; $\theta(x)(t)$ is the temperature of the medium in the section x.

The initial conditions with respect to the time variable are assumed to be zero. The boundary conditions are as follows:

a) for the function φ,

$$\begin{cases} \dfrac{\partial\varphi}{\partial x}(0) = 0, \\ \dfrac{\partial^2\varphi}{\partial t^2}(l) = u, \end{cases} \tag{1.2}$$

where u is the control action (with the physical sense of a mechanical (force) action applied to the system);

b) for the function θ,

$$\begin{cases} \left(-\lambda\dfrac{\partial\theta}{\partial x} + \alpha\theta\right)(0) = 0, \\ \left(\lambda\dfrac{\partial\theta}{\partial x} + \alpha\theta\right)(l) = 0, \end{cases} \tag{1.3}$$

where α and λ are positive constants.

2. CALCULATION OF THE VECTOR TRANSFER FUNCTION $u \to (\varphi(x), \theta(x))$

We perform the Laplace transform of Eqs. (1.1) with the boundary conditions ((1.2), (1.3)) to obtain the system of ordinary differential equations

$$\begin{cases} c^2\dfrac{\partial^2\overline{\varphi}}{\partial x^2}(x)(p) - p^2\overline{\varphi}(x)(p) - \beta\dfrac{\partial\overline{\theta}}{\partial x}(x)(p) = 0, \\ a\dfrac{\partial^2\overline{\theta}}{\partial x^2}(x)(p) - p\overline{\theta}(x)(p) - \beta_{\text{therm}}p\dfrac{\partial\overline{\varphi}}{\partial x}(x)(p) = 0, \end{cases} \tag{2.1}$$

$$\begin{cases} \dfrac{\partial\overline{\varphi}}{\partial x}(0) = 0, \\ p^2\overline{\varphi}(l) = \overline{u}, \end{cases} \tag{2.2}$$

$$\begin{cases} \dfrac{\partial\overline{\theta}}{\partial x}(0) = \kappa\overline{\theta}(0), \\ \dfrac{\partial\overline{\theta}}{\partial x}(l) = -\kappa\overline{\theta}(l), \end{cases} \tag{2.3}$$

where $\kappa = \frac{\alpha}{\lambda}$. In this system, the pair of unknown functions $(\overline{\varphi}(x), \overline{\theta}(x))$ consists of the Laplace images of the desired functions $(\varphi(x)(t), \theta(x)(t))$.

Solving the boundary-value problem (2.1)–(2.3) yields the following expressions for the transfer functions in the channels $u \to \varphi(x)$ and $u \to \theta(x)$.

Theorem 1. *The transfer functions of the system in the channels $u \to \varphi(x)$ and $u \to \theta(x)$ are given by*

$$W_{u\to\varphi(x)} = \frac{a_{22}}{\Delta_A}\left[ac^2D_3(x) - b_1pD_1(x)\right] - \beta\frac{a_{21}}{\Delta_A}\left[\kappa aD_1(x) + pD_0(x)\right] \tag{2.4}$$

and

$$W_{u\to\theta(x)} = \frac{a_{22}}{\Delta_A}\beta_{\text{therm}}p^3D_0(x) + \frac{a_{21}}{\Delta_A}\left[-ac^2D_3(x) - \kappa ac^2D_2(x) + b_2pD_1(x) + \kappa ap^2D_0(x)\right], \tag{2.5}$$

respectively. In these formulas, $\Delta_A = a_{11}a_{22} - a_{12}a_{21}$,

$$a_{11} = p^2\left(ac^2 D_3(l) - b_1 p D_1(l)\right),$$

$$a_{12} = \beta p^2\left(\kappa a D_1(l) + p D_0(l)\right),$$

$$a_{21} = \beta_{\text{therm}} p^3\left(D_1(l) + \kappa D_0(l)\right),$$

$$a_{22} = ac^2 D_4(l) + 2\kappa ac^2 D_3(l) + \left(\kappa^2 ac^2 - b_2 p\right) D_2(l) - \kappa p\left(2ap + \beta\beta_{\text{therm}}\right) D_1(l) - \kappa^2 ap^2 D_0(l),$$

