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NONLINEAR SYSTEMS

# Estimating the Hölder Exponents Based on the $\epsilon$-Complexity of Continuous Functions: An Experimental Analysis of the Algorithm

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**Abstract**—This paper describes one method for estimating the Hölder exponent based on the $\epsilon$-complexity of continuous functions, a concept formulated lately. Computational experiments are carried out to estimate the Hölder exponent for smooth and fractal functions and study the trajectories of discrete deterministic and stochastic systems. The results of these experiments are presented and discussed.

*Keywords*: $\epsilon$-complexity of continuous functions, the Hölder exponent

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## 1. INTRODUCTION

In recent time, fractal-type processes, particularly Wiener and fractal Wiener processes, have been actively used to model various events, effects, and trends, including the so-called processes with long-term memory. Such processes are widespread in various branches of physics (e.g., see [1]), biology, and other fields of science. It seems difficult to find a field of science without representatives of this class of processes.

The Hurst exponent is an important characteristic of stochastic random processes. For its definition, we refer, e.g., to the book [2]. Estimating this exponent from an observed realization of a random process is a rather difficult problem. An estimation method was proposed in the recent paper [3].

One feature of fractal functions is that, being continuous, they are not differentiable anywhere. However, a large class of such functions satisfies the Hölder condition. Estimating the Hölder exponent (see the rigorous definition below) for fractal functions plays the same role as estimating the Hurst exponent in the stochastic case: as is well known [4], for Wiener and fractal Wiener processes with probability 1, the local Hölder exponent is arbitrarily close to the Hurst constant. On the other hand, the Hölder exponent has a close relationship to the Hausdorff dimension or the scaling exponent, which describes the persistence of the geometry or statistical characteristics under scaling. Due to the considerations above, estimating the Hölder exponent is a topical problem interesting for numerous applications.

This paper presents an experimental analysis of a fundamentally new approach to the problem of estimating the Hölder exponent (as mentioned above, the Hurst constant as well). The approach involves the $\epsilon$-complexity of continuous functions, an original concept developed by the authors

recently. It ideologically agrees with A.N. Kolmogorov's complexity of an object. Note that the concept of object's complexity is used in control and, moreover, in a wide range of other applications [5]. On the other hand, applications of the $\epsilon$-complexity of continuous functions in the last few years have attracted interest of leading peer-reviewed journals, including Q1 and Q2 journals indexed by the Web of Science and Scopus; see [6–9]. Therefore, we aim to introduce the new concept of $\epsilon$-complexity to the readers and stimulate new lines of scientific and practical research in this area.

The main results on the $\epsilon$-complexity of continuous functions and the method for estimating the Hölder exponent based on this concept can be found in [8, 10]. For ease of reading, Section 2 briefly recalls the relevant results from these publications. In Section 3, we describe the estimation procedure and the objects of analysis, i.e., some smooth and fractal continuous functions as well as sequences generated by stochastic or deterministic chaotic mechanisms. The results of computational experiments are summarized in Section 4.

## 2. $\epsilon$-COMPLEXITY AND HÖLDER EXPONENT ESTIMATION

Without loss of generality, we consider continuous functions defined on the interval $[0, 1]$. Informally speaking, $\epsilon$-complexity estimates the number of uniform discrete samples of such a function required to reconstruct it by a given set of *approximation methods* with a given accuracy. In other words, this value estimates the minimum amount of information (*in the language of approximation theory*) needed to describe such a function. In this respect, $\epsilon$-complexity agrees with the object's complexity proposed by A.N. Kolmogorov in the mid-1960s. The main idea of the Kolmogorov approach (e.g., see [11]) is as follows: *describing a complex object requires much information, whereas describing a simple one little information.* In other words, it is reasonable to evaluate the complexity of an object through the minimum amount of information required for its description.

Let a continuous function have a finite set of discrete samples on a uniform lattice with some step $1 > h > 0$. (The number of such samples is determined by the value $h$.) By a method for reconstructing (approximating) this function we mean a Borel function that maps this set of samples into a bounded function on $[0, 1]$. (The space of bounded functions is endowed with the uniform metric.) We fix arbitrary countable sets of Borel functions taking values in the space of bounded functions and depending on $1, 2, 3, \ldots$ arguments, respectively. The union of these countable sets is called a *list*. It contains a countable set of reconstruction methods for all $h > 0$. Let $\mathbb{F}$ be a fixed list of reconstruction methods. In what follows, the symbol $\mathcal{F}$ denotes a non-empty subset of $\mathbb{F}$ containing some sets of Borel functions of $1, 2, 3, \ldots$ arguments. The sets $\mathcal{F}$ (and, accordingly, the lists $\mathbb{F}$ if $\mathcal{F} = \mathbb{F}$) are said to be *admissible* if they contain methods of approximation by piecewise constant (step) functions and power polynomials. We introduce

$$\delta^{\mathcal{F}}(h) = \inf_{\hat{x}_h(\cdot)\in\mathcal{F}} \sup_{t\in[0,1]} |\hat{x}_h(t) - x(t)|,$$

where $\hat{x}_h(\cdot) \in \mathcal{F}$ are estimates of the function $x(\cdot)$ from its discrete samples with step $h$ yielded by methods of the family $\mathcal{F}$. In the case $\mathcal{F} = \mathbb{F}$, estimation involves all the functions from $\mathbb{F}$.

