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Exponentially Stable Adaptive Control. Part II. Switched Systems

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Abstract—An adaptive state-feedback control system for a class of linear systems with piecewise-constant unknown parameters is proposed. The solution ensures a global exponential stability of a closed-loop system under condition that a regressor is finitely exciting after each parameters switch, and does not require neither any knowledge of a plant input matrix, nor the switching time instants. The obtained theoretical results are corroborated by numerical simulations.

Keywords: adaptive control, switched systems, piecewise-constant parameters, parameter error, finite excitation, identification, exponential stability

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1. INTRODUCTION

Being applied to uncertain plants with time-invariant parameters, classical algorithms of reference model adaptive control ensure asymptotic convergence of the tracking error [1–3]. However, as far as practical scenarious are concerned, real physical systems are often described by models with time-varying or piecewise-constant unknown parameters. Under such conditions, the above-mentioned solutions guarantee the asymptotic stability only if the unknown parameters variation is represented as a function that satisfies some special requirements [1, 2, 4]. The rate of the time-varying parameters change must be significantly lower than the one of the state vector elements transients (quasi-stationarity requirement). The time range between two consequitive changes of piecewise-constant parameters must be sufficiently long (the regularity requirement). In detail the problems of application of classic reference model-based adaptive systems for plants with time-varying or piecewise-constant unknown parameters have been discussed and experimentally demonstrated in [1, p. 552–554, p. 732–734; 2, p. 337–345].

Modern composite modifications [5–7] of classic algorithms of model reference adaptive control are aimed at relaxation of the well-known requirement of regressor persistent excitation, which, considering mentioned algorithms, is necessary and sufficient for exponential stability of the tracking error [8]. Composite algorithms, in their majority, are based on inertial schemes of measured signal processing, which allow one to reduce the problem of adaptive control to the one of identification of linear regression equation unknown parameters. The persistent excitation requirement is relaxed by application of special intelligent algorithms or various filters with memory to store previously measured signals values in a data stack, so that the parameters of the control law are adjusted even after the end of the excitation period [9]. A common drawback of the above-considered modified adaptation algorithms is the requirement that the unknown parameters of the plant are to be time-invariant, which is necessary to prevent the mixture of data on different values of the unknown parameters in the data stack [10]. The problems of application of the composite model reference-based adaptive control systems for plants with time-varying or piecewise-constant unknown parameters have been discussed in detail and experimentally demonstrated in [6, Fig. 7, Fig. 8; 7, Fig. 2; 10, Fig. 4].

Thus, nowadays, the development of model reference adaptive control methods for plants with time-varying or piecewise-constant unknown parameters remains important and actual problem. Without pretending to provide an exhaustive review, below we focus our attention on key methods to solve the adaptive control problems for a class of systems with piecewise-constant unknown parameters.

The motivation to consider the switched systems control problems, first of all, is related to the popular in applications technique of linearization of physical systems nonlinear models in the neighborhood of operating points [11, p. 13; 12]. The classic model with parameters switches obtained with the above-mentioned technique consists of a continuous part, which includes a differential equation of a known order, and a discrete part that defines the logic of the equation parameters changes. Such logic describes when the plant trajectories enter a certain state space polyhedral region associated with a new operating point. The number of these regions coincides with the number of linear models with unknown parameters, by which the initial nonlinear model can be approximated with sufficient accuracy. Since usually the parameters of each model are unknown or known approximately, the design of control laws for switched systems is to be based on the adaptive control methods.

The pioneering studies by Tao [12–15], in which a unified adaptive control system for switched systems was proposed and its advantages over classic algorithms of adaptive control for time-invariant plants were demonstrated, became the starting point for the development of the adaptive control design procedures for the plants with piecewise-constant parameters. The switching signal is assumed to be known, and as many control laws with adjustable parameters are introduced as many polyhedral regions are defined in the state space of the original nonlinear system. Switching between control laws is made synchronously with switching of the plant model parameters values. The parameters of each control law are adjusted by its own adaptive law and only when it is in use (active). The reference model that defines the desired control quality can be implemented as a system with time-invariant parameters or as a switched system. Moreover, in order to improve the control quality, switches of the reference model parameters values can be made asynchronously with the switches of plant parameters. The asymptotic stability of such hybrid adaptive control strategy and the boundedness of all signals can be proved using either common Lyapunov function [16] or multiple Lyapunov function [16]. The first approach is used if there exists a general solution of the Lyapunov equation for all state matrices of the switched reference model; the second method is applied in the opposite case. It is important to note that, if a multiple Lyapunov function is used, the system asymptotic stability is guaranteed only if the regressor persistent excitation condition is met, whereas, if a common Lyapunov function is applied, such condition is required only to provide an exponential convergence rate. The disadvantage of the results [12–15] is that the tracking error exponential stability is ensured only when the condition of the regressor persistent excitation is satisfied, which results in unsatisfactory reference model tracking quality if it is not met and the plant parameters are frequently switched.

The disadvantages of solutions [12–15] have been overcome by the application of composite adaptive laws that relax the regressor persistent excitation condition. Based on the composite learning algorithm [5], in [17–21] the adaptive laws are proposed that guarantee exponential stability of a closed-loop with switched system if the regressor finite excitation requirement is met after each switch. These laws make it possible to adjust the parameters of inactive control laws if an information matrix of full rank has been obtained over the time period when they were active.

Owing to this technique, the global exponential stability of the tracking error and the convergence of all parametric errors are proved. The disadvantage of [17–21] is the use of nontrivial off-line procedures for monitoring and processing of signals measured from the plant in order to obtain full-rank information matrix after each switch of the plant parameters.

The considered solutions [12–15, 17–21] are based on the assumption that the switching logic/signal is known and related to fact that the plant trajectories enter certain regions of the state space. However, in practice, firstly, the linearization points, boundaries of polyhedral regions and, consequently, the switching logic/signal may be unknown or not known with required accuracy, and secondly, the parameter switches may be caused not only by the plant trajectories, but also by other events of a discrete nature, including the influence of unaccounted nonlinearities, external parametric disturbances, actuator failures or damages. Therefore, the problem of design of adaptive control algorithms that detect the switching time instants and simultaneously adjust the control law parameters is actual.

Two different detection algorithms are proposed in [22, 23], which in the presence of external perturbations allow one to identify time moments of discontinuous changes of the plant switching state with sufficient accuracy. Ideologically, the detection algorithms are based on indirect comparison of current plant parameters values with previous ones, data about which is stored in a special array. If, in the sense of the chosen metric, the indirect information about current parameters differs sufficiently from the indirect information about previous ones, then the plant has changed its switching state. After detection, a new data array is created and filled with indirect information about the plant current parameters values. Subsequently, for detection purposes, indirect comparison of current plant parameters is performed with the information about all previous plant switching states stored in the arrays. The main difference between solutions [22, 23] and [12–15, 17–21] is that there is no need to know *a priori* both the plant parameters switching logic/signal and the number of linearization points of the initial nonlinear model, and therefore as many adjustable control laws are introduced as many plant switching states are identified in the course of the detection process. At the same time, the solutions [22, 23] are based on the composite learning concept, which, as in [17–21], allows one to adjust the control laws parameters for all switched states simultaneously using the stored data. The disadvantages of [22, 23] algorithms are off-line data manipulation and the fact that only indirect adaptive laws can be designed with the well-known involved difficulties [1–3, 21]. A more detailed review of modern and classic methods of identification and adaptive control of switched systems can be found in [11] and statement sections of studies [12–15, 17–23].

In general, all the above-considered algorithms of switched systems adaptive control have common drawbacks, the main of which are, firstly, the discontinuous behavior of the control signal when the system switches to the control law for a particular region of the nonlinear system state space, and secondly, the fact that an excessive number of structurally identical adaptive laws to adjust the controller parameters are used.

Both disadvantages are related to the idea that the switched system is controlled using an appropriate control law with parameters switches (switched control). Considering the adaptive control problem, it is stated in [12, 24] that, if we have many control and adaptive laws and switch between them, then such control system improves the parameter adjustment rate and provides better control quality in comparison with the one based on the common control and common adaptive laws. Moreover, often the motivation to apply several adaptive laws is based on the need to use a switched reference model in case when a common Lyapunov function does not exist [11]. However, the concept of switched control contradicts the basic principle of model reference adaptive control, according to which continuous adjustment of a single control law to meet the current plant parameters [1–3] is required to control a system with parametric uncertainty. The rejection of this fundamental principle and the use of the switched control concept are caused by the disadvantages of the classic adaptive law and, first of all, by the slow convergence rate and lack of the capability of unknown piecewise-constant parameters tracking.

Thus, summarizing the aforesaid, the aim of this study is to develop a new adaptive control system for plants with piecewise-constant unknown parameters, which uses a common control law and common adaptive law for all possible plant switching states. Under such problem statement, the switching signal/logic is assumed to be unknown, and the reference model is chosen to be common for all regions of the system state space.

The main result of this study, which allows one to achieve the stated goal, is based on the combination of the exponentially stable adaptive control approach [25] proposed in the first part of this article series with the recently developed identification law of the unknown piecewise-constant parameters of the linear regression equation [26]. The differences between the proposed adaptive control system for plants with piecewise-constant unknown parameters and the algorithms discussed in the above-given review and others, which the interested reader can find in the references lists of [11–15, 17–23], are summarized as follows:

1) the control law without parameter switches is used for switched systems;
2) the control law parameters are adjusted by one new direct adaptive law, which is capable of tracking unknown ideal piecewise-constant parameters of the ideal control law;
3) off-line procedures to process arrays of data measured from the plant are not used;
4) *a priori* information about the values/sign of the plant input matrix is not required;
5) plant parameters switches are caused either by plant trajectories that cross borders between polyhedral regions of state space, or by various unknown events of discrete nature;
6) global exponential stability of the closed-loop system and exponential convergence of the adjustable controller parameters to their true values are guaranteed if a sufficiently weak condition of the regressor finite excitation is met after each switch of plant parameters.

Definitions

The definition of the regressor finite excitation and the corollary of the Kalman–Jakubovich–Popov lemma will be used to prove the theorems and propositions.

Definition 1. A regressor $\omega(t)$ is finitely exciting $\omega(t) \in \text{FE}$ over the time range $[t_r^+; t_e]$ if there exists $t_r^+ \geqslant 0$, $t_e > t_r^+$ and α such that the following inequality holds:

$$\int\limits_{t_r^+}^{t_e} \omega(\tau)\,\omega^{\mathrm{T}}(\tau)\,d\tau \geqslant \alpha I_{n\times n}, \tag{1.1}$$

where $\alpha > 0$ is the excitation level, $I_{n\times n}$ stands for an identity matrix.

Corollary 1. *For any matrix $D > 0$, controllable pair (A, B) with a Hurwitz matrix $A \in R^{n\times n}$ and $B \in R^{n\times m}$ there exists $P = P^{\mathrm{T}} > 0$, $Q \in R^{n\times m}$, $K \in R^{m\times m}$ and a scalar $\mu > 0$ such that:*

$$\begin{gathered} A^{\mathrm{T}}P + PA = -QQ^{\mathrm{T}} - \mu P, \quad PB = QK, \\ K^{\mathrm{T}}K = D + D^{\mathrm{T}}. \end{gathered} \tag{1.2}$$

2. PROBLEM STATEMENT

A class of continuous linear systems with piecewise-constant parameters is considered:

$$\begin{gathered} \forall t \geqslant t_0^+, \quad \dot{x}(t) = \Theta_{\kappa(t)}^{\mathrm{T}}\Phi(t) = A_{\kappa(t)}x(t) + B_{\kappa(t)}u(t), \quad x\left(t_0^+\right) = x_0, \\ \Phi(t) = \begin{bmatrix} x^{\mathrm{T}}(t) & u^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}, \quad \Theta_{\kappa(t)}^{\mathrm{T}} = \begin{bmatrix} A_{\kappa(t)} & B_{\kappa(t)} \end{bmatrix}, \end{gathered} \tag{2.1}$$

where $x(t) \in R^n$ is the plant state with unknown initial conditions x_0, $u(t) \in R^m$ stands for a control signal, $A_{\kappa(t)} \in R^{n\times n}$ denotes an unknown state matrix, $B_{\kappa(t)} \in R^{n\times m}$ is an unknown input matrix, $\kappa(t) \in \Xi = \{1, 2, \ldots, N\}$ stands for unknown discrete function that defines plant parameters switching time instants, t_0^+ denotes known initial time instant, N is a number of values, which the parameters $\Theta_{\kappa(t)}$ can take. The pair $\left(A_{\kappa(t)}, B_{\kappa(t)}\right)$ is controllable, $\forall t > t_0^+$ the vector $\Phi(t) \in R^{n+m}$ is measurable, and the matrix $\Theta_{\kappa(t)} \in R^{(n+m)\times n}$ is unknown.

To be specific, it is assumed that $\kappa(t)$ and $\Theta_{\kappa(t)}$ are right-continuous:

$$\forall t \geqslant t_0^+ \quad \kappa(t) = \lim_{\tau \to t_i^+} \kappa(\tau), \quad \Theta_{\kappa(t)} = \lim_{\tau \to t_i^+} \Theta_{\kappa(\tau)}, \tag{2.2}$$

where t_i^- is a time instant correspondent to the function value on the left from a jump, t_i^+ is a time instant correspondent to the function value on the right from a jump.

In a general case, the signal $\kappa(t)$ encodes a switching sequence

$$\begin{gathered} \Sigma = \left\{ \left(j_0, t_0^+\right), \ldots, \left(j_{i-1}, t_{i-1}^+\right), \left(j_i, t_i^+\right), \ldots \mid j_i \in \Xi, j_i \neq j_{i+1}, t_i^+ \in \Im, i \in \mathbb{N} \right\}, \\ \Im = \left\{ t_0^+, t_1^+, \ldots, t_{i-1}^+, t_i^+, \ldots \middle| \, i \in \mathbb{N} \right\}, \end{gathered} \tag{2.3}$$

which defines that $\forall t \in \left[t_i^+; t_{i+1}^+\right)$, $\kappa(t) = j_i$, $\Theta_{\kappa(t)} = \Theta_{j_i}$ (over the ith time interval the parameter $\Theta_{\kappa(t)}$ equals to the value correspondent to the jth element of the set Ξ).

The function $\kappa(t)$ is either uniquely determined by the trajectories of states $x(t)$ and control $u(t)$ of the system (2.1), or it changes its values depending on various unknown events of discrete nature:

$$\kappa(t) = j_i \Leftrightarrow \Phi(t) \in \Pi_j = \left\{ \Phi(t) \in R^{n+m} \middle| \, \mathrm{H}_j \Phi(t) \leqslant_{[j]} 0 \right\}, \tag{2.4a}$$

or

$$\kappa(t) = j_i \Leftrightarrow t \in \left[t_i^+; t_{i+1}^+\right), \tag{2.4b}$$

where Π_j is the jth polyhedral region R^{n+m}, $\mathrm{H}_j \in R^{(n+m)\times(n+m)}$ stands for a matrix that defines a region Π_j, $\leqslant_{[j]}$ denotes comparison operators ($<$ or $\leqslant$), which ensure that the following equalities hold: $\cup_j^N \Pi_j = R^{m+n}$, $\Pi_i \cap \Pi_j = \emptyset \ \forall j \neq i$.

For the sake of brevity and at the same time generality, the plant (2.1) parameters that exist over the time range $\left[t_i^+; t_{i+1}^+\right)$ are denoted as ϑ_i ($\forall t \in \left[t_i^+; t_{i+1}^+\right)$ $\vartheta_i = \Theta_{\kappa(t)} = \Theta_{j_i}$) in order to write (2.1) in the following form regardless whether the parameters change is caused by (2.4a) or (2.4b):

$$\begin{gathered} \forall t \geqslant t_0^+, \ \dot{x}(t) = \vartheta^{\mathrm{T}}(t)\Phi(t) = \begin{cases} A_0 x(t) + B_0 u(t), & t \in \left[t_0^+; t_1^+\right) \\ \vdots \\ A_i x(t) + B_i u(t), & t \in \left[t_i^+; t_{i+1}^+\right), \end{cases} \\ \vartheta(t) = \vartheta_i = \vartheta_0 + \sum_{q=1}^{i} \Lambda_q h\left(t - t_q^+\right), \quad \dot{\vartheta}(t) = \sum_{q=1}^{i} \Lambda_q \delta\left(t - t_q^+\right), \end{gathered} \tag{2.5}$$

where $\Lambda_i = \vartheta_i - \vartheta_{i-1} = \Theta_{j_i} - \Theta_{j_{i-1}}$ is the amplitude of ϑ_i change at time instant t_i^+, $h\left(t - t_i^+\right)$ stands for the unit step function at t_i^+, $\delta\left(t - t_i^+\right)$ denotes the Dirac delta function at t_i^+.