$$D_j(x) = \frac{1}{R}\left(\rho_1^{(j-1)/2}\sinh\left(x\sqrt{\rho_1}\right) - \rho_2^{(j-1)/2}\sinh\left(x\sqrt{\rho_2}\right)\right) \quad (j = 0; 2; 4),$$

$$D_j(x) = \frac{1}{R}\left(\rho_1^{(j-1)/2}\cosh\left(x\sqrt{\rho_1}\right) - \rho_2^{(j-1)/2}\cosh\left(x\sqrt{\rho_2}\right)\right) \quad (j = 1; 3),$$

$$\rho_{1,2} = \frac{p\left(ap + b_1\right) \pm R}{2ac^2}, \quad R = ap\sqrt{(p+\mu)^2 + y^2}, \quad \mu = \frac{\beta\beta_{\text{therm}} - c^2}{a}, \quad y = 2\frac{c}{a}\sqrt{\beta\beta_{\text{therm}}},$$

$$b_1 = \beta\beta_{\text{therm}} + c^2, \quad b_2 = \beta\beta_{\text{therm}} + ap.$$

3. ASYMPTOTIC BEHAVIOR OF TRANSFER FUNCTIONS AS $p \to 0$

We study the dynamic properties of the system, beginning with the asymptotic behavior of its transfer functions in the neighborhood of the origin on the complex plane **C**.

Theorem 2. *In the neighborhood of the origin on the plane* **C**, *the transfer function* $W_{u\to\varphi(x)}$ *can be represented as*

$$W_{u\to\varphi(x)} = \frac{1}{p^2}\left(1 + O(p)\right) \tag{3.1}$$

and the transfer function $W_{u\to\theta(x)}$ *as*

$$W_{u\to\theta(x)} = \frac{\beta_{\text{therm}}}{6ac^2}p\left[x^3 - l^2\frac{3+\kappa l}{2+\kappa l}\left(x + \frac{1}{\kappa}\right) + O(p)\right]. \tag{3.2}$$

Here, $O(p)$ *denotes a function* $f(p)$ $(p \in \mathbf{C})$ *with a bounded ratio* $\frac{f(p)}{p}$ *as* $p \to 0$.

Thus, the system has the double integrating property in the channel $u \to \varphi(x)$ *and the differentiating property in the channel* $u \to \theta(x)$.

Remark. According to (3.2), the asymptotic formula for the transfer function $W_{u\to\theta(x)}$ as $p \to 0$ includes the ratio $\frac{1}{\kappa} = \frac{\lambda}{\propto}$. Therefore, the case $\kappa = 0$ (no heat exchange with the environment) should be considered separately; see the next section.

4. THE CASE $\kappa = 0$

$$W_{u\to\varphi(x)} = \frac{a_{22}}{\Delta_A}\left(ac^2 D_3(x) - b_1 p D_1(x)\right) - \beta\frac{a_{21}}{\Delta_A}pD_0(x), \tag{4.1}$$

$$W_{u\to\theta(x)} = \frac{a_{22}}{\Delta_A}\beta_{\text{therm}} p^3 D_0(x) + \frac{a_{21}}{\Delta_A}\left(-ac^2 D_3(x) + b_2 p D_1(x)\right). \tag{4.2}$$

In this case, the functions a_{jk} $(j,\, k = 1, 2)$ have the form

$$a_{11} = p^2\left(ac^2 D_3(l) - b_1 p D_1(l)\right), \quad a_{12} = \beta p^3 D_0(l), \quad a_{21} = \beta_{\text{therm}} p^3 D_1(l), \tag{4.3}$$

$$a_{22} = ac^2 D_4(l) - b_2 p D_2(l). \tag{4.4}$$

Theorem 3. *In the case $\kappa = 0$, in the neighborhood of the origin on the complex plane* **C**, *the transfer function $W_{u\to\varphi(x)}$ admits the same representation* (3.1) *as in the general case (see Theorem 2); the transfer function $W_{u\to\theta(x)}$ in this neighborhood can be represented as*

$$W_{u\to\theta(x)} = -\beta_{\text{therm}}\frac{l}{2c^2}\left(1 + O\left(p\right)\right). \tag{4.5}$$

Thus, in the case $\kappa = 0$, the transfer function of the system in the channel $u \to \theta(x)$ has a finite nonzero limit as $p \to 0$. This property can be called static.