The following result is true.

*Let $\mathbb{F}$ be any fixed admissible list. The set of continuous functions that cannot be exactly reconstructed finitely many discrete samples by methods from $\mathbb{F}$ is dense in the space of all continuous functions on* $[0, 1]$.

Functions that *cannot be exactly reconstructed* by methods of any non-empty admissible subset $\mathcal{F} \subseteq \mathbb{F}$ are said to be $\mathcal{F}$-nontrivial.

Let $\mathbb{F}$ be a fixed admissible list and $\mathcal{F} \subseteq \mathbb{F}$ be any non-empty admissible subset. We take an $\mathcal{F}$-nontrivial function $x(t)$ such that $\max\limits_{t\in[0,1]} |x(t)| = R$. For a sufficiently small number $\epsilon > 0$, we

introduce

$$h_x^*(\epsilon,\mathcal{F}) = \inf\left\{h \leqslant 1 : \frac{\delta^{\mathcal{F}}(h)}{R} > \epsilon\right\}.$$

*The $(\epsilon,\mathcal{F})$-complexity of a continuous function $x(\cdot)$ is defined as the value* $\mathbb{S}_x(\epsilon,\mathcal{F}) = -\log h_x^*(\epsilon,\mathcal{F})$.

If a function is not $\mathcal{F}$-nontrivial (i.e., it can be exactly reconstructed from finitely many discrete samples), its $(\epsilon,\mathcal{F})$-complexity equals 0. Therefore, according to the assertion above, "almost all" continuous functions have a nonzero $(\epsilon,\mathcal{F})$-complexity for any non-empty admissible subfamily $\mathcal{F} \subseteq \mathbb{F}$ of any fixed admissible list $\mathbb{F}$.

Note that $h_x^*(\epsilon,\mathcal{F}) > 0$ for $\epsilon > 0$ and $\lim_{\epsilon\to 0} h_x^*(\epsilon,\mathcal{F}) = 0$ for an $\mathcal{F}$-nontrivial function. On the other hand, $\lim\limits_{h\to 0} \delta^{\mathcal{F}}(h) = 0$. Therefore, for any (sufficiently small) $\epsilon > 0$, there exists some $\eta(\epsilon) > 0$ such that $\eta(\epsilon) \to 0$ as $\epsilon \to 0$ and $\delta^{\mathcal{F}}\left(h_x^*(\cdot)\right) \leqslant \eta(\epsilon)$.

The value $1/h_x^*(\epsilon,\mathcal{F})$ estimates the number of discrete samples of a function. Hence, the $(\epsilon,\mathcal{F})$-complexity of a function is the logarithm of the number of its discrete samples needed to reconstruct this function by methods of the family $\mathcal{F}$ with a maximum relative error of $R^{-1}\eta(\epsilon)$. In other words, this is the *shortest description of the function* by given methods with a given accuracy; see the discussion at the beginning of Section 2.

Let $x(t) : [0,1] \to \mathbb{R}^1$, $\max\limits_{t\in[0,1]} |x(t)| = R$, be a continuous function. Suppose that there exist constants $K > 0$, $0 < p \leqslant 1$ (*the Hölder exponent*) such that $\sup\limits_{(t_1,t_2)\in[0,1],t_1\neq t_2} |t_1 - t_2|^{-p}|x(t_1) - x(t_2)| \leqslant K$. (If this inequality holds for $|t_1 - t_2| \leqslant \epsilon$ with some $\epsilon > 0$, the matter concerns *the local Hölder exponent*.)

We fix some admissible list $\mathbb{F}$ of reconstruction methods and consider the set $\mathcal{T}$ of Hölder functions *not exactly reconstructible by methods of* $\mathbb{F}$ *from finitely many samples.* Let a function $x(\cdot)$ belong to some dense subset $T_0 \subseteq \mathcal{T}$ and *be determined by its n samples* (i.e., its trace) on a uniform lattice of the interval $[0,1]$. The main result on the concept of $\epsilon$-complexity implies the following: for all admissible families $\mathcal{F} : \mathcal{F}^* \subset \mathcal{F} \subseteq \mathbb{F}$ of approximation methods, where $\mathcal{F}^* \subset \mathbb{F}$ is a sufficiently rich set, and sufficiently small numbers $\epsilon > 0$, there exists a lattice step $h_x^*(\epsilon,\mathcal{F})$ such that the complexity $\mathbb{S}_x(\epsilon,\mathcal{F}) = -\log h_x^*(\epsilon,\mathcal{F})$ can be effectively described.

We now fix a (sufficiently small) number $\epsilon > 0$ and consider a set $\mathcal{F} \supseteq \mathcal{F}^*$ of reconstruction methods. Suppose that the number $n$ of samples determining the function satisfies the condition $n \geqslant n_0$, where $[h_x^*(\epsilon,\mathcal{F})n_0] = C$ with $C \gg 1$. (Throughout this paper, $[a]$ denotes the integer part of a value $a$.) This means that keeping only one of $[h_x^*(\cdot)n]$ samples (discarding the others) will be sufficient for reconstructing the function with an accuracy of $R^{-1}\eta(\epsilon)$.

Choosing $0 < S < 1$, we discard some of the initial $n$ samples of the function $x(\cdot)$ so that the remaining $[Sn]$ samples are arranged *uniformly*.