The required control quality for a closed-loop control system for the plant (2.5) is defined with the help of the reference model with time-invariant parameters:

$$\forall t \geqslant t_0^+, \quad \dot{x}_{ref}(t) = A_{ref}x_{ref}(t) + B_{ref}r(t), \quad x_{ref}\left(t_0^+\right) = x_{0ref}, \tag{2.6}$$

where $x_{ref}(t) \in R^n$ is the reference model state vector with initial conditions x_{0ref}, $r(t) \in R^m$ stands for the reference signal, $A_{ref} \in R^{n\times n}$ denotes a Hurwitz state matrix of the reference model, $B_{ref} \in R^{n\times m}$ is a reference model input matrix.

The ideal model following conditions (Erzberger's matching conditions) are assumed to be met for the plant (2.5) and the reference model (2.6).

Assumption 1. There exist matrices $K_i^x \in R^{m\times n}$ and $K_i^r \in R^{m\times m}$ such that the following holds:

$$A_i + B_iK_i^x = A_{ref}, \quad B_iK_i^r = B_{ref}. \tag{2.7}$$

Taking Assumption 1 into consideration, the error equation obtained as the difference between the plant (2.5) and the reference (2.6) models is written as:

$$\begin{aligned}\dot{e}_{ref}(t) &= A_{ref}e_{ref}(t) + B_iu(t) - (A_{ref} - A_i)\,x(t) - B_{ref}r(t)\\ &= A_{ref}e_{ref}(t) + B_i\left[u(t) - K_i^x x(t) - K_i^r r(t)\right]\\ &= A_{ref}e_{ref}(t) + B_i\left[u(t) - \theta^{\mathrm{T}}(t)\omega(t)\right],\end{aligned} \tag{2.8}$$

where

$$e_{ref}(t) = x(t) - x_{ref}(t), \quad \omega(t) = \begin{bmatrix}x^{\mathrm{T}}(t) & r^{\mathrm{T}}(t)\end{bmatrix}^{\mathrm{T}} \in R^{n+m},$$

$$\theta_i = \begin{bmatrix}K_i^x & K_i^r\end{bmatrix}^{\mathrm{T}} \in R^{(n+m)\times m},$$

$$\theta(t) = \theta_i = \theta_0 + \sum_{q=1}^{i}\Delta_q^\theta h\left(t - t_q^+\right), \quad \dot{\theta}_i = \sum_{q=1}^{i}\Delta_q^\theta \delta\left(t - t_q^+\right), \quad \Delta_i^\theta = \theta_i - \theta_{i-1}.$$

As the parameters $\theta(t)$ and sets $\Xi, \Sigma, \Im$ are unknown, the following continuous control law with adjustable parameters is introduced:

$$u(t) = \hat{\theta}^{\mathrm{T}}(t)\omega(t), \tag{2.9}$$

where $\hat{\theta}(t) \in R^{(n+m)\times m}$ is the estimate of $\theta(t)$.

Equation (2.9) is substituted into (2.8) to obtain:

$$\dot{e}_{ref}(t) = A_{ref}e_{ref}(t) + B_i\left[\hat{\theta}^{\mathrm{T}}(t) - \theta^{\mathrm{T}}(t)\right]\omega(t) = A_{ref}e_{ref}(t) + B_i\tilde{\theta}^{\mathrm{T}}(t)\omega(t), \tag{2.10}$$

where $\tilde{\theta}(t) = \hat{\theta}(t) - \theta(t)$ is error of estimation of $\theta(t)$.

The following assumptions are adopted with respect to the parameters $\theta(t)$ and the regressor $\Phi(t)$ excitation.

Assumption 2. Let $\exists\overline{\Delta}_\theta > 0$, $T_{\min} > \min\limits_{\forall i\in\mathbb{N}} T_i > 0$ such that $\forall i \in \mathbb{N}$ it simultaneously holds that:

1) $t_{i+1}^+ - t_i^+ \geqslant T_{\min}, \quad \|\theta_i - \theta_{i-1}\| = \|\Delta_i^\theta\| \leqslant \overline{\Delta}_\theta$;
2) $\Phi(t) \in \mathrm{FE}$ *over* $\left[t_i^+; t_i^+ + T_i\right]$ *with excitation level* α_i;
3) $\Phi(t) \in \mathrm{FE}$ *over* $\left[\hat{t}_i^+; t_i^+ + T_i\right]$ *with excitation level* $\overline{\alpha}_i$,

   where $\alpha_i > \overline{\alpha}_i > 0, \quad \hat{t}_i^+ \in \left[t_i^+; t_i^+ + T_i\right)$.

Assumption 3. There exists a known parameter $l > 0$ such that:

$$\Phi(t) \in \text{FE} \Rightarrow \overline{\varphi}(t) = \begin{bmatrix} \int\limits_{\hat{t}_i^+}^{t} e^{-l(t-\tau)}\Phi^{\text{T}}(\tau)\,d\tau & e^{-l\left(t-t_i^+\right)} \end{bmatrix}^{\text{T}} \in \text{FE}.$$

Then the main goal of the study is to design an algorithm to obtain estimates $\hat{\theta}(t)$ that ensures that the following condition is met:

$$\lim_{t\to\infty} \|\xi(t)\| = 0\,(\exp)\,, \tag{2.11}$$

where $\xi(t) = \begin{bmatrix} e_{ref}^{\text{T}}(t) & vec^{\text{T}}\left(\tilde{\theta}(t)\right) \end{bmatrix}^{\text{T}}$ is an augmented tracking error.

Remark 1. As far as the theory of model reference adaptive control is concerned, Assumption 1 is classical one (the interested reader is referred to [27, 28] to become familiar with the recently proposed new methods to relax Assumption 1 for linear time-invariant plants).

The first part of Assumption 2 requires a finite frequency and amplitude of the plant unknown parameters switches, which are conventional requirements for switched systems [11, 16] and identification [1–3] theories, respectively. The second and third parts of Assumption 2 represent a necessary and sufficient condition of true values identifiability for all elements of the ith matrix of unknown parameters [29].

Assumption 3 corresponds to the identifiability conditions of the parameter vector $\overline{\vartheta}^{\text{T}}(t) = \begin{bmatrix} A_i & B_i & x\left(\hat{t}_i^+\right) \end{bmatrix}$ and requires that the algebraic spectrum of the matrix A does not contain $-l$. If the initial conditions of the system $x(t_0^+)$ are known, then Assumption 3 is not required. The necessity and strictness of Assumption 3 have been commented in more detail in Section 3.4 of [28].

3. PRELIMINARY RESULTS

In this section the exponentially stable control problem (2.11) is considered under the condition that $\kappa(t)$ and $\Theta_{\kappa(t)}$ are known.

As the matrix A_{ref} is a Hurwitz one, then the control law $u(t) = \hat{\theta}^{\text{T}}(t)\omega(t)$, $\hat{\theta}(t) = \theta(t)$, ensures that for all $t \geqslant t_0^+$ $\tilde{\theta}(t) = 0$ and the tracking error $e_{ref}(t)$ is exponentially stable [16]. However, in such case, the control signal suffers from the discontinuities of the first kind when the plant switches its parameters, which may be unacceptable for practical scenarios.

Another way to choose the control signal is to use the following filtering algorithm:

$$\dot{\hat{\theta}}(t) = -\gamma_1\left(\hat{\theta}(t) - \theta(t)\right) = -\gamma_1\tilde{\theta}(t), \quad \hat{\theta}\left(t_0^+\right) = \hat{\theta}_0, \tag{3.1}$$

where $\gamma_1 > 0$ is a gain factor that defines the convergence rate of $\tilde{\theta}(t)$.

The following proposition holds for the system (2.9) with (3.1).

Proposition 1. *If the value of $\gamma_1 > 0$ is sufficiently large, and at least one of the following conditions is met:*

1) $i \leqslant i_{\max} < \infty$,

2) $\forall q \in \mathbb{N}\ \left\|\Delta_q^\theta\right\| \leqslant c_q\phi\left(t_q^+, t_0^+\right), \quad c_q > c_{q+1}, \quad \phi\left(t_q^+, t_0^+\right) = e^{-\gamma_1\left(t_q^+ - t_0^+\right)}$,

then the control law (2.9) *with* (3.1) *ensures that* $\forall t \geqslant t_0^+ \quad \lim_{t\to\infty} \|\xi(t)\| = 0\,(\exp)$.

Proof of Proposition 1 is postponed to Appendix.

The exponential stability conditions from Proposition 1 are equivalent to norm boundedness of the sum of all changes Δ_q^θ:

$$i \leqslant i_{\max} < \infty \Leftrightarrow \|\theta(t)\| \leqslant \|\theta_0\| + \sum_{q=1}^{i_{\max}} \left\|\Delta_q^\theta\right\| h\left(t - t_q^+\right) < \infty,$$

$$\left\|\Delta_q^\theta\right\| \leqslant c_q \phi\left(t_q^+, t_0^+\right) \Leftrightarrow \|\theta(t)\| \leqslant \|\theta_0\| + \sum_{q=1}^{i} c_q \phi\left(t_q^+, t_0^+\right) h\left(t - t_q^+\right) < \infty,$$

which, unlike the control law discontinuity in case $\hat{\theta}(t) = \theta(t)$, is not restrictive.

Thus, in case the parameters $\theta(t)$ are known, the exponentially stable control problem (2.11) can be solved by continuous control signal (2.9) with filtering (3.1). This result motivates us to implement filtering (2.11) indirectly using available measured signals $\Phi(t)$ to solve the problem (3.1) in case $\kappa(t)$ and $\Theta_{\kappa(t)}$ are unknown.

4. MAIN RESULT

Following the method of exponentially stable adaptive control [25], for implicit implementation of (3.1), first of all, a regression equation is obtained that relates the unknown parameters $\theta(t)$ with the measured signals $\Phi(t)$.

The result of such parametrization is written as the following proposition, in which $\hat{t}_i^+$ is assumed to be an estimate of t_i^+.

Proposition 2. *On the basis of states of the filtering with resetting*

$$\dot{\overline{\Phi}}(t) = -l\overline{\Phi}(t) + \Phi(t), \quad \overline{\Phi}\left(\hat{t}_i^+\right) = 0_{m+n}, \tag{4.1}$$

normalization procedures

$$\begin{gathered} \overline{z}_n(t) = n_s(t)\left[x(t) - l\overline{x}(t)\right], \quad \overline{\varphi}_n(t) = n_s(t)\overline{\varphi}(t), \\ n_s(t) = \frac{1}{1 + \overline{\varphi}^{\mathrm{T}}(t)\overline{\varphi}(t)}, \quad \overline{x}(t) = \left[\begin{array}{cc} I_{n\times n} & 0_{n\times m}\end{array}\right]\overline{\Phi}(t), \end{gathered} \tag{4.2}$$

extension ($\sigma > 0$)

$$\dot{z}(t) = e^{-\sigma\left(t-\hat{t}_i^+\right)}\overline{\varphi}_n(t)\overline{z}_n^{\mathrm{T}}(t), \quad z\left(\hat{t}_i^+\right) = 0_{(n+m+1)\times n}, \tag{4.3a}$$

$$\dot{\varphi}(t) = e^{-\sigma\left(t-\hat{t}_i^+\right)}\overline{\varphi}_n(t)\overline{\varphi}_n^{\mathrm{T}}(t), \quad \varphi\left(\hat{t}_i^+\right) = 0_{(n+m+1)\times(n+m+1)}, \tag{4.3b}$$

mixing

$$\begin{gathered} Y(t){:} = \operatorname{adj}\left\{\varphi(t)\right\} z(t), \\ \Delta(t){:} = \det\left\{\varphi(t)\right\}, \end{gathered} \tag{4.4}$$

elimination

$$\begin{gathered} z_A(t) = Y^{\mathrm{T}}(t)\mathfrak{L}, \quad \mathfrak{L} = \left[\begin{array}{cc} I_{n\times n} & 0_{n\times(m+1)}\end{array}\right]^{\mathrm{T}} \in R^{(n+m+1)\times n}, \\ z_B(t) = Y^{\mathrm{T}}(t)\mathfrak{e}_{n+m+1}, \quad \mathfrak{e}_{n+m+1} = \left[\begin{array}{ccc} 0_{m\times n} & I_{m\times m} & 0_{m\times 1}\end{array}\right]^{\mathrm{T}} \in R^{(n+m+1)\times m}, \end{gathered} \tag{4.5}$$

substitution

$$\mathcal{Y}(t) := \begin{bmatrix} \text{adj}\left\{z_B^{\mathrm{T}}(t)z_B(t)\right\} z_B^{\mathrm{T}}(t)\left(\Delta(t)A_{ref} - z_A(t)\right) \\ \text{adj}\left\{z_B^{\mathrm{T}}(t)z_B(t)\right\} z_B^{\mathrm{T}}(t)\Delta(t)B_{ref} \end{bmatrix}, \quad (4.6)$$

$$\mathcal{M}(t) := \det\left\{z_B^{\mathrm{T}}(t)z_B(t)\right\},$$

and smoothing $(k = k_0\gamma_1, \quad k_0 \geqslant 1)$

$$\dot{\Upsilon}(t) = -k\left(\Upsilon(t) - \mathcal{Y}(t)\right), \quad \Upsilon\left(t_0^+\right) = 0_{(n+m)\times n}, \quad (4.7a)$$

$$\dot{\Omega}(t) = -k\left(\Omega(t) - \mathcal{M}(t)\right), \quad \Omega\left(t_0^+\right) = 0, \quad (4.7b)$$

the disturbed regression equation with respect to $\theta(t)$ *is obtained:*

$$\Upsilon(t) = \Omega(t)\theta(t) + w(t), \quad (4.8)$$

where the functions $\Upsilon(t), \Omega(t)$ *are computed using* $\Phi(t)$*, and additionally:*

a) *if Assumptions 1–3 are met, then* $\forall t \geqslant t_0^+ + T_0$ *it holds that* $0 < \Omega_{\text{LB}} \leqslant \Omega(t) \leqslant \Omega_{\text{UB}} < \infty$.

b) *if* $\tilde{t}_i^+ = \hat{t}_i^+ - t_i^+ = 0$*, then* $\|w(t)\| \leqslant w_{\max}\phi\left(t, t_0^+ + T_0\right) \leqslant w_{\max}$.

Proof of Proposition 2 and definitions of $w(t), w_{\max}$ are given in Appendix.

Temporarily assuming time-invariance of the parameters $\vartheta(t) = \vartheta$ and $\theta(t) = \theta$, we briefly explain the purpose of the above-given procedures. The filtering (4.1) allows one to obtain a measured regression equation $x(t) - l\overline{x}(t) = \overline{\vartheta}^{\mathrm{T}}\overline{\varphi}(t)$ with respect to the plant (2.1) parameters using the measured signals $\Phi(t)$. The normalization (4.2) ensures that all the signals used in the following procedures belong to L_∞. The extension and mixing procedures (4.3a), (4.3b), (4.4) transform the regression obtained in (4.1), (4.2) into the form $Y(t) = \Delta(t)\overline{\vartheta}$, where $\Delta(t) \in R$ is a scalar regressor (see proof of Proposition 2 and [9]). In addition, the integral-based filtering (4.3a), (4.3b) allow one to ensure that $\forall t \geqslant t_0^+ + T_0$ the condition $\Delta(t) \geqslant \Delta_{\text{LB}} > 0$ is met [26]. Owing to $\Delta(t) \in R$, the elimination (4.5) makes it possible to consider separately the regression equations $z_A(t) = \Delta(t)A$, $z_B(t) = \Delta(t)B$ with respect to matrices A and B. The substitution (4.6) of (4.5) into the matching condition (2.7) results in the transition from equations with respect to A and B to the equation $\mathcal{Y}(t) = \mathcal{M}(t)\theta$ with respect to θ (see proof of Proposition 2 and [25, 27, 28]). Smoothing (4.7a), (4.7b) allows one to separate $\Omega(t)$ from zero for all $t \geqslant t_0^+ + T_0$ and ensure sufficient smoothness of $\Upsilon(t)$ and $\Omega(t)$.

