ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2023, Vol. 84, No. 3, pp. 270–284.
Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, 2023.
Russian Text © The Author(s), 2023, published in Avtomatika i Telemekhanika, 2023, No. 2, pp. 103–121.

Matrix Inequalities in the Stability Theory: New Results Based on the Convolution Theorem

V. A. Kamenetskiy

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
e-mail: vlakam@ipu.ru

Received July 14, 2022
Revised November 7, 2022
Accepted November 30, 2022

Abstract—Using Pyatnitskiy's convolution theorem, the circle criterion of absolute stability for Lurie systems with several nonlinearities is obtained without use of the S-lemma. For connected systems with switching between three linear subsystems, a new criterion for the existence of a quadratic Lyapunov function is proposed. On the basis of the convolution theorem, two theorems are proved which lead to a substantial reduction in the dimensionality of connected systems of linear matrix inequalities. Issues of improving the circle criterion for Lurie systems with two nonlinearities are also discussed.

Keywords: switched systems, Lurie systems, stability, Lyapunov functions, matrix inequalities, circle criterion

DOI: 10.25728/arcRAS.2023.79.86.001

1. INTRODUCTION

The stability theory of switched systems [1] and the theory of absolute stability [2] are the basic tools for studying the stability of systems with uncertainty [3]. An important result in this area is the circle criterion, a sufficient condition for the existence of a quadratic Lyapunov function (QLF) for Lurie systems with several nonlinearities [2, 4, 5][1]. The sufficiency of the circle criterion stems from the use of a special technique, the so-called S-lemma [6] which, in this case, leads only to sufficient conditions. A theorem originally obtained by Pyatnitskiy appeared (with indication to the authorship) in [7] and later in [8] as a tool for eliminating this deficiency of the S-lemma. The existence of QLFs in the case of several nonlinearities is guaranteed by the feasibility of a system of linear matrix inequalities (LMIs) [8]. Pyatnitskiy's theorem shows how to obtain a single matrix inequality equivalent to a system of two inequalities. Based on this theorem, the transition from two inequalities to a single equivalent one is called convolution in [8]. Using the convolution operation, necessary and sufficient conditions for the existence of a QLF are obtained in the case of two [7] and several [8] nonlinearities. The theorem by Pyatnitskiy should not be considered as an alternative to the S-lemma, but the fields of applications of these techniques are closely correlated. Thus, Section 2 shows how a circle criterion in the case of several nonlinearities can be dealt with via use of the convolution operation and not the S-lemma.

In Section 3, for linear systems with switching between three subsystems, a new frequency criterion is proposed for the existence of a QLF, simpler than the similar criterion from [9].

Section 4 shows how Pyatnitskiy's theorem can be used to significantly reduce the number of inequalities in connected [8, 9] systems of LMIs which ensure the existence of QLFs for linear systems with switchings between an arbitrary number of subsystems.

[1] In [4], the term "circle criterion" was not used, and in [5], the circle criterion was formulated for both the stability problem and the instability problem.

The issues of improving the circle criterion for Lurie systems with two nonlinearities are considered in Section 5, where a numerical example of such an improvement is presented for a sixth-order system.

The goal of the paper is both to demonstrate the capabilities of the Pyatnitskiy's theorem in obtaining a new proof of the classical result, and to obtain on this way new, more efficient conditions for the existence of QLFs for a wide class of Lurie systems and systems with switching.

2. CONVOLUTION THEOREM AND THE CIRCLE CRITERION FOR SYSTEMS WITH SEVERAL NONLINEARITIES

In this paper, Pyatnitskiy's theorem, which is the basis of the convolution operation, will be referred to as the theorem on convolution; we use it in the following form [8, 9].

Theorem 1. *For the system of the two matrix inequalities*

$$I_1 < 0, \quad I_2 < 0, \quad (I_2 - I_1 = Q = pq^\top + qp^\top, \quad p, q \in \mathbb{R}^n), \tag{2.1}$$

to be valid, it is necessary and sufficient that there exists a number $\widetilde{\varepsilon} > 0$ such that the following single inequality holds:

$$I_1 + Q^+(\widetilde{\varepsilon}) = I_2 + Q^-(\widetilde{\varepsilon}) < 0, \;\; Q^\pm(\varepsilon) = \frac{\varepsilon^2}{2} u^\pm (u^\pm)^\top, \;\; u^\pm(\varepsilon) = p \pm \frac{1}{\varepsilon^2} q. \tag{2.2}$$

Clearly, the feasibility of (2.1) follows from the feasibility of (2.2) for some arbitrary $\varepsilon > 0$.