The proofs of Theorems 1–3 are given in Appendices A–C, respectively.

5. CONCLUSIONS

As has been demonstrated by this study, the thermomechanical controlled plant subjected to a mechanical (force) external action possesses the following dynamic properties: double integration in the channel from the force action to the displacement of the one-dimensional medium and (but only under heat exchange with the environment) differentiation in the channel from the force action to the temperature.

The resulting conclusions should be considered when designing control systems for thermomechanical plants with dynamic properties described by (1.1)–(1.3).

According to the results of [10] and this paper, the intrinsic feedback of the plant (from the displacement of the sections to the temperature) complicates the description of its dynamic properties compared to the case of no feedback, which was investigated in [11].

APPENDIX A

Proof of Theorem 1.

1. We apply the Laplace transform to Eqs. (2.1) with respect to the spatial coordinate x considering the first boundary condition in (2.2). (For details, see [12, item 80, formulas (6) and (7)].) As a result,

$$\begin{cases} \left(c^2q^2 - p^2\right)\overline{\overline{\varphi}}(q) - \beta q\overline{\overline{\theta}}(q) = z_1(q), \\ -\beta_{\text{therm}}qp\overline{\overline{\varphi}}(q) + \left(aq^2 - p\right)\overline{\overline{\theta}}(q) = z_2(q), \end{cases} \tag{A.1}$$

where $z_1(q) = c^2q\overline{\varphi}(0) - \beta\overline{\theta}(0)$ and $z_2(q) = aq\overline{\theta}(0) + a\frac{\partial\overline{\theta}}{\partial x}(0) - \beta_{\text{therm}}p\overline{\varphi}(0)$.

In view of these expressions for $z_i(q)$, the solution of system (A.1) in $\left(\overline{\overline{\varphi}}(q), \overline{\overline{\theta}}(q)\right)$ is given by

$$\overline{\overline{\varphi}}(q) = \frac{ac^2q^3\overline{\varphi}(0) + B_1q + \beta p\overline{\theta}(0)}{\Delta(q)}, \tag{A.2}$$

$$\overline{\overline{\theta}}(q) = \frac{ac^2q^3\overline{\theta}(0) + ac^2q^2\frac{\partial\overline{\theta}}{\partial x}(0) + B_2p^2 - b_2qp\overline{\theta}(0)}{\Delta(q)}, \tag{A.3}$$

where

$$\Delta(q) = ac^2\left(q^2 - \rho_1\right)\left(q^2 - \rho_2\right), \quad \rho_{1,2} = \frac{p\left(ap + b_1\right) \pm R}{2ac^2},$$

$$R = ap\sqrt{\left(p + \mu\right)^2 + y^2}, \quad \mu = \frac{\beta\beta_{\text{therm}} - c^2}{a}, \quad y = 2\frac{c}{a}\sqrt{\beta\beta_{\text{therm}}},$$

$$b_1 = \beta\beta_{\text{therm}} + c^2, \quad b_2 = \beta\beta_{\text{therm}} + ap,$$

$$B_1 = \beta a\frac{\partial\overline{\theta}}{\partial x}(0) - b_1p\overline{\varphi}(0), \quad B_2 = \beta_{\text{therm}}p\overline{\varphi}(0) - a\frac{\partial\overline{\theta}}{\partial x}(0).$$