Now we return to the consideration of system with the piecewise-constant unknown parameters (2.1). In this case the disturbance $w(t)$ is nonzero due to the violation of the commutativity of the filters (4.1), (4.3a), and (4.7a).

If $\hat{t}_i^+$ is chosen arbitrarily, the disturbance $w(t)$ is not a vanishing function due to the fact that (4.3a), (4.3b) are integral-based. However, as it follows from the part b) of Proposition 2, if $\hat{t}_i^+ = t_i^+$, i.e. the filters (4.1) and (4.3a), (4.3b) are reset after each switch of the system (2.1) parameters, then $w(t)$ is ensured to be exponentially vanishing.

According to the problem statement, the time instants t_i^+ of the system (2.1) switching state change are unknown, thus the following switching detection algorithm is introduced.

Proposition 3. *If Assumptions 2 and 3 are met, and, using the following function*

$$\epsilon(t) = \Delta(t)\overline{\varphi}_n(t)\overline{z}_n^{\mathrm{T}}(t) - \overline{\varphi}_n(t)\overline{\varphi}_n^{\mathrm{T}}(t)Y(t), \quad (4.9)$$

the estimate $\hat{t}_i^+$ *is obtained in accordance with the switching detection algorithm:*

$$\begin{gathered}\text{Initialization: } i \leftarrow 1,\ t_{\text{up}} = \hat{t}_{i-1}^+ \\ \text{IF } \ t - t_{up} \geqslant \Delta_{pr} \ \text{ AND } \ \|\epsilon(t)\| > 0, \\ \text{THEN } \ \hat{t}_i^+ \colon = t + \Delta_{pr},\ t_{up} \leftarrow t,\ i \leftarrow i+1,\end{gathered} \tag{4.10}$$

then, if Δ_{pr} *is chosen as* $\min\limits_{\forall i \in \mathbb{N}} T_i > \Delta_{pr} \geqslant 0$*, it is ensured that* $\tilde{t}_i^+ = \Delta_{pr} \leqslant T_i$.

Proof of Proposition 3 is presented in Appendix.

Having at hand the regression equation (4.8) based only on measured signals $\Phi(t)$ and the switching detection algorithm (4.10) that guarantees $\tilde{t}_i^+ = \Delta_{pr} \geqslant 0$, the filtration (3.1) can be implicitly implemented, and the stated goal (2.11) can be achieved with the help of the adaptive control framework.

Theorem 1. *Let* $\Delta_{pr} = 0$*, Assumptions 1–3 be met, and additionally at least one of the following conditions hold:*

1) $i \leqslant i_{\max} < \infty$,
2) $\forall q \in \mathbb{N},\ \left\|\Delta_q^\theta\right\| \leqslant c_q \phi^{k_0}\left(t_q^+, t_0^+\right), c_q > c_{q+1}, \phi\left(t_q^+, t_0^+\right) = e^{-\gamma_1\left(t_q^+ - t_0^+\right)}$,

then the adaptive law:

$$\begin{gathered}\dot{\hat{\theta}}(t) = -\gamma(t)\Omega(t)\left(\Omega(t)\hat{\theta}(t) - \Upsilon(t)\right) \\ = -\gamma(t)\Omega^2(t)\tilde{\theta}(t) + \gamma(t)\Omega(t)w(t), \quad \hat{\theta}\left(t_0^+\right) = \hat{\theta}_0, \\ \gamma(t) = \begin{cases} 0, & \text{if } \Omega(t) < \Omega_{LB}, \\ \dfrac{\gamma_1}{\Omega^2(t)} & \text{otherwise,}\end{cases}\end{gathered} \tag{4.11}$$

in case $k_0 \geqslant 1$ *and* $\gamma_1 > 0$ *is sufficiently large, ensures that:*

i) $\forall t \geqslant t_0^+ \ \ \xi(t) \in L_\infty$
ii) $\forall t \geqslant t_0^+ + T_0 \ \ \lim\limits_{t\to\infty} \|\xi(t)\| = 0\,(\exp)$.

Proof of Theorem 1 is presented in Appendix.

The block diagram of the proposed adaptive control algorithm for plants with unknown piecewise-constant parameters is shown in Fig. 1.

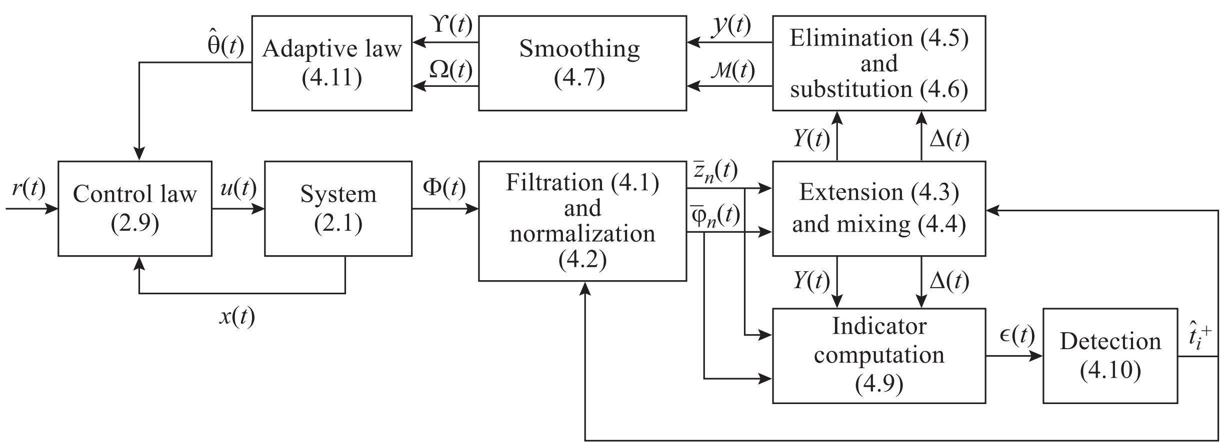

Fig. 1. Block diagram of proposed adaptive control algorithm.

Therefore, the developed adaptive control system for plants with piecewise-constant parameters consists of the control law (2.9), the adaptive law (4.11), a set of procedures (4.1)–(4.8) of measured signal processing and the detection algorithm (4.9)–(4.10) of plant (2.1) parameters switches. In contrast to existing methods of adaptive control of switched systems, the proposed approach (*i*) does not require any information about the plant input matrices B_i, (*ii*) does not employ a discontinuous control signal, (*iii*) is equally applicable to plants with parameters switches caused by both discrete events (2.4b) and state trajectory behavior (2.4a), and (*iv*) ensures a global exponential convergence of the error $\xi(t)$ to zero, provided that the regressor is finitely exciting after each parameters switch.

4.1. Robustness

Any control system designed without consideration of external disturbances must guarantee at least boundedness of all signals when perturbations exist.

The robustness of the proposed adaptive control system (2.9), (4.10), (4.11) in the sense of the error $\xi(t)$ boundedness depends on the robustness of both the adaptive law (4.11) and the detection algorithm (4.10).

When the law (4.11) is in use and the external perturbations affect the plant (2.1) or the measured signals $\Phi(t)$, then the parametric error $\tilde{\theta}(t)$ is represented by the following linear differential equation:

$$\begin{aligned}\dot{\tilde{\theta}}(t) &= -\gamma\Omega(t)\left(\Omega(t)\hat{\theta}(t) - \Upsilon(t)\right) - \dot{\theta}(t)\\ &= -\gamma\Omega^2(t)\tilde{\theta}(t) + \gamma\Omega(t)\left(w(t) - \delta_w(t)\right) - \dot{\theta}(t),\end{aligned} \tag{4.12}$$

where $\delta_w(t) \in L_\infty$ is an external perturbation caused by propagation of the disturbance that affect the plant and measured signals through (4.1)–(4.8).

Owing to $\gamma(t) > 0$, $\Omega(t) \in L_\infty$, and $\forall t \geqslant t_0 + T_0 \;\; \Omega(t) \geqslant \Omega_{LB} > 0$, equation (4.12) is bounded input–bounded output stable. Therefore, if the function $\Upsilon(t)$ is affected by a bounded external perturbation $\delta_w(t) \in L_\infty$, then the law (4.11) guarantees that the parametric error $\tilde{\theta}(t)$ converges to a bounded set of equilibrium point. If the size of such set is sufficient to ensure boundedness of the plant states $x(t)$, then the boundedness of $\xi(t)$ is additionally guaranteed.

Since, in case external perturbations affect the plant, erroneous or even worse persistent resets of the filters (4.1), (4.3a), (4.3b) can lead to a significant deterioration of the identification quality or a complete loss of the identification ability by the law (4.11), along with providing boundedness of $\tilde{\theta}(t)$, it is also important to prevent errors in switching time instants detection.

To this end, the following robust version of the algorithm (4.10) has been proposed in [26, 30]:

$$\begin{gathered}\text{Initialization:}\;\; i \leftarrow 1,\; t_{\text{up}} = \hat{t}^+_{i-1}\\ \text{IF}\;\; t - t_{up} \geqslant \Delta_{pr} \;\; \text{AND} \;\; \left\|\mathrm{E}\left\{\epsilon(t)\right\}\right\| > \left\|0.9\sqrt{\mathrm{var}\left\{(t)\right\}}\right\| + \left\|\rho(t)\right\|,\\ \text{THEN}\;\; \hat{t}^+_i\colon= t + \Delta_{pr}, \quad t_{up} \leftarrow t, \quad i \leftarrow i+1,\end{gathered} \tag{4.13}$$

where $\rho(t)$ is an arbitrary parameter of the robust algorithm, $\mathrm{E}\left\{.\right\}$ is the mean, $\mathrm{var}\left\{.\right\}$ stands for the variance.

The parameter $\rho(t)$ of (4.13) allows one to adjust the detection accuracy and adapt the algorithm to a particular class of external disturbance. For example, if the perturbation is white noise with zero mean, then, according to the results of [26, 30], the appropriate choice is $\rho(t) = 0$. In general, it is recommended to pick $\rho(t)$ as follows:

$$\rho(t) = E\left\{\overline{\varphi}_n(t)\left(\Delta(t)\rho_1 - \rho_2\overline{\varphi}_n^{\mathrm{T}}(t)\ \mathrm{adj}\{\varphi(t)\}\int\limits_{\hat{t}_i^+}^{t} e^{-\sigma\left(\tau-\hat{t}_i^+\right)}\overline{\varphi}_n(\tau)\,d\tau\right)\right\}, \tag{4.14}$$

where $\rho_1 > 0, \rho_2 > 0$ are some constants.

In disturbance-free scenario, the properties of the algorithm (4.13) coincide with those of (4.10). In vice versa case, the algorithm (4.13) avoids detection errors if the parameter $\rho(t)$ is chosen correctly. More details about the algorithm (4.13) can be found in [26, 30].

5. NUMERICAL EXPERIMENTS

In Matlab/Simulink numerical experiments have been conducted, in which the proposed method was applied to control plants with parameters switches caused by both discrete unknown events (2.4b) and the fact that the plant state moved from one polyhedral region of the state space into another one (2.4a). The simulation was conducted using numerical integration by the Euler method with a fixed step size $\tau_s = 10^{-4}$ second.

5.1. Switches Caused by Unknown Discrete-Time Events

The aim was to validate that the proposed system was applicable to the plants with parameters switches caused by unknown discrete-time events. Both cases with and without external disturbances were considered.

5.1.1. Disturbance free scenario

A plant (2.5) with three switches was considered:

$$\forall t \geqslant 0,\ \dot{x}(t) = \begin{cases} A_0x(t) + B_0u(t), \text{ if } t \in [0;5) \\ A_1x(t) + B_1u(t), \text{ if } t \in [5;10) \\ A_2x(t) + B_2u(t), \text{ if } t \geqslant 10, \end{cases}$$

$$A_0 = A_2 = \begin{bmatrix} 0 & 1 \\ -6 & -8 \end{bmatrix};\quad B_0 = B_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}; \tag{5.1.1}$$

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix};\quad B_1 = \begin{bmatrix} 0 \\ -4 \end{bmatrix}.$$

The reference model and reference for (5.1.1) were set as follows:

$$\forall t \geqslant 0,\quad \dot{x}_{ref}(t) = \begin{bmatrix} 0 & 1 \\ -8 & -4 \end{bmatrix} x_{ref}(t) + \begin{bmatrix} 0 \\ 8 \end{bmatrix},\quad x_{ref}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \tag{5.1.2}$$

The respective matrices of the plant and reference model had the same structure, therefore, it was ensured that Assumption 1 was met.

The initial condition of the plant (5.1.1), parameters of the filters (4.1), (4.3), (4.7), adaptive law (4.11) and detection algorithm (4.10) were set as:

$$\begin{gathered} x(0) = \begin{bmatrix} -1 & 0 \end{bmatrix}^{\mathrm{T}},\quad \hat{\theta}(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}},\quad l = 10,\quad \sigma = 5, \\ k_0 = 100,\quad \gamma_0 = 1,\quad \gamma_1 = 1,\quad \Delta_{pr} = 0.1. \end{gathered} \tag{5.1.3}$$

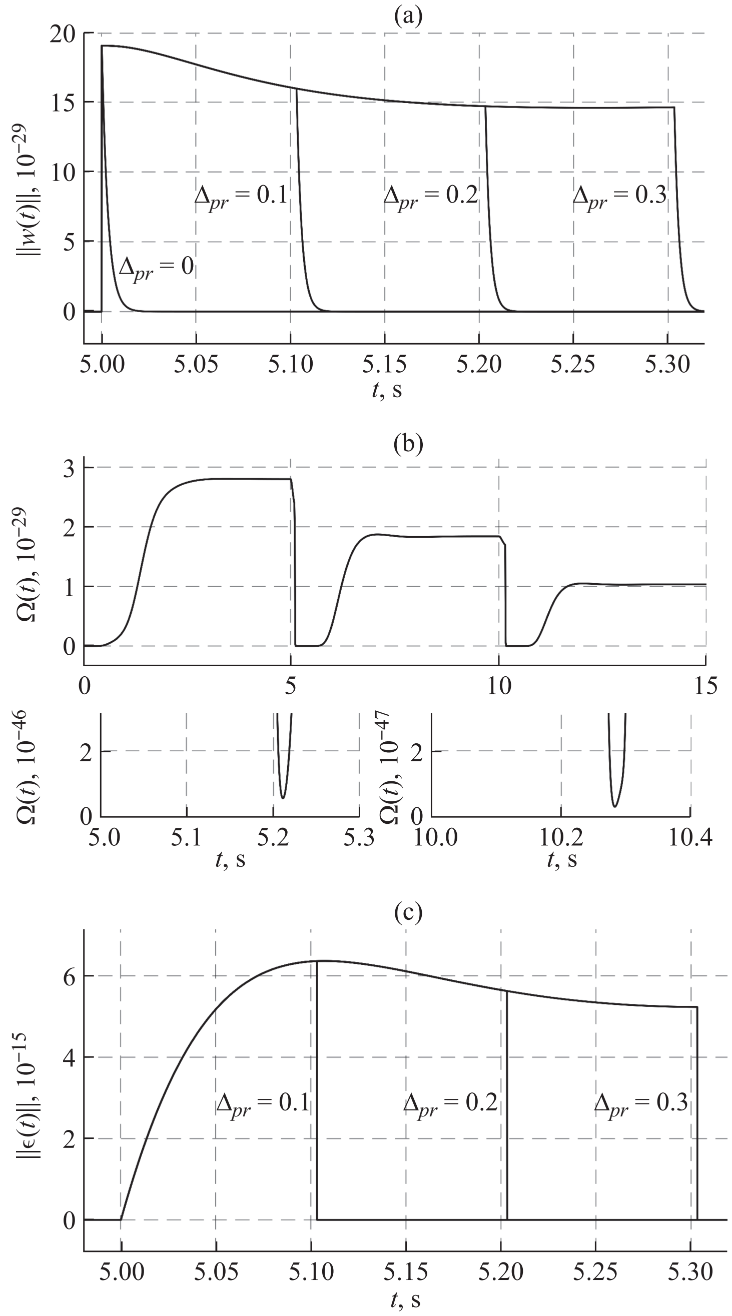

Fig. 2. Transient behavior of: (a) $\|w(t)\|$ for different Δ_{pr}; (b) regressor $\Omega(t)$ when $\Delta_{pr} = 0.1$; (c) $\|\epsilon(t)\|$ for different Δ_{pr}.

It was checked whether Assumptions 2–3 and premises of Theorem 1 and Propositions 2, 3 were met.