In [8], the convolution operation based on Theorem 1 was called rank 2 convolution, or r_2-convolution. In the original paper [7], Pyatnitskiy's theorem was given in a more general form, for an arbitrary matrix Q.

Next, Lurie systems with several nonlinearities are those of the form

$$\dot{x} = Ax + \sum_{j=1}^{m} b_j \varphi_j(t, \sigma_j), \quad \sigma_j = \langle c_j, x \rangle, \quad A \in \mathbb{R}^{n\times n}, \quad b_j, c_j \in \mathbb{R}^n, \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$, and the nonlinearities $\varphi_j(t, \sigma_j)$ satisfy the existence conditions of the absolutely continuous solution $x(t)$, $j = \overline{1, m}$. System (2.3) is said to be absolutely stable in the class N_φ of nonlinearities $\varphi = \|\varphi_j\|_{j=1}^m$ satisfying the sector conditions

$$0 \leqslant \varphi_j \sigma_j \leqslant \sigma_j^2, \;\; j = \overline{1, m}, \tag{2.4}$$

if it is asymptotically stable in the whole for all such nonlinearities.

In the form suitable for the exposition to follow, we briefly recall the considerations based on the S-lemma that yield the circle criterion for system (2.3) with arbitrary finite m; see [4, 5]. The S-lemma is a special tool for dealing with quadratic forms. Specifically, given an inequality on a quadratic form which is to be satisfied over a domain specified by other quadratic constraints, the S-lemma provides conditions that lead to yet another inequality on the quadratic form which is to be satisfied over the whole space; i.e., to a matrix inequality (MI). There exist various formulations of the S-lemma. In this paper, we consider it in the form which establishes the relation between the following two conditions:

$$x^\top G_0 x < 0 \;\text{ for }\; x^\top G_1 x \geqslant 0, \ldots, x^\top G_m x \geqslant 0, \quad x \neq 0, \tag{2.5}$$

$$\text{there exist } \tau_j > 0 \; (j = \overline{1, m})\colon x^\top G_0 x + \sum_{j=1}^{m} \tau_j \, x^\top G_j x < 0, \; x \in \mathbb{R}^n, \; x \neq 0, \tag{2.6}$$

where $G_j \in \mathbb{R}^{n\times n}$, $G_j^\top = G_j$, $j = \overline{0, m}$. Obviously, inequality (2.6) implies inequality (2.5); i.e., it is sufficient for (2.5) to hold. For $m = 1$, the so-called losslessness of the S-lemma can be proved (e.g., see [10, p. 135]), which means that, in this special case, condition (2.6) is also a necessary one.

Various formulations of the S-lemma, the history of appearance of this trick and the term itself, a detailed explanation of the terms "lossness" and "losslessness", etc., can be found in [2, 3, 6, 10].

The sector conditions (2.4) are equivalent to the quadratic constraints

$$F_j(x, \varphi_j) = \varphi_j(\langle c_j, x\rangle - \varphi_j) \geqslant 0, \quad j = \overline{1, m}. \tag{2.7}$$

For the derivative $\dot{v}(x)$ of the Lyapunov function $v(x) = x^\top Lx$, $L \in \mathbb{R}^{n\times n}$, $L^\top = L$, due to system (2.3) we have the inequality

$$x^\top(A^\top L + LA)x + 2\sum_{j=1}^{m} \varphi_j\langle Lb_j, x\rangle < 0, \quad (x, \varphi) \neq 0, \tag{2.8}$$

which must hold for all (x, φ) satisfying (2.7). Consider the quadratic form

$$x^\top(A^\top L + LA)x + 2\sum_{j=1}^{m} \varphi_j\langle Lb_j, x\rangle + \sum_{j=1}^{m} \tau_j\varphi_j\left(\langle c_j, x\rangle - \varphi_j\right) < 0, \tag{2.9}$$

where $\tau_j > 0$, $j = \overline{1, m}$, are free parameters.

In the matrix form, inequality (2.8) is written as

$$(Ax + B\varphi)^\top Lx + x^\top L(Ax + B\varphi) < 0, \;\; (x, \varphi) \neq 0,$$

where $B = \begin{pmatrix} b_1 & b_2 & \ldots & b_m \end{pmatrix}$, and the constraint function $F(x, \varphi, \mathcal{T})$ can be represented in the form

$$F(x, \varphi, \mathcal{T}) = \sum_{j=1}^{m} \tau_j\varphi_j\left(\langle c_j, x\rangle - \varphi_j\right) = \begin{pmatrix} x \\ \varphi \end{pmatrix}^\top \begin{pmatrix} 0 & C\mathcal{T}/2 \\ \mathcal{T}C^\top/2 & -\Gamma \end{pmatrix} \begin{pmatrix} x \\ \varphi \end{pmatrix},$$

where

$$C = \begin{pmatrix} c_1 & c_2 & \ldots & c_m \end{pmatrix}, \; \Gamma = \mathcal{T} = \operatorname{diag}\left\{\tau_1, \ldots, \tau_m\right\}.$$