Based on the relation

$$\frac{1}{\Delta(q)} = \frac{1}{R}\left(\frac{1}{q^2-\rho_1} - \frac{1}{q^2-\rho_2}\right), \tag{A.4}$$

from (A.2) and (A.3) we pass to the original functions with respect to the coordinate x:

$$\overline{\varphi}(x) = ac^2 D_3(x)\overline{\varphi}(0) + D_1(x)B_1 + \beta p D_0(x)\overline{\theta}(0), \tag{A.5}$$

$$\overline{\theta}(x) = ac^2 D_3(x)\overline{\theta}(0) + ac^2 D_2(x)\frac{\partial\overline{\theta}}{\partial x}(0) - b_2 p D_1(x)\overline{\theta}(0) + p^2 D_0(x)B_2, \tag{A.6}$$

where

$$D_j(x) = \frac{1}{R}\left(\rho_1^{(j-1)/2}\sinh\left(x\sqrt{\rho_1}\right) - \rho_2^{(j-1)/2}\sinh\left(x\sqrt{\rho_2}\right)\right) \quad (j=0;2),$$
$$D_j(x) = \frac{1}{R}\left(\rho_1^{(j-1)/2}\cosh\left(x\sqrt{\rho_1}\right) - \rho_2^{(j-1)/2}\cosh\left(x\sqrt{\rho_2}\right)\right) \quad (j=1;3).$$

(The details can be found in [12, item 80, formula (4)].)
Due to (A.5) and the first condition in (2.3), the second boundary condition in (2.2) takes the form

$$p^2\left[\left(ac^2 D_3(l) - b_1 p D_1(l)\right)\overline{\varphi}(0) + \beta\left(pD_0(l) + a\kappa D_1(l)\right)\overline{\theta}(0)\right] = \overline{u}. \tag{A.7}$$

According to (A.6), the second condition in (2.3) reduces to

$$\begin{aligned} &ac^2\left(D_4(l) + \kappa D_3(l)\right)\overline{\theta}(0) + ac^2\kappa\left(D_3(l) + \kappa D_2(l)\right)\overline{\theta}(0) \\ &- b_2 p\left(D_2(l) + \kappa D_1(l)\right)\overline{\theta}(0) + p^2\left(D_1(l) + \kappa D_0(l)\right)\left(\beta_{\text{therm}} p\overline{\varphi}(0) - \kappa a\overline{\theta}(0)\right) = 0, \end{aligned} \tag{A.8}$$

where

$$D_4(l) = \frac{1}{R}\left(\rho_1^{3/2}\sinh\left(x\sqrt{\rho_1}\right) - \rho_2^{3/2}\sinh\left(x\sqrt{\rho_2}\right)\right).$$

Conditions (A.7) and (A.8) make up the following system of equations in the vector $\begin{pmatrix}\overline{\varphi}(0)\\ \overline{\theta}(0)\end{pmatrix}$:

$$A\begin{pmatrix}\overline{\varphi}(0)\\ \overline{\theta}(0)\end{pmatrix} = \begin{pmatrix}\overline{u}\\ 0\end{pmatrix}, \tag{A.9}$$

where $A = (a_{jk};\ j,k=1,2)$, $a_{11} = p^2\left(ac^2 D_3(l) - b_1 p D_1(l)\right)$,

$$a_{12} = \beta p^2\left(\kappa a D_1(l) + pD_0(l)\right), \quad a_{21} = \beta_{\text{therm}} p^3\left(D_1(l) + \kappa D_0(l)\right),$$
$$\begin{aligned} a_{22} = {} & ac^2 D_4(l) + 2\kappa ac^2 D_3(l) + \left(\kappa^2 ac^2 - b_2 p\right) D_2(l) \\ & - \kappa p\left(2ap + \beta\beta_{\text{therm}}\right) D_1(l) - \kappa^2 a p^2 D_0(l). \end{aligned}$$