The following result is true.

*Let a Hölder function from a dense subset $\mathcal{T}_0$ be given by its n discrete samples on a uniform lattice on the interval* $[0,1]$. *If $\mathcal{F}$ is a sufficiently rich set of approximation methods, n is sufficiently large, and $\epsilon > 0$ is sufficiently small, then*

$$\log\epsilon = A_n - p\log S + \varphi(\epsilon,n), \tag{1}$$

where $\lim\limits_{n\to\infty} \frac{A_n}{\log n} = -p$ and $\varphi(\cdot)$ is a bounded function.

The Hölder exponent is estimated by the following algorithm. Several values $S = S_k$, $k = 1,\ldots,m$, of the parameter $0 < S < 1$ are specified; for each $S_k$, the least approximation error $\epsilon_k$ of the function is calculated over the list $\mathcal{F}$ of approximation methods at the discarded points.

(The function is given by its trace on the lattice.) Then the standard least squares method is applied to the set of pairs $(\log \epsilon_k, \log S_k)$ to determine the estimates $\hat{A}_n$ of the coefficients in (1). Let $\hat{A}_n = A_n + \rho(\epsilon, n, m)$, where $\rho(\epsilon, n, m)$ is the estimation error of the least squares method. Due to the boundedness of the function $\varphi(\cdot)$, the error $\rho(\epsilon, n, m)$ is bounded as well. Therefore,

$$\lim_{n\to\infty} \frac{\hat{A}_n}{\log n} = -p. \tag{2}$$

## 3. THE ESTIMATION PROCEDURE. THE OBJECTS OF ANALYSIS

### *3.1. Algorithmic Procedure Description*

Let $\{x(k)\}_{k=1}^n$ be the trace of a continuous function on some uniform lattice. According to the previous considerations, the minimum approximation error can be estimated as

$$\log \epsilon \approx A_n + B_n \log S, \tag{3}$$

where $\approx$ means the approximate reconstruction of a linear dependence of $\log \epsilon$ on $\log S$ within the linear regression remainders (omitted in this expression). The coefficients $A_n$ and $B_n$ are called *the complexity coefficients*. The algorithmic procedure is intended to estimate them.

We choose some value of the parameter $S$ from the following series: $S = 1/2$, $1/3$, $1/4$, $1/5$, $1/6$, $1/10$. The kept values of the observed sequence $\{x(k)\}$ (those not discarded) have different possible arrangements depending on the value $S$. For example, for $S = 1/2$, we may keep the values $x(1), x(3), x(5), \ldots$ or the values $x(2), x(4), x(6), \ldots$. For other values of the parameter $S$, the kept discrete samples of the function have even more possible arrangements.

The sequence of computations is as follows:

(1) For a chosen value $S$ and each possible arrangement of the kept points of the sequence $\{x(k)\}$, the minimum error $\epsilon(S, i)$ of reconstructing the function at the discarded points from the kept ones is determined over all available approximation methods. Here the number $i$ is associated with possible arrangements. Then, it is necessary to calculate $\epsilon(S) = \min\limits_i \epsilon(S, i)$, i.e., the minimum reconstruction error over all possible arrangements of the kept points for the value $S$.

(2) The described process is repeated for each value $S$ from the series above.

(3) Computations in steps 1 and 2 yield the set $(\log \epsilon(S), \log S)$. The standard least squares method is applied to this set to find the complexity coefficients in (3) for the sample length $n$. Then all computations are repeated to study the limiting behavior of these coefficients as $n \to \infty$.

Due to finite dimension, the choice of an appropriate norm for calculating the approximation error is not crucial. We adopted the norm $l_1$ and piecewise polynomials up to degree 4 inclusive as the approximation methods for computational experiments.

### *3.2. The Objects of Analysis*

The following functions were studied in computational experiments.

(1) The smooth function

$$x(t) = \sum_{i=1}^{k} a_i \sin \omega_i t,$$

where $\omega_i$ are incommensurate frequencies. Although such a function looks like "complex," it is differentiable and its Hölder exponent equals 1.

(2) The Weierstrass function ([4])

$$x(t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \sin(\lambda^k t),$$

where $1 < \lambda$ and $1 < s < 2$. Its Hölder exponent is $(2 - s)$.

(3) The standard Wiener process (Brownian motion). As is known (e.g., see [12]), this process has the following trajectory with probability 1:

$$x(t) = \frac{1}{\sqrt{\pi}} C_0 t + \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} C_k \frac{\sin kt}{k},$$

where $C_k$ are independent standard Gaussian random variables. The *local Hölder exponent* of this process is arbitrarily close to 1/2 with probability 1; for example, see [13].

For the functions given by series, it was necessary to estimate a sufficient number of series terms: only finite sums can be simulated. The following rule seems reasonable: the error of replacing a series by a finite sum (with respect to the signal scale) should be at least two orders of magnitude smaller than the error of approximating the function at the discarded points with the available set of methods. For a stochastic process, the error is the standard deviation value and all computational results must be averaged over a large number of realizations.

We also considered the trajectories of discrete dynamic systems, both stochastic and deterministic, of the chaotic type. Any finite-length numerical sequence can be treated as the trace of some continuous function on a finite interval. Indeed, for example, a polynomial of appropriate degree passes through all values of a given sequence; hence, the set of continuous functions whose trace coincides with this numerical sequence is non-empty. Adhering to this viewpoint, we can apply the concept of $\epsilon$-complexity to any finite numerical sequence and use the basic relation (3).