Figure 2 depicts: (a) comparison of $\|w(t)\|$ for different Δ_{pr}; (b) transient behavior of the regressor $\Omega(t)$ when $\Delta_{pr} = 0.1$; (c) comparison of $\|\epsilon(t)\|$ for different Δ_{pr}.

Low amplitudes of signals in Fig. 2 are explained by application of the mixing procedure (4.4) and the fact that $\varphi(t)$ is ill-conditioned: $\lambda_{\max}(\varphi(t)) \gg \lambda_{\min}(\varphi(t)) > 0 \Rightarrow \Delta(t) = \prod\limits_{i=1}^{n+m+1} \lambda_i(\varphi(t)) \to 0$. The computational elimination of signals was discussed in more detail in [25]. In general, the

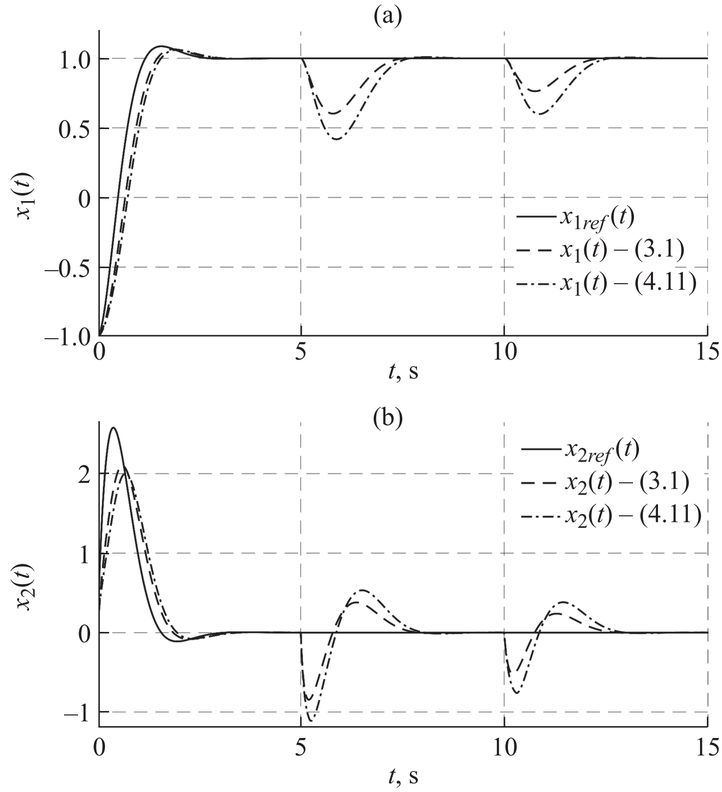

Fig. 3. Transient behavior of reference model $x_{ref}(t)$ and plant $x(t)$ states for control systems (2.9) with (3.1) and (4.11).

simulation results confirmed that all assumptions made in the theoretical analysis were met in the course of the experiment:

—plant parameters switches caused the regressor finite excitation over $\left[t_i^+;\, t_i^+ + T_i\right]$ and $\left[\hat{t}_i^+;\, t_i^+ + T_i\right]$ (statements 2)–3) of Assumption 2);

—appropriate choice of l allowed us to ensure preservation of the regressor excitation and its further propagation in the parametrization (Assumption 3).

In addition to this, the obtained results validated theoretical conclusions of Propositions 2–3:

—the regressor $\Omega(t)$ was bounded away from zero $\forall t \geqslant t_0^+ + T_0$;

—when Δ_{pr} was close to zero, the disturbance was the exponentially vanishing function;

—the indicator $\epsilon(t)$ was non-zero only over the time range $\left[t_i^+;\, \hat{t}_i^+\right]$;

—when the detection algorithm (4.10) was used, the inequality $\tilde{t}_i^+ \leqslant T_i$ held, and the detection error $\tilde{t}_i^+$ was determined by Δ_{pr}.

Thus, all adopted assumptions were met in the course of the experiment, and the results of Propositions 2, 3 were experimentally validated.

Figure 3 presents comparison of the transients of plant $x(t)$ and reference model $x_{ref}(t)$ states for control systems (2.9) with (3.1) and (4.11).

The $x(t)$ curves demonstrate quite high performance of the proposed adaptive control system (2.9), (4.11) in comparison with the ideal continuous law (2.9), (3.1) and confirm the exponential convergence of the error $e_{ref}(t)$ to zero when the number of the plant parameter switches is finite, as it is proved in Proposition 1.

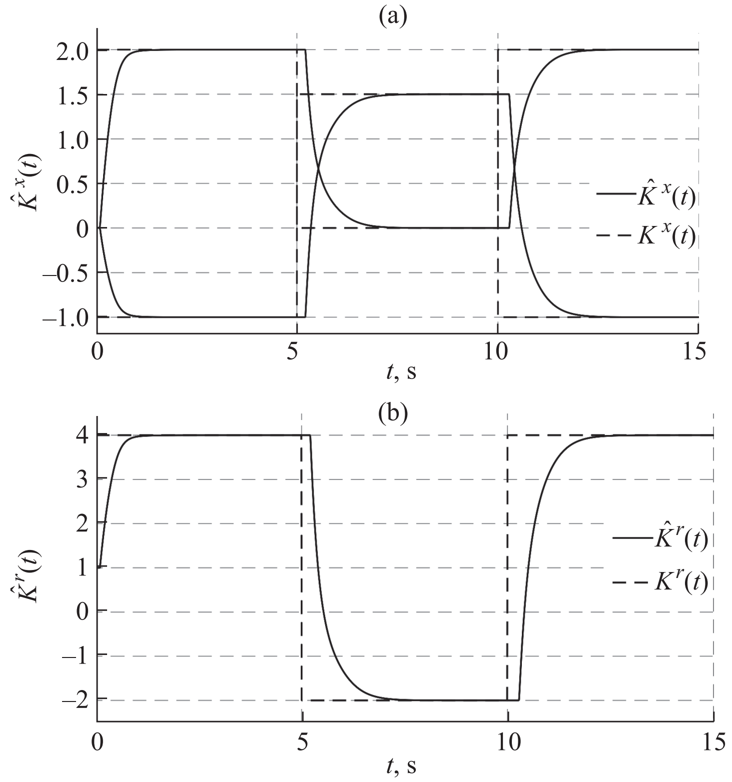

Fig. 4. Transient behavior of estimates $\hat{\theta}(t)$ of unknown parameters $\theta(t)$.

Figure 4 shows the transients of the estimates $\hat{\theta}(t)$ of the unknown parameters $\theta(t)$ and validates the exponential convergence of the error $\tilde{\theta}(t)$ to zero, as it is proved in the theorem.

Thus, the experiment conducted under the condition that the plant switched its parameters at unknown discrete time instants confirmed the theoretical properties of the proposed adaptive control system.

5.1.2. Scenario with external disturbances

The aim was to test the proposed adaptive control system under the condition that the plant was affected by a bounded external disturbance.

The plant (2.5) was implemented as:

$$\forall t \geqslant 0,\ \dot{x}(t) = \begin{cases} A_0 x(t) + B_0(u(t) + 0.25\mathrm{sgn}(\sin(2.5t))), \text{ if } t \in [0;\, 5) \\ A_1 x(t) + B_1(u(t) + 0.25\mathrm{sgn}(\sin(2.5t))), \text{ if } t \in [5;\, 10) \\ A_2 x(t) + B_2(u(t) + 0.25\mathrm{sgn}(\sin(2.5t))), \text{ if } t \geqslant 10, \end{cases} \tag{5.1.4}$$

where A_i, B_i were matrices defined in (5.1.1), $0.25\,\mathrm{sgn}\,(\sin\,(2.5t))$ standed for an external matched bounded disturbance.

All initial conditions and parameters of the adaptive system were set according to (5.1.3). The robust algorithm (4.13) was used to detect plant parameters switches, for which the parameter $\rho(t)$ was chosen according to (4.14) with $\rho_1 = 1$, $\rho_2 = 10^{-1}$.

Figure 5 depicts transient behavior of:

(a) $\|\mathrm{E}\,\{\epsilon(t)\}\|$ and $\left\|0.9\sqrt{\mathrm{var}\,\{(t)\}}\right\| + \|\rho(t)\|$;

(b) the state $x_1(t)$ for control systems (2.9) with (3.1) and (4.11);

(c) the estimates $\hat{\theta}(t)$ of the unknown parameters $\theta(t)$.

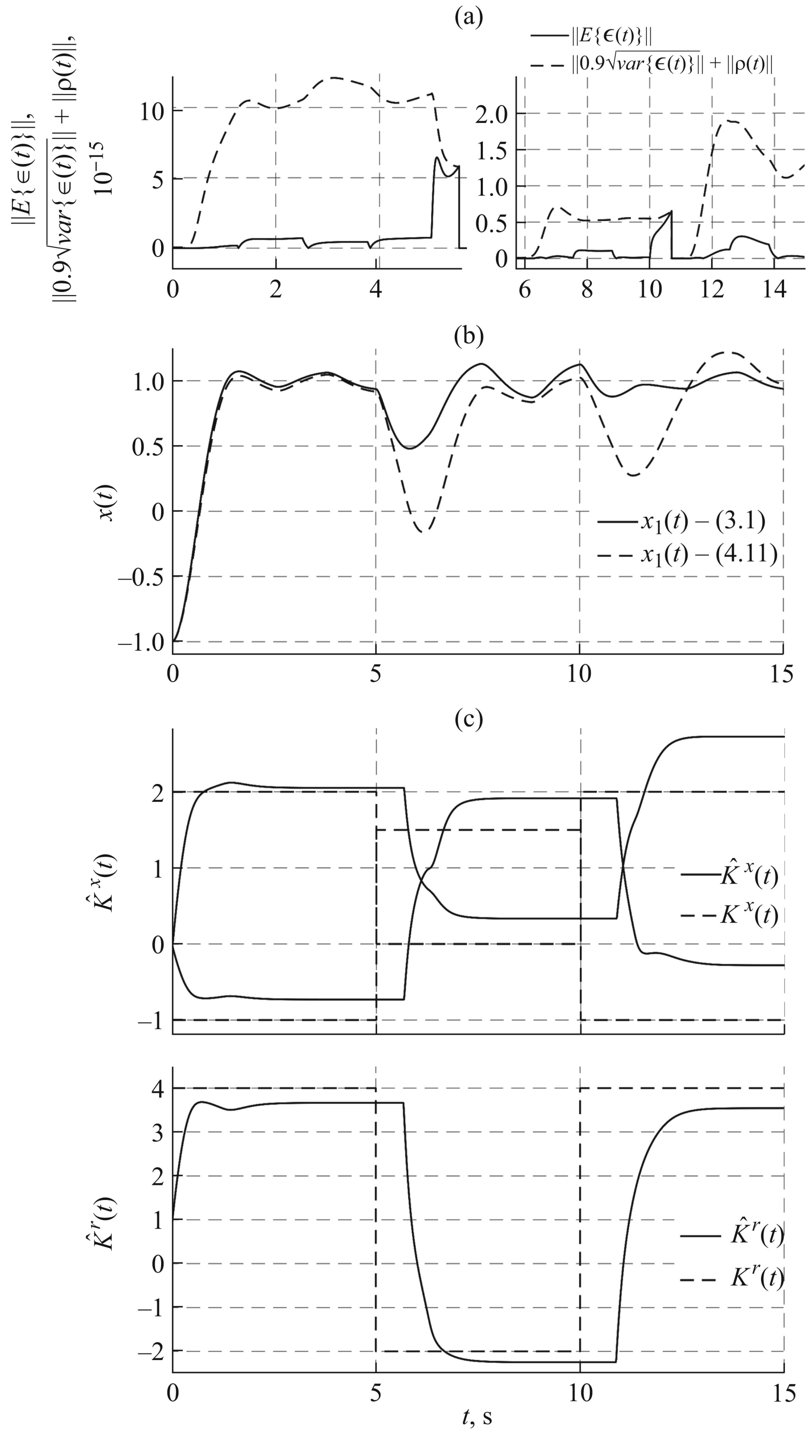

Fig. 5. Transient behavior of: (a) $\|\mathrm{E}\{\epsilon(t)\}\|$ and $\left\|0.9\sqrt{\mathrm{var}\{\epsilon(t)\}}\right\| + \|\rho(t)\|$; (b) state $x_1(t)$ for systems (2.9) with (3.1) and (4.11); (c) estimates $\hat{\theta}(t)$ of parameters $\theta(t)$.

The obtained results validated conclusions made after analytical discussion of robustness:
—the parametric error equation(4.12) was indeed bounded input—bounded output stable;
—if the parametric error $\tilde{\theta}(t)$ converged to a sufficiently small neighbourhood of zero, then the boundedness of $x(t)$ and $e_{ref}(t)$ was ensured;
—if the parameter $\rho(t)$ was chosen appropriately, then the robust algorithm (4.13) detected plant parameters switches even in case the plant was affected by bounded external disturbances.

Therefore, the experiment validated the robustness of the adaptive law (4.11) and the detection algorithm (4.13) to external bounded disturbances.

5.2. Parameter Switches Caused by State Trajectory Behavior

In this experiment, it was validated that the proposed adaptive control system was able to control the plant with parameter switches caused by the state trajectory behavior.

The following implementation of the plant (2.1) was considered:

$$\forall t \geqslant 0, \quad \dot{x}(t) = \begin{cases} \begin{bmatrix} 0 & 1 \\ -1 & 0.2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u(t), \text{ if } x_1(t) \geqslant 0 \\ \begin{bmatrix} 0 & 1 \\ -1.5 & -0.2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \text{ if } x_1(t) < 0. \end{cases} \tag{5.2.1}$$

The reference model and the reference for (5.2.1) were defined as follows:

$$\forall t \geqslant 0, \quad \dot{x}_{ref}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix} x_{ref}(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} r(t),$$
$$r(t) = \begin{cases} 1, & \text{if } 0 \leqslant t < 10 \\ -1, & \text{if } 10 \leqslant t < 20 \\ 1, & \text{if } 20 \leqslant t < 30 \\ -1, & \text{if } 30 \leqslant t < 40. \end{cases} \tag{5.2.2}$$

The respective matrices of the plant and reference model had the same structure, therefore, it was ensured that Assumption 1 was met.

The plant (5.2.1) and reference model (5.2.2) initial conditions, the parameters of filters (4.1), (4.3), (4.7), adaptive law (4.11) and detection algorithm (4.10) were set as:

$$x(0) = \begin{bmatrix} -2 & 2 \end{bmatrix}^{\mathrm{T}}, \quad x_{ref}(0) = \begin{bmatrix} -1 & 0 \end{bmatrix}^{\mathrm{T}}, \quad \hat{\theta}(0) = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}^{\mathrm{T}},$$
$$l = 10, \quad \sigma = 5, \quad k_0 = 100, \quad \gamma_0 = 1, \quad \gamma_1 = 1, \quad \Delta_{pr} = 0.1. \tag{5.2.3}$$

Figure 6 depicts transient behavior of: (a) $x(t)$ and $x_{ref}(t)$; (b) $\hat{\theta}(t)$ and $\theta(t)$.

The simulation results confirmed the results of Propositions 2, 3 and the theorem, and validated the fact that the developed adaptive control system was able to control the plants with parameter switches caused by the fact that the plant trajectories (2.4a) entered certain regions of the state space.

6. CONCLUSION

To solve the control problems of linear plants with unknown piecewise-constant parameters, a new adaptive law was proposed, which was equally applicable to the systems with different nature of parameter switching and ensured exponential stability of the augmented tracking error $\xi(t)$ if the regressor was finitely exciting after each parameter switch. In contrast to existing solutions, the developed adaptive control system did not: (*i*) require the sign/values of the plant input matrices and switching time instants t_i^+ to be *a priori* known, and (*ii*) use off-line data manipulation procedures.