The solution of system (A.9) is given by

$$\begin{pmatrix}\overline{\varphi}(0)\\ \overline{\theta}(0)\end{pmatrix} = \begin{pmatrix}a_{22}\\ -a_{21}\end{pmatrix}\frac{\overline{u}}{\Delta_A}, \tag{A.10}$$

where $\Delta_A = a_{11}a_{22} - a_{12}a_{21}.$

Substituting (A.10) into (A.5) and (A.6) and using the formula for B_i and the first condition in (2.3), we finally arrive at the following expressions for $\overline{\varphi}(x)$ and $\overline{\theta}(x)$:

$$\overline{\varphi}(x) = \left\{a_{22}\left[ac^2 D_3(x) - b_1 p D_1(x)\right] - \beta a_{21}\left[\kappa a D_1(x) + p D_0(x)\right]\right\}\frac{\overline{u}}{\Delta_A}, \tag{A.11}$$

$$\overline{\theta}(x) = \Big\{a_{22}\beta_{\text{therm}} p^3 D_0(x) + a_{21}\left[-ac^2 D_3(x) - \kappa a c^2 D_2(x) + b_2 p D_1(x) + \kappa a p^2 D_0(x)\right]\Big\}\frac{\overline{u}}{\Delta_A}. \tag{A.12}$$

APPENDIX B

Proof of Theorem 2.

1. In the neighborhood of the origin on the plane $\mathbf{C}$, the functions R and ρ_j ($j = 1, 2$; see the explanations for (2.4) and (2.5)) can be represented as

$$R = b_1 p\left(1 + O(p)\right), \quad \rho_1 = \frac{b_1 p}{ac^2}\left(1 + O(p)\right), \quad \rho_2 = O\left(p^2\right). \tag{B.1}$$

2. In the neighborhood of the origin on the plane $\mathbf{C}$, the functions $D_j(x)$ ($j = 0 \div 4$) can be represented as follows:

$$D_0(x) = \frac{x^3}{6}\frac{\rho_1 - \rho_2}{R} + O\left(p^2\right) = \frac{x^3}{6ac^2} + O\left(p^2\right), \tag{B.2}$$

$$D_1(x) = \frac{x^2}{2}\frac{\rho_1 - \rho_2}{R} + O\left(p^2\right) = \frac{x^2}{2ac^2} + O\left(p^2\right), \tag{B.3}$$

$$D_2(x) = x\frac{\rho_1 - \rho_2}{R} + O\left(p^2\right) = \frac{x}{ac^2} + O\left(p^2\right), \tag{B.4}$$

$$D_3(x) = \frac{\rho_1 - \rho_2}{R} + O\left(p^2\right) = \frac{1}{ac^2} + O\left(p^2\right), \tag{B.5}$$

$$D_4(x) = x\frac{\rho_1^2 - \rho_2^2}{R} + O\left(p^2\right) = x\frac{\rho_1 + \rho_2}{ac^2} + O\left(p^2\right) = x\frac{b_1}{a^2c^4}p + O\left(p^2\right). \tag{B.6}$$

3. In the neighborhood of the origin on the plane $\mathbf{C}$, the functions a_{jk} ($j, k = 1, 2$; see the explanations for (A.9)) can be represented as follows:

$$a_{11} = p^2\left(1 - \frac{b_1 l^2}{2ac^2}p + O\left(p^2\right)\right) = p^2\left(1 + O(p)\right), \tag{B.7}$$

$$a_{12} = \beta p^2\left(\frac{l^3}{6ac^2}p + \kappa\frac{l^2}{2c^2} + O\left(p^2\right)\right) = \beta\kappa\frac{l^2}{2c^2}p^2\left(1 + O(p)\right), \tag{B.8}$$

$$a_{21} = \beta_{\text{therm}} p^3\left(\frac{l^2}{2ac^2} + \kappa\frac{l^3}{6ac^2} + O\left(p^2\right)\right) = \beta_{\text{therm}} p^3\frac{l^2}{2ac^2}\left(1 + \kappa\frac{l}{3}\right)\left(1 + O\left(p^2\right)\right), \tag{B.9}$$

$$a_{22} = \frac{b_1 l}{ac^2}p + 2\kappa + l\left(\kappa^2 - \frac{b_2}{ac^2}p\right) - \kappa p\frac{l^2}{c^2}\left(p + \frac{\beta\beta_{\text{therm}}}{2a}\right) - \kappa^2 p^2\frac{l^3}{6c^2} + O\left(p^2\right) = \kappa\left(2 + \kappa l\right)\left(1 + O(p)\right). \tag{B.10}$$