According to rich experimental evidence, the relation (3) works for numerical sequences generated by discrete dynamic systems of both stochastic and deterministic type. We refer to the numerous experiments, particularly with real data, published in the our recent works. However, it is especially interesting to study the behavior of the complexity coefficients for different sequences when increasing their length.

## 4. THE RESULTS OF COMPUTATIONAL EXPERIMENTS

### *4.1. Smooth Functions*

In the first experiment, the sum of sinusoids with coprime frequencies was analyzed; see the graph in Fig. 1.

Figure 2 plots the dependence of the calculated $\epsilon$-complexity coefficients on the number of points on the interval $[0, 1]$. All results in this section are presented as two graphs: the value $A(n)/\ln(n)$ (on the left) and the coefficient $B(n)$ (on the right).

According to the graphs in Fig. 2, both converged to $-1$ with increasing $n$, which fully matches the hypothesis on the Hölder exponent.

### *4.2. The Weierstrass Function*

The Weierstrass function was simulated using a finite number $k_0$ of terms; see the formula in Subsection 3.2. We present the formula for estimating the marginal residual sum provided that $\lambda^{(s-2)} < 1$ for $1 < \lambda$ and $1 < s < 2$:

$$S_{rest} = \sum_{k=k_0+1}^{\infty} \lambda^{(s-2)k} \sin(\lambda^k t) \leqslant \sum_{k=k_0+1}^{\infty} \lambda^{(s-2)k} = \frac{\lambda^{(s-2)(k_0+1)}}{1 - \lambda^{(s-2)}} = \frac{A}{2} \lambda^{(s-2)k_0}.$$

Here, the constant $A = \frac{2}{\lambda^{(2-s)}-1}$ serves for normalizing the signal scale.

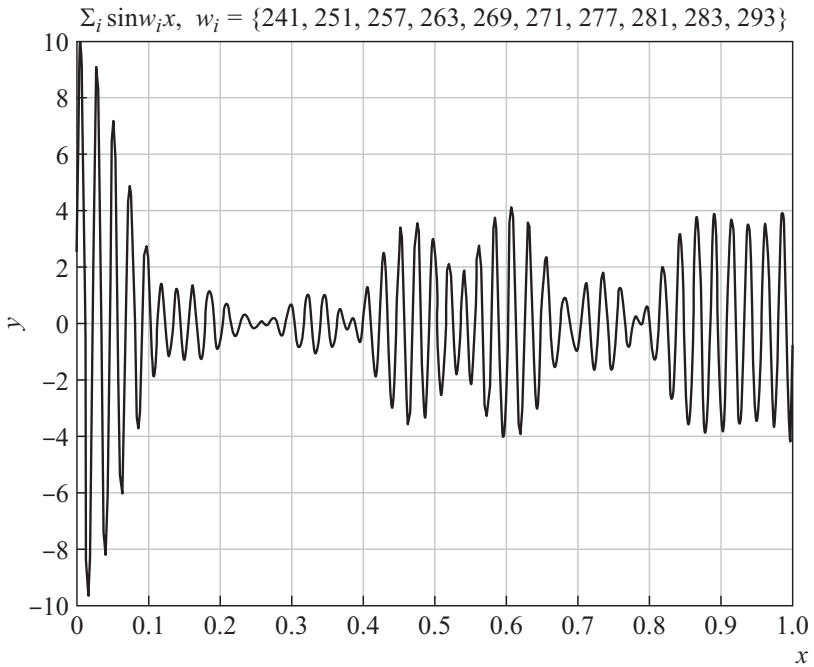


**Fig. 1.** An example of smooth functions.

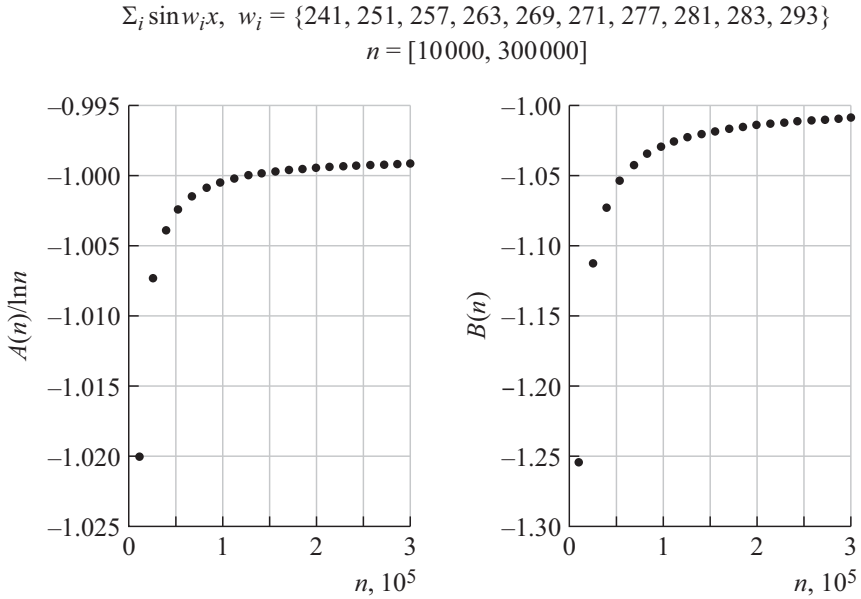


**Fig. 2.** Computational results for the smooth function.

We studied two modifications of the Weierstrass function with the values $s = 1.1$ and $s = 1.3$ (Fig. 3), i.e., with the Hölder exponents $\alpha = 0.9$ and $\alpha = 0.7$, respectively. The computational results for the Weierstrass function are demonstrated in Fig. 4.