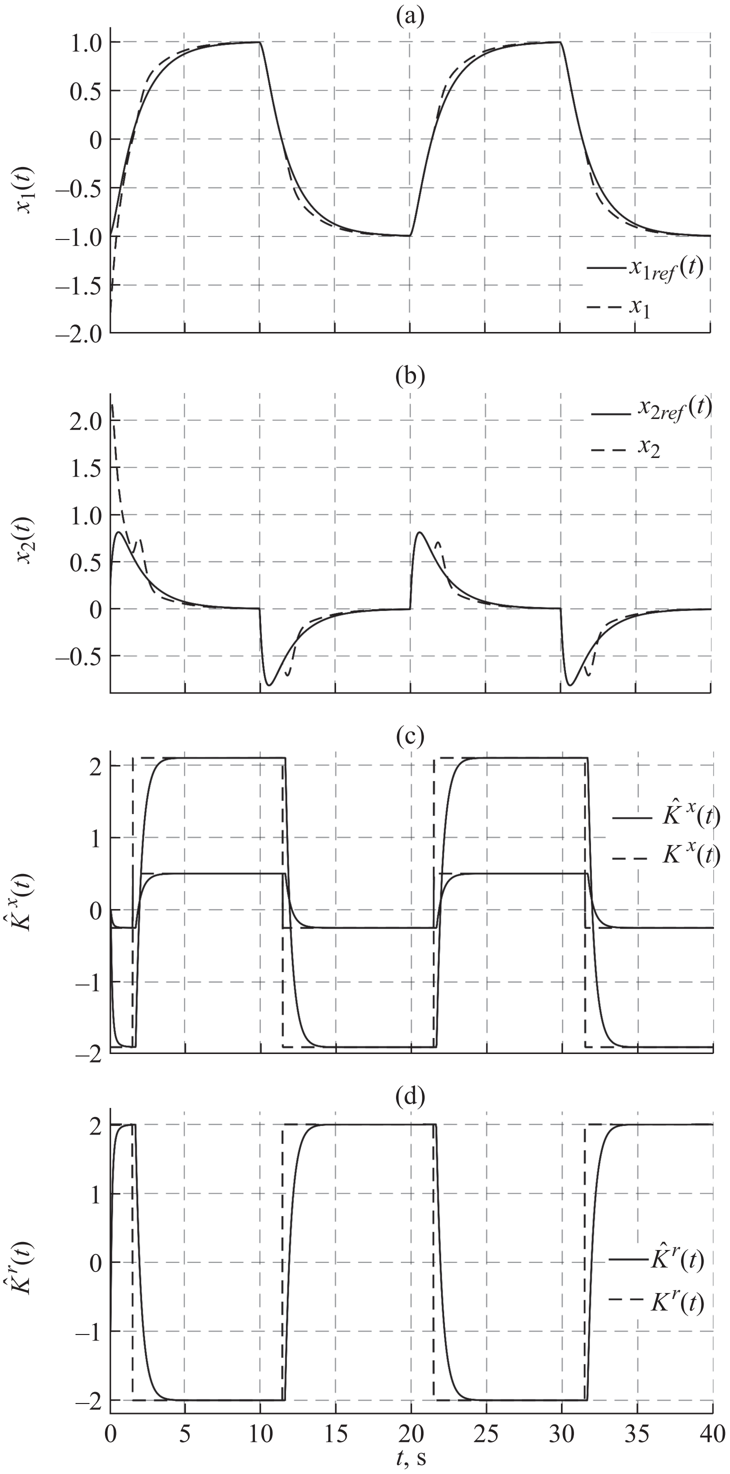

Fig. 6. Transient behavior of: (a) $x(t)$ and $x_{ref}(t)$; (b) $\hat{\theta}(t)$ and $\theta(t)$.

The scope of further research is to extend the obtained results to *a*) output-feedback control of linear systems with piecewise-constant parameters; *b*) state-feedback control of plants with unmatched parametric uncertainty (for example, application of the proposed approach to schemes used in [27, 28]).

The third paper of the current series will be devoted to the development of an exponentially stable adaptive control method for systems with time-varying parameters.

APPENDIX

Proof of Proposition 1. The proof of $\xi(t)$ exponential stability is divided into two steps. The first one is to show that $\tilde{\theta}(t)$ exponentially converges to zero without regard to boundedness of $e_{ref}(t)$ and $\omega(t)$. Using the obtained result, the second step is to show convergence of $e_{ref}(t)$.

Step 1. The equation for $\tilde{\theta}(t) = \hat{\theta}(t) - \theta(t)$ obtained from (3.1) is solved:

$$\tilde{\theta}(t) = \phi\left(t, t_0^+\right)\tilde{\theta}\left(t_0^+\right) - \int\limits_{t_0^+}^{t} \phi(t, \tau) \sum_{q=1}^{i} \Delta_q^\theta \delta\left(\tau - t_q^+\right) d\tau, \tag{A.1}$$

where $\phi(t, \tau) = e^{-\int_\tau^t \gamma_1 d\tau}$.

Using the sifting property of the Dirac function:

$$\int\limits_{t_0^+}^{t} f(\tau)\delta\left(\tau - t_q^+\right) d\tau = f\left(t_q^+\right) h\left(t - t_q^+\right), \forall f(t), \tag{A.2}$$

it is obtained from (A.1):

$$\begin{gathered}\left\|\tilde{\theta}(t)\right\| \leqslant \phi\left(t, t_0^+\right)\left\|\tilde{\theta}\left(t_0^+\right)\right\| + \sum_{q=1}^{i} \phi\left(t, t_q^+\right)\left\|\Delta_q^\theta\right\| h\left(t - t_q^+\right) \\ = \underbrace{\left(\left\|\tilde{\theta}\left(t_0^+\right)\right\| + \sum_{q=1}^{i} \phi\left(t_0^+, t_q^+\right)\left\|\Delta_q^\theta\right\| h\left(t - t_q^+\right)\right)}_{\beta(t)} \phi\left(t, t_0^+\right),\end{gathered} \tag{A.3}$$

where $\phi\left(t_0^+, t_q^+\right) = \phi^{-1}\left(t_q^+, t_0^+\right) = \phi^{-1}\left(t, t_0^+\right)\phi\left(t, t_q^+\right) = \phi\left(t_0^+, t\right)\phi\left(t, t_q^+\right)$.

To prove the exponential stability of $\tilde{\theta}(t)$ it remains to show that $\beta(t)$ is bounded. If the number of paramerers swithces is finite: $i \leqslant i_{\max} < \infty$, then as:

a) when i is finite, the time instants t_i^+ are also finite (we do not consider the case of switches at infinite time: $\forall i \ \ t_i^+ \neq \infty$),

b) $\phi\left(t_0^+, t_q^+\right)$ is bounded in case t_q^+ is finite,

we have the following upper bounds:

$$\beta(t) \leqslant \left\|\tilde{\theta}\left(t_0^+\right)\right\| + \sum_{q=1}^{i_{\max}} \phi\left(t_0^+, t_q^+\right)\left\|\Delta_q^\theta\right\| h\left(t - t_q^+\right) = \beta_{\max}. \tag{A.4}$$

If $\forall q \in \mathbb{N} \ \left\|\Delta_q^\theta\right\| \leqslant c_q \phi\left(t_q^+, t_0^+\right)$, $c_q > c_{q+1}$, then even in case of unbounded i it holds that:

$$\beta(t) \leqslant \left\|\tilde{\theta}\left(t_0^+\right)\right\| + \sum_{q=1}^{i} c_q h\left(t - t_q^+\right) = \beta_{\max}. \tag{A.5}$$

The series from (A.5) is constant sign one, and all its subsums are bounded owing to monotonicity of $0 < c_{q+1} < c_q$, therefore, $\sum\limits_{q=1}^{\infty} c_q h\left(t - t_q^+\right) < \infty$, which results in $\beta(t) \leqslant \beta_{\max}$.

It immediately follows from the boundedness of (A.4) or (A.5) that:

$$\left\|\tilde{\theta}(t)\right\| \leqslant \beta_{\max}\phi\left(t, t_0^+\right) = \beta_{\max}e^{-\gamma_1\left(t-t_0^+\right)} < \beta_{\max}. \tag{A.6}$$

The next aim is to analyze the behaviour of the tracking error $e_{ref}(t)$.

Step 2. The following quadratic form is introduced:

$$\begin{gathered} V_{e_{ref}} = e_{ref}^{\mathrm{T}} P e_{ref} + \frac{2a_0^2}{\gamma_1}e^{-\gamma_1\left(t-t_0^+\right)}, \quad H = \text{blockdiag}\left\{P, \frac{2a_0^2}{\gamma_1}\right\}, \\ \underbrace{\lambda_{\min}\left(H\right)}_{\lambda_m}\left\|\overline{e}_{ref}\right\|^2 \leqslant V\left(\left\|\overline{e}_{ref}\right\|\right) \leqslant \underbrace{\lambda_{\max}\left(H\right)}_{\lambda_M}\left\|\overline{e}_{ref}\right\|^2, \end{gathered} \tag{A.7}$$

where $\overline{e}_{ref}(t) = \left[e_{ref}^{\mathrm{T}}(t) \quad e^{-\frac{\gamma_1}{2}\left(t-t_0^+\right)}\right]^{\mathrm{T}}$, $a_0 > 0$, and P is a solution of the below-given set of equations when $K = I_{n\times n}$:

$$\begin{gathered} A_{ref}^{\mathrm{T}} P + P A_{ref} = -QQ^{\mathrm{T}} - \mu P, \quad PI_{n\times n} = QK, \\ K^{\mathrm{T}}K = D + D^{\mathrm{T}}, \end{gathered}$$

which is equivalent to the Riccati equation $A_{ref}^{\mathrm{T}} P + P A_{ref} + PP^{\mathrm{T}} + \mu P = 0_{n\times n}$.

The derivative of (A.7) is written as:

$$\begin{aligned} \dot{V}_{e_{ref}} &= e_{ref}^{\mathrm{T}}\left(A_{ref}^{\mathrm{T}} P + P A_{ref}\right)e_{ref} - 2a_0^2 e^{-\gamma_1\left(t-t_0^+\right)} + 2e_{ref}^{\mathrm{T}} P I_n B_i \tilde{\theta}^{\mathrm{T}}\omega \\ &= -\mu e_{ref}^{\mathrm{T}} P e_{ref} - e_{ref}^{\mathrm{T}} QQ^{\mathrm{T}} e_{ref} - 2a_0^2 e^{-\gamma_1\left(t-t_0^+\right)} + tr\left(2B_i\tilde{\theta}^{\mathrm{T}}\omega e_{ref}^{\mathrm{T}} QK\right). \end{aligned} \tag{A.8}$$

As $KK^{\mathrm{T}} = K^{\mathrm{T}}K = I_{n\times n}$, equation (A.8) is rewritten as:

$$\begin{aligned} \dot{V}_{e_{ref}} &= -\mu e_{ref}^{\mathrm{T}} P e_{ref} - 2a_0^2 e^{-\gamma_1\left(t-t_0^+\right)} - e_{ref}^{\mathrm{T}} QKK^{\mathrm{T}}Q^{\mathrm{T}} e_{ref} + tr\left(2B_i\tilde{\theta}^{\mathrm{T}}\omega e_{ref}^{\mathrm{T}} QK\right) \\ &= -\mu e_{ref}^{\mathrm{T}} P e_{ref} - 2a_0^2 e^{-\gamma_1\left(t-t_0^+\right)} + tr\left(-K^{\mathrm{T}}Q^{\mathrm{T}} e_{ref} e_{ref}^{\mathrm{T}} QK + 2B_i\tilde{\theta}^{\mathrm{T}}\omega e_{ref}^{\mathrm{T}} QK\right). \end{aligned} \tag{A.9}$$

Completing the square

$$\begin{aligned} &K^{\mathrm{T}}Q^{\mathrm{T}} e_{ref} e_{ref}^{\mathrm{T}} QK - 2B_i\tilde{\theta}^{\mathrm{T}}\omega e_{ref}^{\mathrm{T}} QK + B_i\tilde{\theta}^{\mathrm{T}}\omega\omega^{\mathrm{T}}\tilde{\theta}B_i^{\mathrm{T}} \\ &= \left(B_i\tilde{\theta}^{\mathrm{T}}\omega - K^{\mathrm{T}}Q^{\mathrm{T}} e_{ref}\right)\left(B_i\tilde{\theta}^{\mathrm{T}}\omega - K^{\mathrm{T}}Q^{\mathrm{T}} e_{ref}\right)^{\mathrm{T}} \geqslant 0, \end{aligned} \tag{A.10}$$

we have:

$$\begin{gathered} \dot{V}_{e_{ref}} \leqslant -\mu e_{ref}^{\mathrm{T}} P e_{ref} - 2a_0^2 e^{-\gamma_1\left(t-t_0^+\right)} \\ + tr\left(-K^{\mathrm{T}}Q^{\mathrm{T}} e_{ref} e_{ref}^{\mathrm{T}} QK + 2B_i\tilde{\theta}^{\mathrm{T}}\omega e_{ref}^{\mathrm{T}} QK \pm B_i\tilde{\theta}^{\mathrm{T}}\omega\omega^{\mathrm{T}}\tilde{\theta}B_i^{\mathrm{T}}\right) \\ \leqslant -\mu e_{ref}^{\mathrm{T}} P e_{ref} - 2a_0^2 e^{-\gamma_1\left(t-t_0^+\right)} + tr\left(B_i\tilde{\theta}^{\mathrm{T}}\omega\omega^{\mathrm{T}}\tilde{\theta}B_i^{\mathrm{T}}\right) \\ \leqslant -\mu\lambda_{\min}\left(P\right)\left\|e_{ref}\right\|^2 - 2a_0^2 e^{-\gamma_1\left(t-t_0^+\right)} + b_{\max}^2\lambda_{\max}\left(\omega\omega^{\mathrm{T}}\right)\left\|\tilde{\theta}\right\|^2 \\ \leqslant -\mu\lambda_{\min}\left(P\right)\left\|e_{ref}\right\|^2 - 2a_0^2 e^{-\gamma_1\left(t-t_0^+\right)} + b_{\max}^2\beta_{\max}^2\lambda_{\max}\left(\omega\omega^{\mathrm{T}}\right)\phi^2\left(t, t_0^+\right) \\ \leqslant -\mu\lambda_{\min}\left(P\right)\left\|e_{ref}\right\|^2 - 2a_0^2 e^{-\gamma_1\left(t-t_0^+\right)} + b_{\max}^2\beta_{\max}^2\lambda_{\max}\left(\omega\omega^{\mathrm{T}}\right)e^{-\gamma_1\left(t-t_0^+\right)}e^{-\gamma_1\left(t-t_0^+\right)}, \end{gathered} \tag{A.11}$$

where $\forall i \in \mathbb{N}$ $\|B_i\| \leqslant b_{\max}$ follows from the fact that the pair (A_i, B_i) is controllable.

The exponential vanishing of the third term of (A.11) is required to unsure the exponential stability of the tracking error $e_{ref}(t)$, which, in its turn, requires:

$$\chi(t) = \lambda_{\max}\left(\omega(t)\omega^{\mathrm{T}}(t)\right) e^{-\gamma_1\left(t-t_0^+\right)} \leqslant \chi_{\mathrm{UB}}, \tag{A.12}$$

where $\chi_{\mathrm{UB}} > 0$.

The growth rate of $\lambda_{\max}\left(\omega(t)\omega^{\mathrm{T}}(t)\right)$ is estimated via introduction of $L_{e_{ref}} = e_{ref}^{\mathrm{T}} P e_{ref}$:

$$\begin{aligned}\dot{L}_{e_{ref}} &= e_{ref}^{\mathrm{T}}\left(A_{ref}^{\mathrm{T}}P + PA_{ref}\right)e_{ref} + 2e_{ref}^{\mathrm{T}}PB_i\tilde{\theta}^{\mathrm{T}}\omega \\ &\leqslant -\mu e_{ref}^{\mathrm{T}}Pe_{ref} + 2e_{ref}^{\mathrm{T}}PB_i\tilde{K}_x x + 2e_{ref}^{\mathrm{T}}PB_i\tilde{K}_r r \\ &\leqslant -\mu\lambda_{\min}(P)\|e_{ref}\|^2 + 2\lambda_{\max}(P)\,b_{\max}\|e_{ref}\|\,\|\tilde{\theta}\|\,\|x\| \\ &\qquad + 2\lambda_{\max}(P)\,b_{\max}\|e_{ref}\|\,\|\tilde{\theta}\|r_{\max} \\ &\leqslant -\mu\lambda_{\min}(P)\|e_{ref}\|^2 + 2\lambda_{\max}(P)\,b_{\max}\|e_{ref}\|^2\|\tilde{\theta}\| \\ &\qquad + 2\lambda_{\max}(P)\,b_{\max}\left(x_{ref}^{\mathrm{UB}} + r_{\max}\right)\|e_{ref}\|\,\|\tilde{\theta}\| \\ &\leqslant \left(-\mu\lambda_{\min}(P) + 2\lambda_{\max}(P)\,b_{\max}\|\tilde{\theta}\|\right)\|e_{ref}\|^2 \\ &\qquad + 2\lambda_{\max}(P)\,b_{\max}\left(x_{ref}^{\mathrm{UB}} + r_{\max}\right)\|e_{ref}\|\,\|\tilde{\theta}\|,\end{aligned} \tag{A.13}$$

where $\|x_{ref}(t)\| \leqslant x_{ref}^{\mathrm{UB}}$ is an upper bound of the reference model states norm.