4. In the neighborhood of the origin on the plane $\mathbf{C}$, the function Δ_A (see the explanations for (A.10)) and the ratios $\frac{a_{2j}}{\Delta_A}$ ($j = 1, 2$; see (2.4) and (2.5)) can be represented as

$$\Delta_A = p^2\kappa\,(2+\kappa l)\,(1+O\,(p)) - p^5\kappa\frac{\beta\beta_{\text{therm}}l^4}{4ac^4}\left(1+\kappa\frac{l}{3}\right)(1+O\,(p))$$
$$= p^2\kappa\,(2+\kappa l)\,(1+O\,(p)) \tag{B.11}$$

$$\frac{a_{21}}{\Delta_A} = p\frac{\beta_{\text{therm}}l^2\,(1+\kappa l/3)}{2\kappa ac^2\,(2+\kappa l)}(1+O\,(p)) = p\frac{\beta_{\text{therm}}l^2\,(3+\kappa l)}{6\kappa ac^2\,(2+\kappa l)}(1+O\,(p))\,, \tag{B.12}$$

$$\frac{a_{22}}{\Delta_A} = \frac{1}{p^2}(1+O\,(p))\,. \tag{B.13}$$

5. In the neighborhood of the origin on the plane $\mathbf{C}$, the transfer functions $W_{u\to\varphi(x)}$ and $W_{u\to\theta(x)}$ can be represented as

$$W_{u\to\varphi(x)} = \frac{1}{p^2}\left(1-p\frac{b_1}{2ac^2}x^2\right)\left(1+O\left(p^2\right)\right)$$
$$-\,p\frac{\beta\beta_{\text{therm}}l^2\,(1+\kappa l/3)}{2ac^2\kappa\,(2+\kappa l)}\left(\kappa\frac{x^2}{2c^2}+p\frac{x^3}{6ac^2}+O\left(p^2\right)\right) = \frac{1}{p^2}(1+O\,(p))\,, \tag{B.14}$$

$$W_{u\to\theta}(x) = p\beta_{\text{therm}}\left[\frac{x^3}{6ac^2}-\frac{l^2\,(3+\kappa l)}{6ac^2\kappa\,(2+\kappa l)}\left(1+\kappa x-px^2\frac{b_2}{2ac^2}-p^2x^3\frac{\kappa}{6c^2}\right)\right](1+O\,(p))$$
$$= p\frac{\beta_{\text{therm}}}{6ac^2}\left[x^3-l^2\frac{3+\kappa l}{2+\kappa l}\left(x+\frac{1}{\kappa}\right)\right](1+O\,(p))\,. \tag{B.15}$$

APPENDIX C

Proof of Theorem 3.

1. Due to (3.5) and (3.6), in the neighborhood of the origin on the plane $\mathbf{C}$, the functions a_{jk} ($j, k = 1, 2$) for $\kappa = 0$ can be represented as follows:

$$a_{11} = p^2\,(1+O\,(p)) \quad \text{(coincides with (B.7))}, \tag{C.1}$$

$$a_{12} = \beta p^3\left(\frac{l^3}{6ac^2}+O\left(p^2\right)\right) = \beta p^3\frac{l^3}{6ac^2}\left(1+O\left(p^2\right)\right), \tag{C.2}$$

$$a_{21} = \beta_{\text{therm}}p^3\left(\frac{l^2}{2ac^2}+O\left(p^2\right)\right) = \beta_{\text{therm}}p^3\frac{l^2}{2ac^2}\left(1+O\left(p^2\right)\right), \tag{C.3}$$

$$a_{22} = p\frac{l}{ac^2}(b_1-b_2)+O\left(p^2\right) = p\frac{l}{ac^2}\left(c^2-ap\right)+O\left(p^2\right) = p\frac{l}{a}(1+O\,(p))\,. \tag{C.4}$$

2. Consequently,

$$\Delta_A = p^3\frac{l}{a}(1+O\,(p)) - \beta\beta_{\text{therm}}p^6\frac{l^5}{12a^2c^4}\left(1+O\left(p^2\right)\right) = p^3\frac{l}{a}(1+O\,(p))\,, \tag{C.5}$$

$$\frac{a_{21}}{\Delta_A} = \beta_{\text{therm}}\frac{l}{2c^2}(1+O\,(p))\,, \quad \frac{a_{22}}{\Delta_A} = \frac{1}{p^2}(1+O\,(p))\,. \tag{C.6}$$