With this notation, the negative definiteness of the form (2.9) is equivalent to the MI

$$\begin{pmatrix} A^\top L + AL & LB + C\mathcal{T}/2 \\ B^\top L + \mathcal{T}C^\top/2 & -\Gamma \end{pmatrix} < 0. \tag{2.10}$$

Feasibility conditions for this inequality follow from the frequency theorem (the KYP lemma [6, 10]). A version of this theorem, most suitable for our purposes, is given in Corollary 1; see [10], p. 54. Namely, for a Hurwitz stable A and $\Gamma > 0$, the feasibility of inequality (2.10) is equivalent to the satisfaction of the frequency inequality

$$\Gamma + \operatorname{Re} W(i\omega) > 0, \;\; W(i\omega) = \mathcal{T}C^\top(A - i\omega E_n)^{-1}B, \;\; \omega \in [-\infty, \infty], \tag{2.11}$$

where $\operatorname{Re} W = (W + W^*)/2$, the matrix $W^* = \overline{W}^\top$ is the Hermite conjugate of W, and E_n is the unit $(n \times n)$ matrix. Hence, the circle criterion for system (2.3) with several nonlinearities consists in checking the frequency condition (2.11), which is sufficient for the existence of a QLF for systems of this sort.

Using Theorem 1 it is possible to derive sufficient conditions for the existence of a QLF for system (2.3) which coincide with the circle criterion of the absolute stability of control systems with two nonlinearities; see [9]. Here, a similar result will be obtained for systems with arbitrary finite number of nonlinearities.

It is well known and also stressed in [11] that the absolute stability of the Lurie system (2.3) in the class of nonlinearities N_φ is equivalent to the stability of the system with switching between the linear systems $\dot{x} = A_s x$ with matrices A_s of the following form (see [8]):

$$A_s = A + \sum_{j=1}^{m} h_{sj} b_j c_j^\top, \quad h_s = \|h_{sj}\|_{j=1}^{m}, \quad s = \overline{1, N}, \quad (N = 2^m), \tag{2.12}$$

where the h_{sj}s take the two values 0 or 1 independently of each other. We assume that $h_1 = (0, \ldots, 0)$; i.e., $A_1 = A$. The existence of a QLF $v(x) = x^\top Lx$ for the Lurie system (2.3) is equivalent to the feasibility of the following set of LMIs (see [8]):

$$I_s = A_s^\top L + LA_s < 0, \quad s = \overline{1, N}. \tag{2.13}$$

From (2.12) we have

$$I_s = A^\top L + LA + \sum_{j=1}^{m} h_{sj} \left(Lb_j c_j^\top + c_j b_j^\top L \right) = A^\top L + LA + \sum_{j=1}^{m} h_{sj} Q_j,$$

where

$$Q_j = p_j q_j^\top + q_j p_j^\top, \;\; p_j \triangleq Lb_j, \;\; q_j \triangleq c_j, \;\; j = \overline{1, m}. \tag{2.14}$$

Following Theorem 1, we represent the matrices Q_j as the differences $Q_j = Q_j^+ - Q_j^-$. Then the MIs

$$Q_j \leqslant Q_j^+(\varepsilon_j) = \frac{\varepsilon_j^2}{2} u_j^+(\varepsilon_j) u_j^+(\varepsilon_j)^\top, \;\; u_j^+(\varepsilon_j) \triangleq p_j + \frac{1}{\varepsilon_j^2} q_j, \;\; j = \overline{1, m}, \tag{2.15}$$

hold for all $\varepsilon_j > 0$. Consider the following MI:

$$I_{\text{cir}} \triangleq A^\top L + LA + \sum_{j=1}^{m} Q_j^+(\varepsilon_j) < 0. \tag{2.16}$$

Since $h_{sj} = 0$ or $h_{sj} = 1$, we have

$$I_s \leqslant I_{\text{cir}}, \;\; s = \overline{1, N}. \tag{2.17}$$

The latter inequality implies that the feasibility of (2.16) guarantees the feasibility of (2.13).

Substituting expressions (2.14) for p_j and q_j in $Q_j^+(\varepsilon_j)$, re-defining in (2.15) the additional variables

$$\tau_j \triangleq 2/\varepsilon_j^2, \tag{2.18}$$

and using the Schur complement, we arrive at the equivalence of (2.16) and (2.10). In other words, the circle criterion is obtained without use of the S-lemma.