The error $\tilde{\theta}(t)$ is bounded, then, considering the conservative case, it is obtained from (A.13) that:

$$\dot{L}_{e_{ref}} \leqslant c_1\|e_{ref}\|^2 + 2c_2\|e_{ref}\|, \tag{A.14}$$

where

$$\begin{aligned}c_1 &= -\mu\lambda_{\min}(P) + 2\lambda_{\max}(P)\,b_{\max}\beta_{\max} > 0, \\ c_2 &= \lambda_{\max}(P)\,b_{\max}\beta_{\max}\left(x_{ref}^{\mathrm{UB}} + r_{\max}\right).\end{aligned}$$

Applying the Young's inequality $ab \leqslant \frac{1}{2}a^2 + \frac{1}{2}b^2$, we have from (A.14) that:

$$\dot{L}_{e_{ref}} \leqslant \left(c_1 + 2c_2^2\right)\|e_{ref}\|^2 + 0.5 \leqslant \left(c_1 + 2c_2^2\right)\|e_{ref}\|^2 + 1 = \frac{c_1 + 2c_2^2}{\lambda_{\max}(P)}L_{e_{ref}} + 1. \tag{A.15}$$

The equation (A.15) is solved using

$$\lambda_{\min}(P)\|e_{ref}(t)\|^2 \leqslant L_{e_{ref}}(t), \quad L_{e_{ref}}(t) \leqslant \lambda_{\max}(P)\|e_{ref}(t)\|^2:$$

$$\|e_{ref}(t)\| \leqslant \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}e^{\frac{c_1+2c_2^2}{2\lambda_{\max}(P)}\left(t-t_0^+\right)}\left\|e_{ref}\left(t_0^+\right)\right\| + \sqrt{\frac{\lambda_{\max}(P)\,e^{\frac{c_1+2c_2^2}{\lambda_{\max}(P)}\left(t-t_0^+\right)}}{\lambda_{\min}(P)\left(c_1 + 2c_2^2\right)}}. \tag{A.16}$$

Therefore, the growth rate of $x(t)$ does not exceed exponential one, and thus, as $r(t)$ is bounded, it holds that:

$$\begin{aligned}\lambda_{\max}\left(\omega(t)\omega^{\mathrm{T}}(t)\right) &= tr\left(\omega(t)\omega^{\mathrm{T}}(t)\right) = \sum_{i=1}^{n}x_i^2(t) \\ &+ \sum_{i=1}^{m}r_i^2(t) \leqslant \overline{c}_0 e^{\overline{c}_1\left(t-t_0^+\right)}, \quad \overline{c}_0 > 0, \quad \overline{c}_1 > 0.\end{aligned} \tag{A.17}$$

The estimate (A.17) is substituted into (A.12) to obtain that (A.12) holds if $\gamma_1 > 0$ is sufficiently large. Equation (A.12) is used in (A.11) to have:

$$\dot{V}_{e_{ref}} \leqslant -\mu\lambda_{\min}(P)\left\|e_{ref}\right\|^2 - 2a_0^2 e^{-\gamma_1\left(t-t_0^+\right)} + a_0^2 e^{-\gamma_1\left(t-t_0^+\right)} = -\overline{\eta}_{e_{ref}} V_{e_{ref}}, \tag{A.18}$$

where

$$a_0^2 = b_{\max}^2 \beta_{\max}^2 \chi_{\mathrm{UB}}, \quad \overline{\eta}_{e_{ref}} = \min\left\{\frac{\mu\lambda_{\min}(P)}{\lambda_{\max}(P)}, \frac{\gamma_1}{2}\right\}.$$

The differential inequality (A.18) is solved to write:

$$V_{e_{ref}}(t) \leqslant e^{-\overline{\eta}_{e_{ref}}\left(t-t_0^+\right)} V_{e_{ref}}\left(t_0^+\right). \tag{A.19}$$

Therefore, the tracking error $e_{ref}(t)$ exponentially convergences to zero:

$$\left\|e_{ref}(t)\right\| \leqslant \sqrt{\frac{\lambda_M}{\lambda_m}}\left\|e_{ref}\left(t_0^+\right)\right\| e^{-\eta_{e_{ref}}\left(t-t_0^+\right)}, \tag{A.20}$$

where

$$\eta_{e_{ref}} = \frac{1}{2}\overline{\eta}_{e_{ref}}.$$

Having combined (A.20) and (A.6), it is written:

$$\|\xi(t)\| \leqslant \max\left\{\sqrt{\frac{\lambda_M}{\lambda_m}}\left\|e_{ref}\left(t_0^+\right)\right\|, \beta_{\max}\right\} e^{-\eta_{e_{ref}}\left(t-t_0^+\right)}, \tag{A.21}$$

which completes the proof of Proposition 1.

Proof of Proposition 2. The expression $x(t) - l\overline{x}(t)$ is differentiated:

$$\dot{x}(t) - l\dot{\overline{x}}(t) = -l\left(x(t) - l\overline{x}(t)\right) + \vartheta^{\mathrm{T}}(t)\Phi(t). \tag{A.22}$$

The differential equation (A.22) is solved to obtain:

$$\begin{aligned} x(t) - l\overline{x}(t) &= e^{-l\left(t-\hat{t}_i^+\right)} x\left(\hat{t}_i^+\right) + \int_{\hat{t}_i^+}^{t} e^{-l(t-\tau)}\vartheta^{\mathrm{T}}(\tau)\Phi(\tau)\,d\tau \pm \vartheta^{\mathrm{T}}(t)\overline{\Phi}(t) = \\ &= \overline{\vartheta}^{\mathrm{T}}(t)\overline{\varphi}(t) + \int_{\hat{t}_i^+}^{t} e^{-l(t-\tau)}\vartheta^{\mathrm{T}}(\tau)\Phi(\tau)\,d\tau - \vartheta^{\mathrm{T}}(t)\overline{\Phi}(t), \end{aligned} \tag{A.23}$$

where $\overline{\vartheta}^{\mathrm{T}}(t) = \begin{bmatrix} A_i & B_i & x\left(\hat{t}_i^+\right)\end{bmatrix} \in R^{n\times(n+m+1)}$.

Having applied (4.2) to the left- and right-hand parts of (A.23), it is obtained:

$$\begin{aligned} &\forall t \geqslant t_0^+ \quad \overline{z}_n(t) = n_s(t)\left[x(t) - l\overline{x}(t)\right] = \overline{\vartheta}^{\mathrm{T}}(t)\overline{\varphi}_n(t) + \overline{\varepsilon}_0(t), \\ &\overline{\varepsilon}_0(t) = n_s(t)\left(\int_{\hat{t}_i^+}^{t} e^{-l(t-\tau)}\vartheta^{\mathrm{T}}(\tau)\Phi(\tau)\,d\tau - \vartheta^{\mathrm{T}}(t)\overline{\Phi}(t)\right), \end{aligned} \tag{A.24}$$

where $\overline{z}_n(t) \in R^n$, $\overline{\varphi}_n(t) \in R^{n+m+1}$, $\overline{\varepsilon}_0(t) \in R^n$.

Considering (4.4), $z(t)$ is multiplied by adj $\{\varphi(t)\}$ to write:

$$\begin{gathered} Y(t) := \operatorname{adj}\{\varphi(t)\}\left(z(t) \pm \varphi(t)\overline{\vartheta}(t)\right) = \Delta(t)\overline{\vartheta}(t) + \overline{\varepsilon}_1(t), \\ \operatorname{adj}\{\varphi(t)\}\varphi(t) = \det\{\varphi(t)\}I_{(n+m+1)\times(n+m+1)} = \Delta(t)I_{(n+m+1)\times(n+m+1)}, \\ \overline{\varepsilon}_1(t) = \operatorname{adj}\{\varphi(t)\}\left(z(t) - \varphi(t)\overline{\vartheta}(t)\right), \end{gathered} \tag{A.25}$$

where $Y(t) \in R^{(n+m+1)\times n}$, $\Delta(t) \in R$, $\overline{\varepsilon}_1(t) \in R^{(n+m+1)\times n}$.

Owing to $\Delta(t) \in R$, the elimination (4.5) allow one to obtain from (A.25) that:

$$\begin{gathered} z_A(t) = Y^{\mathrm{T}}(t)\mathfrak{L} = \Delta(t)A_i + \overline{\varepsilon}_1^{\mathrm{T}}(t)\mathfrak{L}, \\ z_B(t) = Y^{\mathrm{T}}(t)\mathfrak{e}_{n+m+1} = \Delta(t)B_i + \overline{\varepsilon}_1^{\mathrm{T}}(t)\mathfrak{e}_{n+m+1}, \\ \mathfrak{L} = \begin{bmatrix} I_{n\times n} & 0_{n\times(m+1)} \end{bmatrix}^{\mathrm{T}} \in R^{(n+m+1)\times n}, \\ \mathfrak{e}_{n+m+1} = \begin{bmatrix} 0_{m\times n} & I_{m\times m} & 0_{m\times 1} \end{bmatrix}^{\mathrm{T}} \in R^{(n+m+1)\times m}, \end{gathered} \tag{A.26}$$

where $z_A(t) \in R^{n\times n}$, $z_B(t) \in R^{n\times m}$.

Each equation from (2.7) is left-multiplied by adj $\left\{z_B^{\mathrm{T}}(t)z_B(t)\right\} z_B^{\mathrm{T}}(t)\Delta(t)$. Considering (A.26), equations (4.5) are substituted into the result of multiplication, then the obtained equations are combined to have:

$$\begin{gathered} \mathcal{Y}(t) = \mathcal{M}(t)\theta(t) + d(t) \\ \mathcal{Y}(t) := \begin{bmatrix} \operatorname{adj}\left\{z_B^{\mathrm{T}}(t)z_B(t)\right\} z_B^{\mathrm{T}}(t)\left(\Delta(t)A_{ref} - z_A(t)\right) \\ \operatorname{adj}\left\{z_B^{\mathrm{T}}(t)z_B(t)\right\} z_B^{\mathrm{T}}(t)\Delta(t)B_{ref} \end{bmatrix}, \\ \operatorname{adj}\left\{z_B^{\mathrm{T}}(t)z_B(t)\right\} z_B^{\mathrm{T}}(t)z_B(t) = \det\left\{z_B^{\mathrm{T}}(t)z_B(t)\right\} I_{m\times m} = \mathcal{M}(t)I_{m\times m}, \\ d(t) := -\begin{bmatrix} \operatorname{adj}\left\{z_B^{\mathrm{T}}(t)z_B(t)\right\} z_B^{\mathrm{T}}(t)\left(\overline{\varepsilon}_1^{\mathrm{T}}(t)\mathfrak{L} + \overline{\varepsilon}_1^{\mathrm{T}}(t)\mathfrak{e}_{n+m+1}K_i^x\right) \\ \operatorname{adj}\left\{z_B^{\mathrm{T}}(t)z_B(t)\right\} z_B^{\mathrm{T}}(t)\overline{\varepsilon}_1^{\mathrm{T}}(t)\mathfrak{e}_{n+m+1}K_i^r \end{bmatrix}, \end{gathered} \tag{A.27}$$

where $\mathcal{Y}(t) \in R^{(n+m)\times n}$, $\mathcal{M}(t) \in R$, $d(t) \in R^{(n+m)\times n}$.

Considering (A.27), equation (4.7a) is solved to have the following expression:

$$\begin{gathered} \Upsilon(t) = \int\limits_{t_0^+}^{t} e^{\int_{t_0^+}^{\tau} kd\tau}\mathcal{M}(\tau)\,\theta(\tau)\,d\tau + \int\limits_{t_0^+}^{t} e^{\int_{t_0^+}^{\tau} kd\tau} d(\tau)\,d\tau \pm \Omega(t)\theta(t) = \Omega(t)\theta(t) + w(t), \\ w(t) = \Upsilon(t) - \Omega(t)\theta(t), \end{gathered} \tag{A.28}$$

which proves that (4.8) can be obtained using the procedures (4.1)–(4.7).

To prove the statement (a), equation (4.7b) is solved over both $\left[\hat{t}_i^+;\, t_i^+ + T_i\right]$ and $\left[t_i^+ + T_i;\, \hat{t}_{i+1}^+\right]$:

$$\begin{gathered} \forall t \in \left[\hat{t}_i^+;\, t_i^+ + T_i\right] \quad \Omega(t) = \phi^{k_0}\left(t, \hat{t}_i^+\right)\Omega\left(\hat{t}_i^+\right) + \int\limits_{\hat{t}_i^+}^{t} \phi^{k_0}(t,\tau)\,\mathcal{M}(\tau)\,d\tau, \\ \forall t \in \left[t_i^+ + T_i;\, \hat{t}_{i+1}^+\right] \quad \Omega(t) = \phi^{k_0}(t, t_i^+ + T_i)\Omega(t_i^+ + T_i) + \int\limits_{t_i^+ + T_i}^{t} \phi^{k_0}(t,\tau)\mathcal{M}(\tau)d\tau. \end{gathered} \tag{A.29}$$

It is up to notation proved in [26] that if $\Phi(t) \in \mathrm{FE}$, $\hat{t}_i^+ \geqslant t_i^+$, then $\forall t \in \left[t_i^+ + T_i; \hat{t}_{i+1}^+\right)$ it holds that $\Delta_{\mathrm{UB}} \geqslant \Delta(t) \geqslant \Delta_{LB} > 0$. Then the following holds for the regressor $\mathcal{M}(t)$ over the time ranges considered in (A.29):

$$\begin{gathered}\forall t \in \left[\hat{t}_i^+; t_i^+ + T_i\right] \;\; \mathcal{M}(t) = \det\left\{z_B^{\mathrm{T}}(t) z_B(t)\right\} = \Delta^m(t) \det\left\{B_i^{\mathrm{T}} B_i\right\} \equiv 0, \\ \forall t \in \left[t_i^+ + T_i;\; \hat{t}_{i+1}^+\right] \;\; \Delta_{\mathrm{UB}}^m \det\left\{B_i^{\mathrm{T}} B_i\right\} \geqslant \mathcal{M}(t) \geqslant \Delta_{LB}^m \det\left\{B_i^{\mathrm{T}} B_i\right\} > 0.\end{gathered} \tag{A.30}$$

Having substituted (A.30) into (A.29) and considered $0 \leqslant \phi(t, \tau) \leqslant 1$, the bounds for $\Omega(t)$ are obtained:

$$\begin{aligned}&\forall t \in \left[\hat{t}_0^+; t_0^+ + T_0\right] \;\; \Omega(t) \equiv 0, \\ &\forall i \geqslant 1 \; \forall t \in \left[\hat{t}_i^+; t_i^+ + T_i\right] \;\; \Omega\left(\hat{t}_i^+\right) \geqslant \Omega(t) \geqslant \phi^{k_0}\left(t_i^+ + T_i, \hat{t}_i^+\right) \Omega\left(\hat{t}_i^+\right) > 0, \\ &\forall t \in \left[t_i^+ + T_i; \hat{t}_{i+1}^+\right] \;\; \Omega\left(t_i^+ + T_i\right) + \left(\hat{t}_{i+1}^+ - t_i^+ - T_i\right) \Delta_{\mathrm{UB}}^m \det\left\{B_i^{\mathrm{T}} B_i\right\} \\ &\geqslant \Omega(t) \geqslant \phi^{k_0}\left(\hat{t}_{i+1}^+, t_i^+ + T_i\right) \Big(\Omega\left(t_i^+ + T_i\right) \\ &\qquad + \left(\hat{t}_{i+1}^+ - t_i^+ - T_i\right) \Delta_{LB}^m \det\left\{B_i^{\mathrm{T}} B_i\right\}\Big) > 0.\end{aligned} \tag{A.31}$$

From which we have:

$$\begin{gathered}\forall t \geqslant t_0^+ + T_0 \quad \Omega_{\mathrm{UB}} \geqslant \Omega(t) \geqslant \Omega_{\mathrm{LB}} > 0, \\ \Omega_{\mathrm{LB}} = \min_{\forall i \geqslant 1} \left\{\begin{array}{l}\phi^{k_0}(\hat{t}_{i+1}^+, t_i^+ + T_i)\Big(\Omega\left(t_i^+ + T_i\right) \\ \qquad + \left(\hat{t}_{i+1}^+ - t_i^+ - T_i\right) \Delta_{\mathrm{LB}}^m \det\left\{B_i^{\mathrm{T}} B_i\right\}\Big), \\ \phi^{k_0}\left(t_i^+ + T_i, \hat{t}_i^+\right) \Omega\left(\hat{t}_i^+\right)\end{array}\right\}, \\ \Omega_{\mathrm{UB}} = \max_{\forall i \geqslant 1} \left\{\Omega\left(\hat{t}_i^+\right), \Omega\left(t_i^+ + T_i\right) + \left(\hat{t}_{i+1}^+ - t_i^+ - T_i\right) \Delta_{\mathrm{UB}}^m \det\left\{B_i^{\mathrm{T}} B_i\right\}\right\},\end{gathered} \tag{A.32}$$

which completes the proof of the statement (a).