3. As a result, we obtain

$$W_{u\to\varphi}(x) = \frac{1+O(p)}{p^2}\left[1+O\left(p^2\right) - b_1 p\left(\frac{x^2}{2ac^2}+O\left(p^2\right)\right)\right]$$
$$- p\beta\beta_{\text{therm}}\left(\frac{lx^3}{12ac^4}+O\left(p^2\right)\right) = \frac{1}{p^2}\left(1+O(p)\right), \tag{C.7}$$

$$W_{u\to\theta}(x) = p\beta_{\text{therm}}\frac{x^3}{6ac^2}\left(1+O(p)\right)$$
$$+ \beta_{\text{therm}}\frac{l}{2c^2}\left(1+O(p)\right)\left[pb_2\frac{x^2}{2ac^2} - 1 + O\left(p^2\right)\right] = -\beta_{\text{therm}}\frac{l}{2c^2}\left(1+O(p)\right). \tag{C.8}$$

REFERENCES

1. Kovalenko, A.D., *Vvedenie v termouprugost'* (Introduction to Thermoelasticity), Kiev: Naukova Dumka, 1964.
2. Lord, H. and Shulman, Y., A Generalized Dynamical Theory of Thermoelasticity, *J. Mech. Phys. Solids*, 1967, vol. 15, pp. 299–309.
3. Nayfeh, A.H. and Nemat-Nasser, S., Thermoelastic Waves in Solids with Thermal Relaxation, *Acta Mechanica*, 1971, vol. 12, pp. 53–69.
4. Novacki, W., *Teoria sprężystośći*, Warszawa. Państwowe Wydawnictwo Naukowe, 1970.
5. Jordan, P.M. and Puri, P., On the Propagation of Plane Waves in Type–I11 Thermoelastic media, *Proc. Royal Soc. Lond. A.*, 2004, vol. 460, pp. 3203–3221.
6. Rogovoi, A.A. and Stolbova, O.S., Evolutionary Model of Finite-Strain Thermoelasticity, *J. Appl. Mech. Tech. Phys.*, 2008, vol. 49, no. 3, pp. 500–509.
7. Babenkov, M.B., Propagation of Harmonic Perturbations in a Thermoelastic Medium with Heat Relaxation, *J. Appl. Mech. Tech. Phys.*, 2013, vol. 54, no. 2, pp. 277–286.
8. Markin, A.A. and Sokolova, M.Yu., *Termomekhanika uprugoplasticheskogo deformirovaniya* (Thermomechanics of Elastoplastic Deformation), Moscow: Nauka, 2013.
9. Torsukova, E.B. and Khristich, D.V., Statement of a Coupled Dynamic Thermoelasticity Problem for a Bar, *Vestn. Tul'sk. Gos. Univ. Ser. Differ. Uravn. Prikl. Zadachi*, 2016, no. 1, pp. 88–92.
10. Solnechnyi, E.M., Studying the Dynamic Properties of a Distributed Thermomechanical Controlled Plant with Intrinsic Feedback. I, *Autom. Remote Control*, 2020, vol. 81, no. 4, pp. 614–621. https://doi.org/10.1134/S0005117920040049.
11. Solnechnyi, E.M., The Dynamic Properties Investigation for a Distributed Thermomechanical Control Plant, *Proceedings of 2018 Eleventh International Conference "Management of Large-Scale System Development" (MLSD)*, Moscow, October 1–3, 2018, pp. 1–4. https://doi.org/10.1109/MLSD.2018.8551890.
12. Lavrent'ev, M.A. and Shabat, B.V., *Metody teorii funktsii kompleksnogo peremennogo* (Methods of the Theory of Functions of Complex Variable), Moscow: Lan', 2002.

This paper was recommended for publication by A.G. Kushner, a member of the Editorial Board