Note that under an insufficient number of the series terms (when the residual sum was large), the complexity coefficients converged to $-1$, as in the case of of smooth functions. In other words, the simulated function lost its fractal properties.

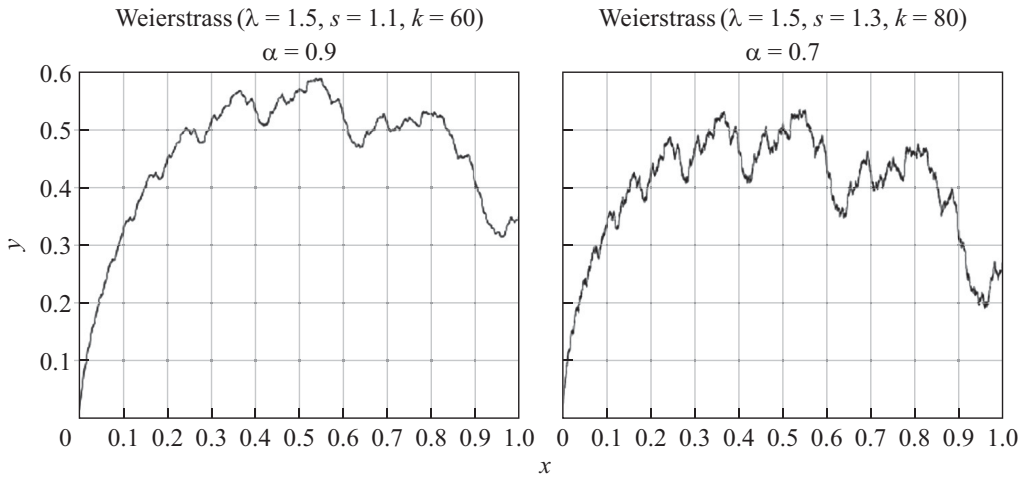


**Fig. 3.** Simulation of the Weierstrass function with $s = 1.1$ and $s = 1.3$.

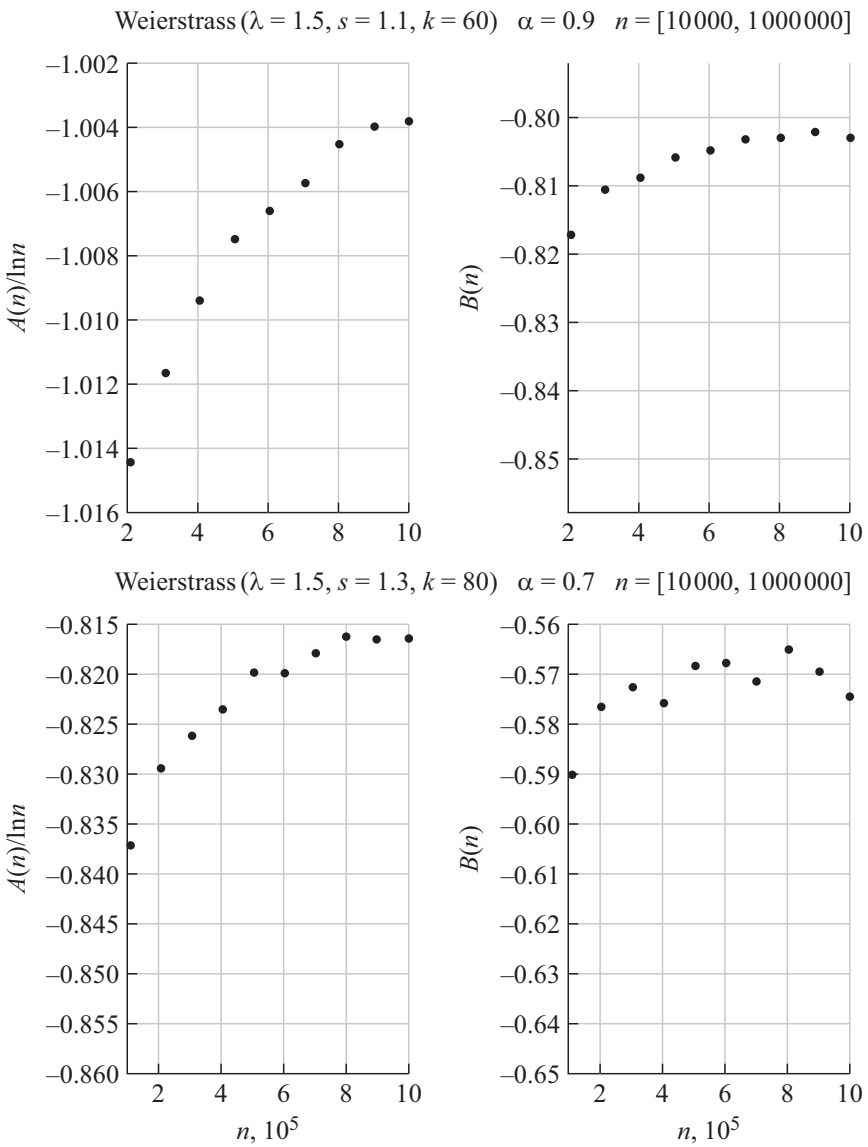


**Fig. 4.** Computational results for the Weierstrass function with $s = 1.1$ and $s = 1.3$.

In both examples considered, the limit values of the coefficients exceeded $-1$. In the experiment with $s = 1.1$, the values of the coefficient $B$ converged to approximately $-0.8$; in the experiment with $s = 1.3$, to $-0.57$. As for the coefficient $A$, due to the limiting nature of the ratio $A(n)/\ln(n)$, the amount of points considered was probably insufficient for convergence to $-p$ according to (1).

### *4.3. Wiener Process*

Wiener processes include a random component. Therefore, in this case, we estimate not the residual sum of the series but its variance. According to the previous section, the sum of all terms starting from some number $k_0$ has the form

$$S_{rest} = \sqrt{\frac{2}{\pi}} \sum_{k=k_0+1}^{\infty} C_k \frac{\sin(kt)}{k},$$

where $C_k \propto \mathcal{N}(0,1)$ are independent random variables with the standard Gaussian distribution.

The variance of this sum is expressed as follows:

$$\mathcal{D}\left[S_{rest}\right] = \mathcal{D}\left[\sqrt{\frac{2}{\pi}} \sum_{k=k_0+1}^{\infty} C_k \frac{\sin(kt)}{k}\right] = \frac{2}{\pi} \sum_{k=k_0+1}^{\infty} \mathcal{D}\left[C_k \frac{\sin(kt)}{k}\right]$$

$$= \frac{2}{\pi} \sum_{k=k_0+1}^{\infty} \frac{\sin^2(kt)}{k^2} \mathcal{D}\left[C_k\right] = \frac{2}{\pi} \sum_{k=k_0+1}^{\infty} \frac{\sin^2(kt)}{k^2} \leqslant \frac{2}{\pi} \sum_{k=k_0+1}^{\infty} \frac{1}{k^2}$$

$$= \frac{2}{\pi} \left(\frac{\pi^2}{6} - \sum_{k=1}^{k_0} \frac{1}{k^2}\right) = \frac{\pi}{3} - \frac{2}{\pi} \sum_{k=1}^{k_0} \frac{1}{k^2}.$$

As a result,