To prove the statement (b) the disturbance $w(t)$ is differentiated:

$$\begin{aligned}\dot{w}(t) &= \dot{\Upsilon}(t) - \dot{\Omega}(t)\theta(t) - \Omega(t)\dot{\theta}(t) \\ &= -k\left(\Upsilon(t) - \mathcal{Y}(t)\right) + k\left(\Omega(t) - \mathcal{M}(t)\right)\theta(t) - \Omega(t)\dot{\theta}(t) \\ &= -k\left(\Upsilon(t) - \mathcal{M}(t)\theta(t) - d(t)\right) + k\left(\Omega(t) - \mathcal{M}(t)\right)\theta(t) - \Omega(t)\dot{\theta}(t) \\ &= -k\left(\Upsilon(t) - \Omega(t)\theta(t)\right) - \Omega(t)\dot{\theta}(t) + kd(t) \\ &= -kw(t) - \Omega(t)\dot{\theta}(t) + kd(t), \quad w\left(t_0^+\right) = 0_{(n+m)\times m}.\end{aligned} \tag{A.33}$$

The next aim is to show that the identical equality $d(t) \equiv 0$ holds when $\tilde{t}_i^+ = 0$. It follows from the definition (A.27) that $\bar{\varepsilon}_1(t) \equiv 0 \Leftrightarrow d(t) \equiv 0$. Let it be assumed that $\forall i \in \mathbb{N}\;\; \hat{t}_i^+ \geqslant t_i^+$, then the definition of $\overline{\varepsilon}_1(t)$ is obtained over the time ranges $\left[\hat{t}_i^+; t_{i+1}^+\right)$ and $\left[t_i^+; \hat{t}_i^+\right)$:

$$\forall t \in \left[\hat{t}_i^+; t_{i+1}^+\right)\ \vartheta(t) = \vartheta_i$$

$$\Updownarrow$$

$$\begin{gathered}\overline{\varepsilon}_1(t) = \operatorname{adj}\{\varphi(t)\} \int\limits_{\hat{t}_i^+}^{t} e^{-\int_{\hat{t}_i^+}^{\tau} \sigma ds} \overline{\varphi}_n(\tau)\, \overline{z}_n^{\mathrm{T}}(\tau)\, d\tau - \Delta(t)\overline{\vartheta}_i \\ = \operatorname{adj}\{\varphi(t)\} \left(\int\limits_{\hat{t}_i^+}^{t} e^{-\int_{\hat{t}_i^+}^{\tau} \sigma ds} \overline{\varphi}_n(\tau)\, \overline{\varphi}_n^{\mathrm{T}}(\tau)\, d\tau \overline{\vartheta}_i + \int\limits_{\hat{t}_i^+}^{t} e^{-\int_{\hat{t}_i^+}^{\tau} \sigma ds} \overline{\varphi}_n(\tau)\, \overline{\varepsilon}_0^{\mathrm{T}}(\tau)\, d\tau \right) \\ -\Delta(t)\overline{\vartheta}_i = \Delta(t)\overline{\vartheta}_i - \Delta(t)\overline{\vartheta}_i + \int\limits_{\hat{t}_i^+}^{t} e^{-\int_{\hat{t}_i^+}^{\tau} \sigma ds} \overline{\varphi}_n(\tau)\, \overline{\varepsilon}_0^{\mathrm{T}}(\tau)\, d\tau = 0_{(n+m+1)\times n}.\end{gathered} \tag{A.34}$$

At the same time:

$$\forall t \in \left[t_{i-1}^+; t_i^+\right)\ \vartheta(t) = \vartheta_{i-1};\ \forall t \in \left[t_i^+; \hat{t}_i^+\right)\ \vartheta(t) = \vartheta_i$$

$$\Updownarrow$$

$$\begin{gathered}\forall t \in \left[t_i^+; \hat{t}_i^+\right),\ \overline{\varepsilon}_1(t) = \operatorname{adj}\{\varphi(t)\} \int\limits_{\hat{t}_{i-1}^+}^{t} e^{-\int_{\hat{t}_{i-1}^+}^{\tau} \sigma ds} \overline{\varphi}_n(\tau)\overline{z}_n^{\mathrm{T}}(\tau)\, d\tau - \Delta(t)\overline{\vartheta}_i \\ = \operatorname{adj}\{\varphi(t)\} \left(\int\limits_{\hat{t}_{i-1}^+}^{t_i^+} e^{-\int_{\hat{t}_{i-1}^+}^{\tau} \sigma ds} \overline{\varphi}_n(\tau)\overline{\varphi}_n^{\mathrm{T}}(\tau)\, d\tau \overline{\vartheta}_{i-1} + \int\limits_{t_i^+}^{t} e^{-\int_{\hat{t}_{i-1}^+}^{\tau} \sigma ds} \overline{\varphi}_n(\tau)\overline{\varphi}_n^{\mathrm{T}}(\tau)\, d\tau \overline{\vartheta}_i \right) \\ + \operatorname{adj}\{\varphi(t)\} \left(\pm \int\limits_{\hat{t}_{i-1}^+}^{t_i^+} e^{-\int_{\hat{t}_{i-1}^+}^{\tau} \sigma ds} \overline{\varphi}_n(\tau)\overline{\varphi}_n^{\mathrm{T}}(\tau)\, d\tau \overline{\vartheta}_i + \int\limits_{\hat{t}_{i-1}^+}^{t} e^{-\int_{\hat{t}_{i-1}^+}^{\tau} \sigma ds} \overline{\varphi}_n(\tau)\overline{\varepsilon}_0^{\mathrm{T}}(\tau)\, d\tau \right) - \Delta(t)\overline{\vartheta}_i \\ = \operatorname{adj}\{\varphi(t)\} \left(\int\limits_{\hat{t}_{i-1}^+}^{t_i^+} e^{-\int_{\hat{t}_{i-1}^+}^{\tau} \sigma ds} \overline{\varphi}_n(\tau)\overline{\varphi}_n^{\mathrm{T}}(\tau)\, d\tau \left(\overline{\vartheta}_{i-1} - \overline{\vartheta}_i\right) + \int\limits_{\hat{t}_{i-1}^+}^{t} e^{-\int_{\hat{t}_{i-1}^+}^{\tau} \sigma ds} \overline{\varphi}_n(\tau)\overline{\varepsilon}_0^{\mathrm{T}}(\tau t)\, d\tau \right).\end{gathered} \tag{A.35}$$

Having combined (A.34) and (A.35), it is written that:

$$\overline{\varepsilon}_1(t) := \begin{cases} \operatorname{adj}\{\varphi(t)\} \left(\int\limits_{\hat{t}_{i-1}^+}^{t_i^+} e^{-\int_{\hat{t}_{i-1}^+}^{\tau} \sigma ds} \overline{\varphi}_n(\tau)\, \overline{\varphi}_n^{\mathrm{T}}(\tau)\, d\tau \left(\overline{\vartheta}_{i-1} - \overline{\vartheta}_i\right) \right. \\ \left. + \int\limits_{\hat{t}_{i-1}^+}^{t} e^{-\int_{\hat{t}_{i-1}^+}^{\tau} \sigma ds} \overline{\varphi}_n(\tau)\, \overline{\varepsilon}_0^{\mathrm{T}}(\tau)\, d\tau \right),\ i > 0,\ \forall t \in \left[t_i^+; \hat{t}_i^+\right) \\ 0_{(n+m+1)\times n},\ \forall t \in \left[\hat{t}_i^+; t_{i+1}^+\right) \end{cases} \tag{A.36}$$

from which it follows that $\overline{\varepsilon}_1(t) \equiv 0$ when $\tilde{t}_i^+ = 0$, and consequently that $d(t) \equiv 0$.

Using (A.2) and considering $d(t) \equiv 0$, equation (A.33) is solved:

$$w(t) = -\int\limits_{t_0^+ + T_0}^{t} \phi^{k_0}(t,\tau)\,\Omega(\tau)\sum_{q=1}^{i}\Delta_q^\theta\delta\left(\tau - t_q^+\right)d\tau$$
$$= -\sum_{q=1}^{i}\phi^{k_0}\left(t, t_q^+\right)\Omega\left(t_q^+\right)\Delta_q^\theta h\left(t - t_q^+\right) \tag{A.37}$$
$$= \left(-\sum_{q=1}^{i}\phi^{k_0}\left(t_0^+ + T_0, t_q^+\right)\Omega\left(t_q^+\right)\Delta_q^\theta h\left(t - t_q^+\right)\right)\phi^{k_0}\left(t, t_0^+ + T_0\right).$$

It should be noted that, owing to Assumption 2, there are no switches over $\left[t_0^+;\, t_0^+ + T_0\right)$, so the summation in (A.37) is from $q = 1$ to i.

If the number of switches is finite: $i \leqslant i_{\max} < \infty$, then, as:

a) finite i means that time instants t_i^+ are also finite (we do not consider the case of switches at infinite time: $\forall i\ \ t_i^+ \neq \infty$);

b) $\forall q \in \mathbb{N}\ \ \phi^{k_0}\left(t_0^+ + T_0, t_q^+\right)$ is finite in case t_q^+ is finite,

c) $k_0 \geqslant 1$,

the following upper bound holds:

$$\|w(t)\| \leqslant \phi\left(t, t_0^+ + T_0\right)\sum_{q=1}^{i_{\max}}\phi^{k_0}\left(t_0^+ + T_0, t_q^+\right)\Omega_{\mathrm{UB}}\left\|\Delta_q^\theta\right\| h\left(t - t_q^+\right)$$
$$= w_{\max}\phi\left(t, t_0^+ + T_0\right) \leqslant w_{\max}. \tag{A.38}$$

If $\forall q \in \mathbb{N}\ \left\|\Delta_q^\theta\right\| \leqslant c_q\phi^{k_0}\left(t_q^+, t_0^+\right)$, $c_q > c_{q+1}$, then we have from (A.37) that:

$$\|w(t)\| \leqslant \phi^{k_0}\left(t, t_0^+ + T_0\right)\Omega_{\mathrm{UB}}\phi^{k_0}\left(t_0^+ + T_0,\, t_0^+\right)\sum_{q=1}^{i} c_q h\left(t - t_q^+\right). \tag{A.39}$$

All subsums of positive terms series from (A.39) are bounded, so $\sum\limits_{q=1}^{i} c_q h\left(t - t_q^+\right) < \infty$, and even if the number of switches is infinite, the following holds:

$$\|w(t)\| \leqslant w_{\max}\phi\left(t, t_0^+ + T_0\right) \leqslant w_{\max}, \tag{A.40}$$

which completes the proof of Proposition 2.

Remark 2. The disturbance $d(t)$, which reflects the difference between the real perturbation $w(t)$ and the estimate (A.40), occurs in the proposed parametrization when $\tilde{t}_i^+ > 0$ over the finite time intervals $\left[t_i^+; \hat{t}_i^+\right]$, and $\forall t \geqslant \hat{t}_i^+$ its contribution into $w(t)$ is an exponentially vanishing function. Thus $d(t)$ affects only the transient quality of $\tilde{\theta}(t)$ and $e_{ref}(t)$, but not the global properties of the tracking error $\xi(t)$. The effect of $d(t)$ can be reduced by improvement of the parameter σ (detailed analysis on that matter is given in Proposition 4 in [26]).

Proof of Proposition 3. According to the results of [26], the algorithm (4.10) ensures that $\tilde{t}_i^+ = \Delta_{pr} \leqslant T_i$ holds if the function $\epsilon(t)$ is an indicator of the system parameters switch:

$$\forall t \in \left[t_i^+;\ \hat{t}_i^+\right)\ f(t) \neq 0,\ \forall t \in \left[\hat{t}_i^+;\ t_{i+1}^+\right)\ f(t) = 0, \tag{A.41}$$

i.e. it is non-zero only over the time range $\left[t_i^+; \hat{t}_i^+\right)$.

Equations (A.25) and (A.24) are substituted into (4.9) to obtain:

$$\begin{aligned}\epsilon(t) = \Delta(t)\overline{\varphi}_n(t)\overline{z}_n^{\mathrm{T}}(t) - \overline{\varphi}_n(t)\overline{\varphi}_n^{\mathrm{T}}(t)Y(t) = \Delta(t)\overline{\varphi}_n(t)\overline{\varphi}_n^{\mathrm{T}}(t)\overline{\vartheta}(t)\\ + \Delta(t)\overline{\varphi}_n(t)\overline{\varepsilon}_0^{\mathrm{T}}(t) - \Delta(t)\overline{\varphi}_n(t)\overline{\varphi}_n^{\mathrm{T}}(t)\overline{\vartheta}(t) - \overline{\varphi}_n(t)\overline{\varphi}_n^{\mathrm{T}}(t)\overline{\varepsilon}_1(t)\\ = \Delta(t)\overline{\varphi}_n(t)\overline{\varepsilon}_0^{\mathrm{T}}(t) - \overline{\varphi}_n(t)\overline{\varphi}_n^{\mathrm{T}}(t)\overline{\varepsilon}_1(t).\end{aligned} \tag{A.42}$$

The error $\epsilon(t)$ satisfies the definition (A.41) if $\overline{\varepsilon}_0^{\mathrm{T}}(t)$ and $\overline{\varepsilon}_1(t)$ meet (A.41). Using the results of Proposition 2 (see (A.36)), the function $\overline{\varepsilon}_1(t)$ is an indicator of the system parameters switch. Then now we need to prove the same thesis for $\overline{\varepsilon}_0^{\mathrm{T}}(t)$. Let it be assumed that $\forall i \in \mathbb{N}\ \hat{t}_i^+ \geqslant t_i^+$, then:

$$\begin{gathered}\forall t \in \left[\hat{t}_i^+; t_{i+1}^+\right)\ \vartheta(t) = \vartheta_i\\ \Updownarrow\\ \forall t \in \left[\hat{t}_i^+; t_{i+1}^+\right)\quad \overline{\varepsilon}_0(t) = n_s(t)\left(\int\limits_{\hat{t}_i^+}^{t} e^{-l(t-\tau)}\dot{x}(\tau)\,d\tau - \vartheta_i^{\mathrm{T}}\overline{\Phi}(t)\right)\\ = n_s(t)\left(\vartheta_i^{\mathrm{T}}\int\limits_{\hat{t}_i^+}^{t} e^{-l(t-\tau)}\Phi(\tau)\,d\tau - \vartheta_i^{\mathrm{T}}\overline{\Phi}(t)\right) = n_s(t)\left(\vartheta_i^{\mathrm{T}}\overline{\Phi}(t) - \vartheta_i^{\mathrm{T}}\overline{\Phi}(t)\right) = 0.\end{gathered} \tag{A.43}$$