In fact, the necessary part of Theorem 1 is only needed in the case $m = 1$ in order to show the equivalence of the set of LMIs (2.13) and the inequality $I_{\text{cir}} < 0$.

Below, the matrix inequality (2.10) as well as the equivalent MI (2.16) is referred to as the matrix inequality of the circle criterion (MICC).

3. STABILITY OF SYSTEMS WITH SWITCHING BETWEEN THREE LINEAR TIME-INVARIANT SUBSYSTEMS

As noted above, the absolute stability of the Lurie system (2.3) is a particular case of the stability problem for the linear switched system

$$\dot{x} = A(t)x, \quad A(t) \in \overline{A} = \{A_1, \dots, A_N\}, \tag{3.1}$$

under arbitrary switchings, where $A_s \in \mathbb{R}^{n\times n}$, and $A(t) : \mathbb{R}_+ \longrightarrow \overline{A}$ is a piecewise constant mapping. For $m = 1$, system (2.3) is associated with system (3.1) with switching between two subsystems. The existence of a QLF for such systems is determined by the feasibility of the two MIs of the form (2.13). The resulting MI (RMI) equivalent to the two MIs above is an LMI, and the frequency condition for its feasibility (circle criterion) is necessary and sufficient for the existence of a QLF. The next natural step would be the formulation of a similar result for system (3.1) with $N = 3$. In that case, the existence of a QLF is determined by the feasibility of the set of three MIs having form (2.13). Hence, in this section we are targeted at formulating the RMI for the set of three MIs (2.13) in the form of an LMI, and a frequency condition of the feasibility of this RMI.

We note that a realization of such a scenario for system (2.3) with $m = 2$ or for system (3.1) with $N = 4$ is obscure.

For $N = 3$, the matrices $\{A_1, A_2, A_3\}$ that define the connected (see [9]) system (3.1) can be represented as follows:

$$A_1 = A, \quad A_2 = A + b_1c_1^\top, \quad A_3 = A + b_2c_2^\top. \tag{3.2}$$

Using notation (2.14) with $m = 2$, the corresponding set of inequalities (2.13) for $N = 3$ takes the form

$$I_1 + p_1q_1^\top + q_1p_1^\top < 0, \quad I_1 < 0, \quad I_1 + p_2q_2^\top + q_2p_2^\top < 0. \tag{3.3}$$

Applying Theorem 1 to the first and second inequalities in (3.3) and then to the second and third ones, we see that this whole set of inequalities is feasible if and only if there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that the pair of MIs

$$\widetilde{I}_1 = I_1 + \frac{\varepsilon_1^2}{2}u_1^+(\varepsilon_1)u_1^+(\varepsilon_1)^\top < 0, \;\; \widetilde{I}_2 = I_1 + \frac{\varepsilon_2^2}{2}u_2^+(\varepsilon_2)u_2^+(\varepsilon_2)^\top < 0 \tag{3.4}$$

is feasible.

In [9], an RMI equivalent to (3.4) was obtained along with a frequency condition for the feasibility of this RMI. However, the RMI in [9] is not an LMI in the extra parameters involved, so that the frequency condition presented in [9] is very bulky. Below we propose a new tool for deriving an RMI for (3.4) in the form of an LMI, which considerably simplifies the frequency condition for the existence of a QLF for system (3.1) with $N = 3$. The key idea is as follows: Instead of the inequalities for $(n \times n)$-matrices in (3.4), we consider equivalent inequalities for $((n + 1) \times (n + 1))$-matrices by using the Schur complement. As a result, we arrive at the set of MIs

$$\widetilde{I}_1 < 0 \cong \widehat{\widetilde{I}}_1 = \begin{pmatrix} I_1 & u_1^+ \\ (u_1^+)^\top & -2/\varepsilon_1^2 \end{pmatrix} < 0, \quad \widetilde{I}_2 < 0 \cong \widehat{\widetilde{I}}_2 = \begin{pmatrix} I_1 & u_2^+ \\ (u_2^+)^\top & -2/\varepsilon_2^2 \end{pmatrix} < 0, \tag{3.5}$$

which is equivalent to (3.4).