$$\delta = 3 * \mathrm{std}\left(S_{rest}\right) = 3\sqrt{\mathcal{D}\left[S_{rest}\right]} \leqslant \sqrt{\frac{\pi}{3} - \frac{2}{\pi} \sum_{k=1}^{k_0} \frac{1}{k^2}}.$$

Due to the Gaussian nature of the process, the value $\delta$ (the triple standard deviation of the residual sum) is used to determine the number of terms left after truncation of the series. For example, $\delta$ takes value $2.4 \times 10^{-2}$, $3.4 \times 10^{-3}$, and $7.5 \times 10^{-4}$ when using $k_0 = 10\,000$, $k_0 = 500\,000$, and $k_0 = 10\,000\,000$ terms, respectively. It seems reasonable to choose $k_0$ so that the triple standard deviation of the residual sum is at least two orders of magnitude smaller than the approximation error when calculating the complexity coefficients.

We used the value $k_0 = 10^7$ for the simulation. Due to the randomness of the process, all results were averaged over $n_{mc} = 100$ realizations. Figures 5 and 6 show one realization of the Wiener process and the computational results, respectively.

According to the computational results, the technology of calculating the complexity coefficients is sensitive to the process type. As in the case of the Weierstrass function, the complexity coefficients in this example converged to a value differing from $-1$. The coefficient $B$ took values near $-0.405$ instead of the theoretical value $-0.5$ corresponding to the Hölder exponent for the Wiener process; see the graph in Fig. 6.

Thus, in the examples under consideration, the experimental error of calculating the Hölder exponent varied from 12 to 19%. Since the tested hypothesis concerns limiting dynamics, it is obviously possible to improve the accuracy of computations by increasing the number $n$ of points. Furthermore, as mentioned above, the experiments involved only piecewise polynomials of degree 4 inclusive as approximation methods. Therefore, expanding this list (say, with splines, wavelet transforms, etc.) will make the computation results more consistent with the theory.