At the same time:

$$\begin{gathered}\forall t \in \left[t_{i-1}^+; t_i^+\right)\ \vartheta(t) = \vartheta_{i-1}; \forall t \in \left[t_i^+; \hat{t}_i^+\right)\ \vartheta(t) = \vartheta_i\\ \Updownarrow\\ \forall t \in \left[t_i^+; \hat{t}_i^+\right),\ \overline{\varepsilon}_0(t) = n_s(t)\left(\int\limits_{\hat{t}_{i-1}^+}^{t} e^{-l(t-\tau)}\dot{x}(\tau)\,d\tau - \vartheta_i^{\mathrm{T}}\overline{\Phi}(t)\right)\\ = n_s(t)\left(e^{-l\left(t-t_i^+\right)}\int\limits_{\hat{t}_{i-1}^+}^{t_i^+} e^{-l\left(t_i^+-\tau\right)}\vartheta_{i-1}^{\mathrm{T}}\Phi(\tau)\,d\tau + \int\limits_{t_i^+}^{t} e^{-l(t-\tau)}\vartheta_i^{\mathrm{T}}\Phi(\tau)\,d\tau\right.\\ \left. - \vartheta_i^{\mathrm{T}}\left(e^{-l\left(t-t_i^+\right)}\int\limits_{\hat{t}_{i-1}^+}^{t_i^+} e^{-l\left(t_i^+-\tau\right)}\Phi(\tau)\,d\tau + \int\limits_{t_i^+}^{t} e^{-l(t-\tau)}\Phi(\tau)\,d\tau\right)\right)\\ = n_s(t)e^{-l\left(t-t_i^+\right)}\left(\vartheta_{i-1}^{\mathrm{T}} - \vartheta_i^{\mathrm{T}}\right)\int\limits_{\hat{t}_{i-1}^+}^{t_i^+} e^{-l\left(t_i^+-\tau\right)}\Phi(\tau)\,d\tau.\end{gathered} \tag{A.44}$$

Having combined (A.43) and (A.44), it is obtained:

$$\overline{\varepsilon}_0(t) := \begin{cases} n_s(t)e^{-l\left(t-t_i^+\right)}\left(\vartheta_{i-1}^{\mathrm{T}} - \vartheta_i^{\mathrm{T}}\right)\int\limits_{\hat{t}_{i-1}^+}^{t_i^+} e^{-l\left(t_i^+-\tau\right)}\Phi(\tau)\,d\tau, & i > 0,\quad \forall t \in \left[t_i^+; \hat{t}_i^+\right)\\ 0_n, \forall t \in \left[\hat{t}_i^+; t_{i+1}^+\right), & \end{cases} \tag{A.45}$$

which, considering (A.36), allows one to write:

$$\forall i \in \mathbb{N}, \quad \epsilon(t) := \begin{cases} \Delta(t)\overline{\varphi}_n(t)\overline{\varepsilon}_0^{\mathrm{T}}(t) - \overline{\varphi}_n(t)\overline{\varphi}_n^{\mathrm{T}}(t)\overline{\varepsilon}_1(t), \ i > 0, \ \forall t \in \left[t_i^+; \hat{t}_i^+\right) \\ 0_{(n+m+1)\times n}, \ \forall t \in \left[\hat{t}_i^+; t_{i+1}^+\right), \end{cases} \tag{A.46}$$

from which $\epsilon(t)$ is an indicator of the system parameters switch, and, following the results from [26], when $\Delta(t) \in \mathrm{FE}$ and $\overline{\varphi}_n(t) \in \mathrm{FE}$ over $\left[\hat{t}_i^+; t_i^+ + T_i\right]$ (which holds as Assumptions 2 and 3 are met), then $\tilde{t}_i^+ = \Delta_{pr} \leqslant T_i$.

Proof of Theorem 1. The proof of theorem is arranged in the same way as the one of Proposition 1.

Two time ranges are considered: $\left[t_0^+; t_0^+ + T_0\right)$ and $\left[t_0^+ + T_0; \infty\right)$. As for $\left[t_0^+; t_0^+ + T_0\right)$, it holds that $\Omega(t) \leqslant \Omega_{LB}$ in the conservative case, so $\dot{\tilde{\theta}}(t) = 0_{(n+m)\times m} \Rightarrow \tilde{\theta}(t) = \tilde{\theta}\left(t_0^+\right)$ (as there are no switches over $\left[t_0^+; t_0^+ + T_0\right)$ according to Assumption 2). Then, taking the proof of Proposition 1 into consideration (see (A.13)–(A.17)), the exponential growth rate of $e_{ref}(t)$ follows from the boundedness of $\tilde{\theta}(t)$, and, as a result, as well the boundedness of $e_{ref}(t)$ by its finite value at the right-hand border of the time interval in question: $\forall t \in \left[t_0^+; t_0^+ + T_0\right) \ e_{ref}(t) \leqslant e_{ref}\left(t_0^+ + T_0\right)$. Therefore, $\xi(t)$ is bounded over the time range $\left[t_0^+; t_0^+ + T_0\right)$.

The next aim is to consider the interval $\left[t_0^+ + T_0; \infty\right)$.

Step 1. The exponential convergence of $\tilde{\theta}(t) \ \forall t \geqslant t_0^+ + T_0$ is to be proved.

Taking into consideration (A.38) or (A.40) and the boundedness of $\Omega(t) \geqslant \Omega_{LB}$, the solution of the equation (4.11) $\forall t \geqslant t_0^+ + T_0$ meets the inequality:

$$\begin{gathered} \tilde{\theta}(t) = \phi\left(t, t_0^+ + T_0\right)\tilde{\theta}\left(t_0^+ + T_0\right) + \int\limits_{t_0^+ + T_0}^{t} \phi(t,\tau)\frac{\gamma_1 w(\tau)}{\Omega(\tau)}d\tau \\ - \int\limits_{t_0^+ + T_0}^{t} \phi(t,\tau)\sum_{q=1}^{i}\Delta_q^\theta \delta\left(\tau - t_q^+\right)d\tau \leqslant \phi\left(t, t_0^+ + T_0\right)\tilde{\theta}\left(t_0^+ + T_0\right) \\ + \frac{\gamma_1 w_{\max}}{\Omega_{LB}}\int\limits_{t_0^+ + T_0}^{t} \phi(t,\tau)\phi\left(\tau, t_0^+ + T_0\right)d\tau - \sum_{q=1}^{i}\phi\left(t, t_q^+\right)\Delta_q^\theta h\left(t - t_q^+\right). \end{gathered} \tag{A.47}$$

As at least one of the following conditions is met:

1) $i \leqslant i_{\max} < \infty$,

2) $\forall q \in \mathbb{N} \ \left\|\Delta_q^\theta\right\| \leqslant c_q \phi^{k_0}\left(t_q^+, t_0^+\right) \leqslant c_q \phi\left(t_q^+, t_0^+\right), \ c_q > c_{q+1}$,

then, by the analogy with (A.3)–(A.5), the following upper bound is obtained from (A.47):

$$\begin{aligned} \left\|\tilde{\theta}(t)\right\| &\leqslant \beta_{\max}\phi\left(t, t_0^+ + T_0\right) + \frac{\gamma_1 w_{\max}}{\Omega_{LB}}\phi\left(t, t_0^+ + T_0\right)\left(t - t_0^+ - T_0\right) \\ &\leqslant \beta_{\max}\phi\left(t, t_0^+ + T_0\right) + \frac{\gamma_1 w_{\max}}{\Omega_{LB}}\chi_1(t)e^{-\frac{\gamma_1}{2}\left(t - t_0^+ - T_0\right)}, \end{aligned} \tag{A.48}$$

where $\chi_1(t)$ is a time-varying parameter:

$$\chi_1(t) = e^{-\frac{\gamma_1}{2}\left(t - t_0^+ - T_0\right)}\left(t - t_0^+ - T_0\right), \quad \chi_1\left(t_0^+ + T_0\right) = 0,$$

and $\beta(t)$ for both cases under consideration is defined as:

$$\beta(t) \leqslant \left\|\tilde{\theta}\left(t_0^+ + T_0\right)\right\| + \sum_{q=1}^{i_{\max}} \phi\left(t_0^+ + T_0,\, t_q^+\right) \left\|\Delta_q^\theta\right\| h\left(t - t_q^+\right) = \beta_{\max}, \tag{A.49}$$

$$\begin{aligned} \beta(t) &\leqslant \left\|\tilde{\theta}\left(t_0^+ + T_0\right)\right\| + \sum_{q=1}^{i} \phi\left(t_0^+ + T_0,\, t_q^+\right) \phi\left(t_q^+, t_0^+\right) c_q h\left(t - t_q^+\right) \\ &= \left\|\tilde{\theta}\left(t_0^+ + T_0\right)\right\| + \sum_{q=1}^{i} \phi\left(t_0^+ + T_0,\, t_0^+\right) c_q h\left(t - t_q^+\right) = \beta_{\max}. \end{aligned} \tag{A.50}$$

If the parameter $\chi_1(t)$ is bounded, then it holds for $\tilde{\theta}(t)$ that:

$$\left\|\tilde{\theta}(t)\right\| \leqslant \left(\beta_{\max} + \frac{\gamma_1 w_{\max}}{\Omega_{LB}} \chi_1^{\mathrm{UB}}\right) e^{-\frac{\gamma_1}{2}\left(t - t_0^+ - T_0\right)}. \tag{A.51}$$

Then $|\chi_1(t)| \leqslant \chi_1^{\mathrm{UB}}$ is to be proved. We differentiate $\chi_1(t)$ with respect to time:

$$\dot{\chi}_1(t) = -\frac{\gamma_1}{2}\chi_1(t) + e^{-\frac{\gamma_1}{2}\left(t - t_0^+ - T_0\right)}. \tag{A.52}$$

The upper bound of the solution of (A.52) is written as:

$$|\chi_1(t)| \leqslant \left| \int\limits_{t_0^+ + T_0}^{t} e^{-\int_\tau^t \frac{\gamma_1}{2} d\tau} \, e^{\frac{-\gamma_1}{2}\left(\tau - t_0^+ - T_0\right)} \, d\tau \right| \leqslant \left| \int\limits_{t_0^+ + T_0}^{t} e^{\frac{-\gamma_1}{2}\left(\tau - t_0^+ - T_0\right)} \, d\tau \right| \leqslant \frac{2}{\gamma_1}, \tag{A.53}$$

which proves the required boundedness $|\chi_1(t)| \leqslant \chi_1^{\mathrm{UB}}$.

The exponential convergence (A.51) immediately follows from boundedness (A.53), which was to be proved at Step 1.

Step 2. The exponential convergence of the error $\xi(t)$ $\forall t \geqslant t_0^+ + T_0$ is to be proved.

To prove the convergence of $\xi(t)$ $\forall t \geqslant t_0^+ + T_0$, owing to the estimate (A.51), it remains to prove the convergence of the tracking error $e_{ref}(t)$ $\forall t \geqslant t_0^+ + T_0$.

The following quadratic form is introduced:

$$\begin{gathered} V_{e_{ref}} = e_{ref}^{\mathrm{T}} P e_{ref} + \frac{4a_0^2}{\gamma_1} e^{-\frac{\gamma_1}{2}\left(t - t_0^+ - T_0^+\right)}, \quad H = \operatorname{blockdiag}\left\{P,\, \frac{4a_0^2}{\gamma_1}\right\}, \\ \underbrace{\lambda_{\min}(H)}_{\lambda_m} \left\|\overline{e}_{ref}\right\|^2 \leqslant V\left(\left\|\overline{e}_{ref}\right\|\right) \leqslant \underbrace{\lambda_{\max}(H)}_{\lambda_M} \left\|\overline{e}_{ref}\right\|^2, \\ \overline{e}_{ref}(t) = \begin{bmatrix} e_{ref}^{\mathrm{T}}(t) & e^{-\frac{\gamma_1}{4}\left(t - t_0^+ - T_0^+\right)} \end{bmatrix}^{\mathrm{T}}. \end{gathered} \tag{A.54}$$

By analogy with the proof of Proposition 1, $\forall t \geqslant t_0^+ + T_0$ the derivative of (A.54) is written as:

$$\dot{V}_{e_{ref}}(t) \leqslant -\mu \lambda_{\min}(P) \left\|e_{ref}(t)\right\|^2 - 2a_0^2 e^{-\frac{\gamma_1}{2}\left(t - t_0^+ - T_0^+\right)} + b_{\max}^2 \lambda_{\max}\left(\omega(t)\omega^{\mathrm{T}}(t)\right) \left\|\tilde{\theta}(t)\right\|^2. \tag{A.55}$$

Using (A.55), the following upper bound is introduced for $b_{\max}^2 \left\|\tilde{\theta}(t)\right\|^2$:

$$b_{\max}^2 \left\|\tilde{\theta}(t)\right\|^2 \leqslant b_{\max}^2 \left(\beta_{\max} + \frac{\gamma_1 w_{\max}}{\Omega_{LB}} \chi_1^{\mathrm{UB}}\right)^2 e^{-\gamma_1\left(t - t_0^+ - T_0\right)}. \tag{A.56}$$

Equation (A.56) is substituted into (A.55):

$$\begin{gathered}\dot{V}_{e_{ref}}(t) \leqslant -\mu\lambda_{\min}(P)\left\|e_{ref}(t)\right\|^2 - 2a_0^2 e^{-\frac{\gamma_1}{2}\left(t-t_0^+-T_0^+\right)} + \\ + b_{\max}^2\left(\beta_{\max} + \frac{\gamma_1 w_{\max}}{\Omega_{LB}}\chi_1^{\text{UB}}\right)^2 \lambda_{\max}\left(\omega(t)\omega^{\text{T}}(t)\right) e^{-\frac{\gamma_1}{2}\left(t-t_0^+-T_0\right)} e^{-\frac{\gamma_1}{2}\left(t-t_0^+-T_0\right)}.\end{gathered} \tag{A.57}$$

The exponential stability of $e_{ref}(t)$ requires the third term of (A.57) to be exponentially vanishing, which demands:

$$\chi(t) = \lambda_{\max}\left(\omega(t)\omega^{\text{T}}(t)\right) e^{-\frac{\gamma_1}{2}\left(t-t_0^+\right)} \leqslant \chi_{\text{UB}}, \tag{A.58}$$

where $\chi_{\text{UB}} > 0$.

The error $\tilde{\theta}(t)$ is bounded (A.51). In such case, following results of Proposition 1, the growth rate of $\lambda_{\max}\left(\omega(t)\omega^{\text{T}}(t)\right)$ does not exceed the exponential one (A.17). So, when $\gamma_1 > 0$ is sufficiently large, then the estimate (A.58) holds.

The equation (A.58) is substituted into (A.57) to obtain:

$$\dot{V}_{e_{ref}}(t) \leqslant -\mu\lambda_{\min}(P)\left\|e_{ref}(t)\right\|^2 - 2a_0^2 e^{-\frac{\gamma_1}{2}\left(t-t_0^+-T_0^+\right)} + a_0^2 e^{-\frac{\gamma_1}{2}\left(t-t_0^+-T_0\right)} \leqslant -\overline{\eta}_{e_{ref}} V_{e_{ref}}(t), \tag{A.59}$$

where

$$a_0^2 = b_{\max}^2\left(\beta_{\max} + \frac{\gamma_1 w_{\max}}{\Omega_{LB}}\chi_1^{\text{UB}}\right)^2 \chi_{\text{UB}}, \quad \overline{\eta}_{e_{ref}} = \min\left\{\frac{\mu\lambda_{\min}(P)}{\lambda_{\max}(P)}, \frac{\gamma_1}{4}\right\}.$$

The differential inequality (A.59) is solved to obtain:

$$V_{e_{ref}}(t) \leqslant e^{-\overline{\eta}_{e_{ref}}\left(t-t_0^+-T_0\right)} V_{e_{ref}}\left(t_0^+ + T_0\right), \tag{A.60}$$

from which we have the exponential convergence of the tracking error $e_{ref}(t)$ to zero:

$$\left\|e_{ref}(t)\right\| \leqslant \sqrt{\frac{\lambda_M}{\lambda_m}}\left\|e_{ref}\left(t_0^+ + T_0\right)\right\| e^{-\eta_{e_{ref}}\left(t-t_0^+-T_0\right)}, \tag{A.61}$$

where

$$\eta_{e_{ref}} = \frac{1}{2}\overline{\eta}_{e_{ref}}.$$

Having combined (A.61) and (A.51), it is obtained:

$$\left\|\xi(t)\right\| \leqslant \max\left\{\sqrt{\frac{\lambda_M}{\lambda_m}}\left\|e_{ref}\left(t_0^+ + T_0\right)\right\|, \beta_{\max} + \frac{\gamma_1 w_{\max}}{\Omega_{LB}}\chi_1^{\text{UB}}\right\} e^{-\eta_{e_{ref}}\left(t-t_0^+-T_0\right)}, \tag{A.62}$$

which, taking into consideration that $\xi(t)$ is bounded over $\left[t_0^+; t_0^+ + T_0\right]$, allows one to make conclusions of both global boundedness of $\xi(t) \in L_\infty$ and the exponential convergence of $\xi(t)$ to zero $\forall t \geqslant t_0^+ + T_0$. The proof of Theorem 1 is complete.

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