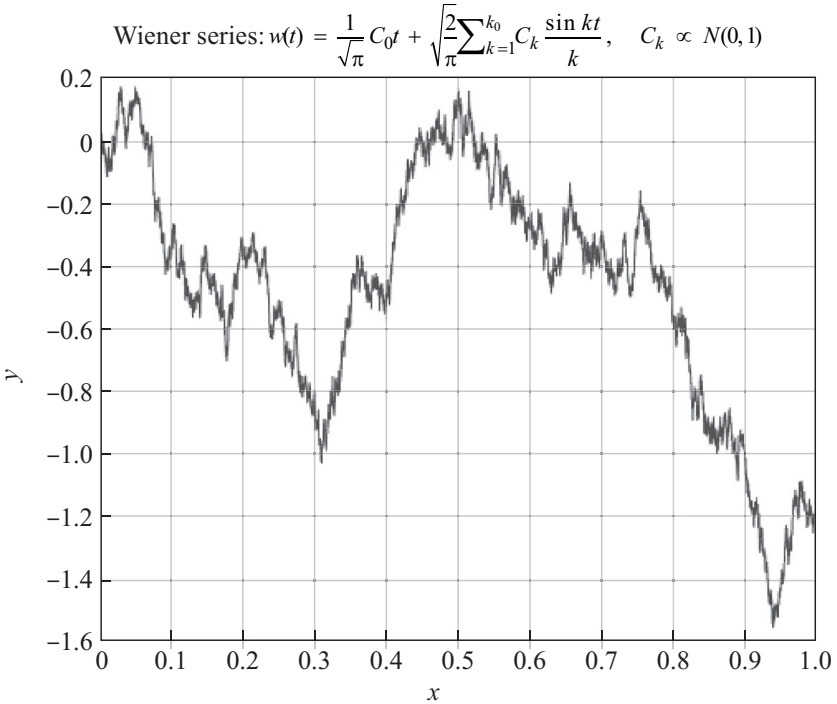


**Fig. 5.** Simulation of the Wiener process.

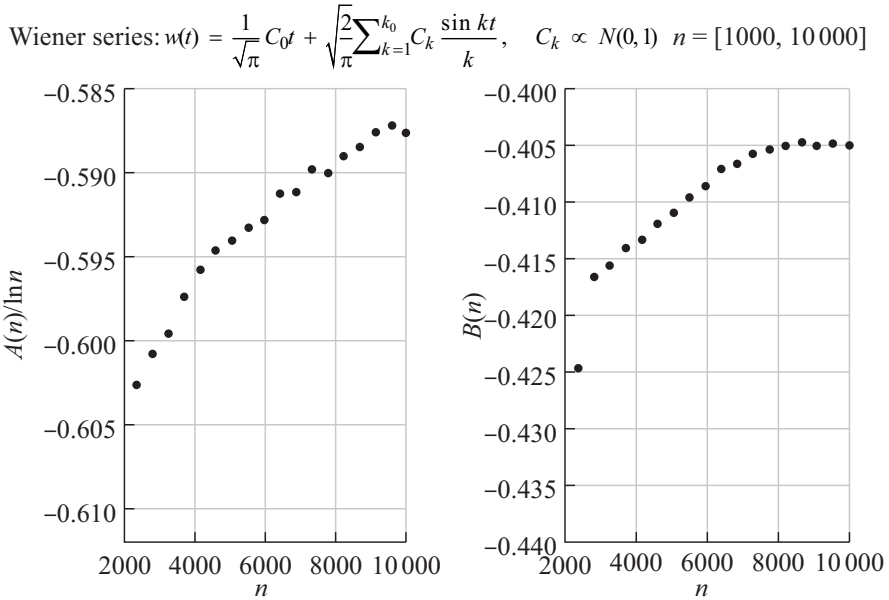


**Fig. 6.** Computational results for the Wiener process.

### *4.4. The Trajectories of Discrete Dynamic Systems*

This series of experiments was intended to study the trajectories of discrete dynamic systems using $\epsilon$-complexity coefficients. In contrast to the previous experiments, the interested was not in the limits but scatter of the coefficients due to increasing the sequence length.

We considered two systems:

1) the classical autoregressive moving-average model $ARMA(p, q)$ with the parameters $p = 3$ and $q = 2$ (Fig. 7);

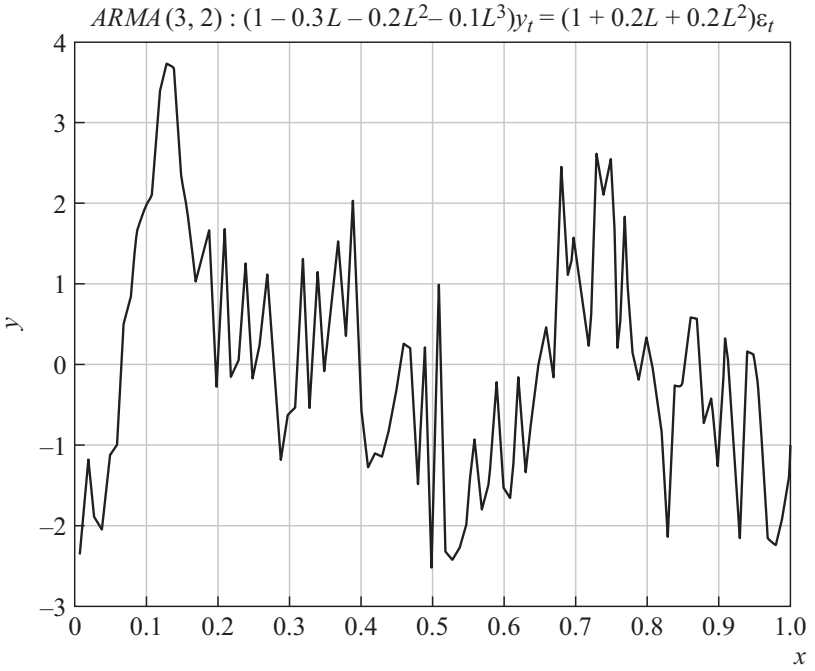


**Fig. 7.** One example of stochastic sequences.

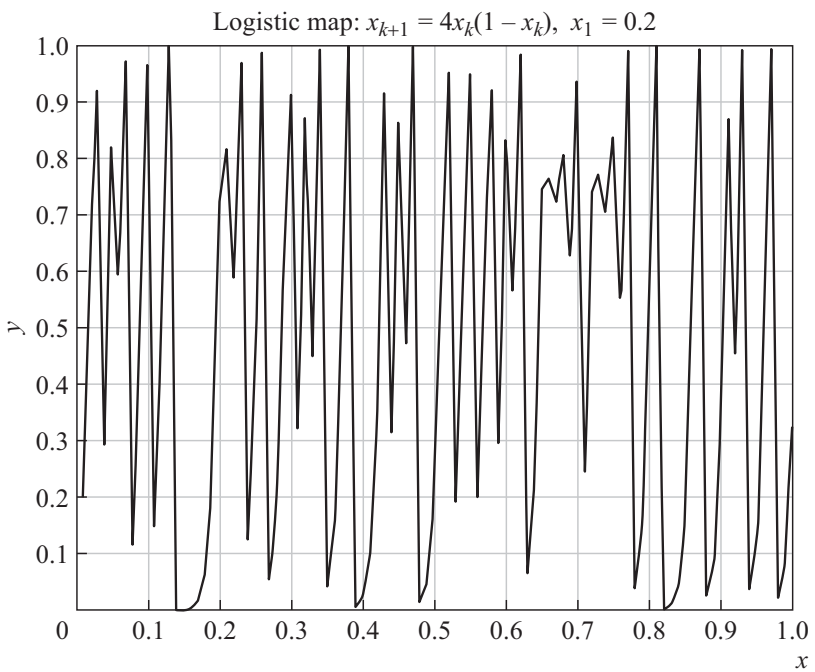


**Fig. 8.** One example of chaotic sequences.

2) a deterministic sequence of chaotic type, the so-called logistic map (Fig. 8):

$$x_{k+1} = 4x_k\,(1 - x_k)\,, \quad x_0 = 0.2.$$

In case 2), the trajectory essentially depends on the initial value, which is an intrinsic property of chaotic sequences. Therefore, during the simulation, we neglected the first 1000 values of the sequence before starting to calculate the complexity coefficients.

Computational results for sequences

| $n$ | 100–1000 | | 1000–10 000 | | 10 000–100 000 | |
|---|---|---|---|---|---|---|
| Sequence | $A_{mean}$<br>$A_{std}$ | $B_{mean}$<br>$B_{std}$ | $A_{mean}$<br>$A_{std}$ | $B_{mean}$<br>$B_{std}$ | $A_{mean}$<br>$A_{std}$ | $B_{mean}$<br>$B_{std}$ |
| $ARMA(3,2)$ | −0.4420<br>0.0071 | −0.1856<br>0.0039 | −0.4458<br>0.0030 | −0.1880<br>0.0015 | –<br>– | –<br>– |
| *Logistic map* | −1.0199<br>0.0173 | −0.0223<br>0.0121 | −1.0432<br>0.0089 | −0.0247<br>0.0048 | −1.0384<br>0.0031 | −0.0263<br>0.0014 |

It seems convenient to present the computation results of these experiments in tabular form instead of graphs. The table contains the mean values of the complexity coefficients and their standard deviations for each process under consideration with different sequence lengths.

According to the calculations results, the relative value of the standard deviation decreased from 1.7–2% to 0.7–0.8% for both coefficients when reaching 10 000 points in the $ARMA$ model as well as down to 0.3% for the coefficient $A$ and 5.3% for the coefficient $B$ when reaching 100 000 points in the chaotic sequence.

Note that the mean and standard deviation statistics were calculated not over different realizations of a random process but over different sequence lengths in the specified ranges. This kind of experiment makes practical sense for analyzing real data when the process model (or even the true value of its Hölder exponent) is unknown.

## 5. CONCLUSIONS

This paper has described a method for estimating the Hölder exponent based on the $\epsilon$-complexity of continuous functions and the corresponding computational experiments. This concept agrees with A.N. Kolmogorov's general idea on measuring the complexity of certain objects. For the class of Hölder functions, $\epsilon$-complexity is effectively described by the so-called *complexity coefficients*. The use of complexity coefficients allows estimating the Hölder exponent, which is widespread in the analysis of fractal-type data in various applications. In particular, the Hölder exponent almost coincides with the Hurst constant for fractal Brownian motion, a model for many physical and financial processes. The estimation method of the Hölder exponent is quite simple to implement and does not require statistical procedures. In the computational experiments carried out, the complexity coefficients has been calculated based on a very short list of approximation methods. (The set of such methods conceptually underlies the $\epsilon$-complexity of continuous functions.) However, the experimental results obtained testify to the effectiveness of the estimation method even under such conditions. The performance of this method can be significantly improved by extending the set of algorithms for approximating continuous functions from their discrete samples. Other experiments have been intended to estimate the behavior of complexity coefficients for the trajectories of discrete dynamic systems. Although there is no rigorous mathematical theory for this case, the complexity coefficients have turned out to possess limits with increasing the sample size. Due to this fact, the technology of calculating complexity coefficients can be applied for arbitrary discrete processes in a sliding window of a relatively small size for data segmentation and classification.

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