The matrix of the difference has the form

$$\widehat{\widehat{I}}_2 - \widehat{\widehat{I}}_1 = \begin{pmatrix} 0_{n\times n} & u_2^+ - u_1^+ \\ (\bullet)^\top & 2/\varepsilon_1^2 - 2/\varepsilon_2^2 \end{pmatrix},$$

where $0_{n\times m}$ denotes the $n \times m$ zero matrix. Also, the symbol "$\bullet$" is used to denote the corresponding symmetric block in a symmetric matrix. Denoting $\widehat{p} \triangleq u_2^+ - u_1^+$ and $\gamma \triangleq 1/\varepsilon_1^2 - 1/\varepsilon_2^2$, we obtain

$$\widehat{\widehat{I}}_2 - \widehat{\widehat{I}}_1 = \widetilde{p}\,\widetilde{q}^\top + \widetilde{q}\,\widetilde{p}^\top, \quad \widetilde{p} = \begin{pmatrix} \widehat{p} \\ \gamma \end{pmatrix}, \quad \widetilde{q} = \begin{pmatrix} 0_{n\times 1} \\ 1 \end{pmatrix}.$$

In other words, Theorem 1 can be applied to (3.5), and we conclude that the feasibility of (3.5) is equivalent to the existence of $\varepsilon_3 > 0$ such that the following single MI is feasible:

$$\widetilde{\widetilde{I}} = \widehat{\widehat{I}}_1 + \frac{\varepsilon_3^2}{2}\left(\widetilde{p} + \frac{1}{\varepsilon_3^2}\widetilde{q}\right)\left(\widetilde{p} + \frac{1}{\varepsilon_3^2}\widetilde{q}\right)^\top < 0. \tag{3.6}$$

By the Schur complement, the MI (3.6) is equivalent to the MI

$$\widetilde{\widetilde{I}} < 0 \cong \overset{\approx}{\widetilde{I}} = \begin{pmatrix} \widehat{\widehat{I}}_1 & \widetilde{p} + (1/\varepsilon_3^2)\widetilde{q} \\ (\bullet)^\top & -2/\varepsilon_3^2 \end{pmatrix} = \begin{pmatrix} I_1 & u_1^+(\varepsilon_1) & \widehat{p} \\ (\bullet)^\top & -2/\varepsilon_1^2 & \gamma + 1/\varepsilon_3^2 \\ (\bullet)^\top & \gamma + 1/\varepsilon_3^2 & -2/\varepsilon_3^2 \end{pmatrix} < 0 \tag{3.7}$$

in the augmented space.

With notation (2.18) for τ_j ($s = \overline{1,3}$), the entries of the matrix $\overset{\approx}{\widetilde{I}}$ are seen to satisfy the relations

$$u_1^+(\tau_1) = p_1 + \frac{\tau_1}{2}q_1, \quad u_2^+(\tau_2) = p_2 + \frac{\tau_2}{2}q_2, \quad \gamma = \frac{1}{2}\left(\tau_1 - \tau_2\right).$$

We present the result obtained above in the following form.

Theorem 2. *Let the inequalities in* (3.3) *be LMIs in the unknown variable ν; i.e., $I_s = I_s(\nu)$, $s = \overline{1,3}$, and $Q_j(\nu) = p_j(\nu)q_j^\top + q_j p_j^\top(\nu)$, where $p_j = p_j(\nu)$ depends linearly on ν, and q_j, $j = 1,2$, do not depend on ν.*

Then (3.3) *is equivalent to the single MI*

$$\overset{\approx}{\widetilde{I}} = \begin{pmatrix} I_1(\nu) & p_1(\nu) + \frac{\tau_1}{2}q_1 & p_2(\nu) - p_1(\nu) + \frac{\tau_2}{2}q_2 - \frac{\tau_1}{2}q_1 \\ (\bullet)^\top & -\tau_1 & (\tau_1 - \tau_2 + \tau_3)/2 \\ (\bullet)^\top & \bullet & -\tau_3 \end{pmatrix} < 0,$$

which is an LMI in $(\nu, \tau_1, \tau_2, \tau_3)$.

Let us express $\overset{\approx}{\widetilde{I}}$ in the original notation by using (2.14) for p_j and q_j:

$$\overset{\approx}{\widetilde{I}} = \begin{pmatrix} A^\top L + LA & Lb_1 + (\tau_1/2)c_1 & L(b_2 - b_1) - (\tau_1/2)c_1 + (\tau_2/2)c_2 \\ (\bullet)^\top & -\tau_1 & (\tau_1 - \tau_2 + \tau_3)/2 \\ (\bullet)^\top & \bullet & -\tau_3 \end{pmatrix} < 0. \tag{3.8}$$

Hence, the MI (3.8) is an LMI with respect to the unknown L and τ_j, $j = \overline{1,3}$, and it can be solved numerically by using the standard software.

Moreover, the MI (3.8) can be represented in the form (2.10) for

$$B = \begin{pmatrix} b_1 & b_2 - b_1 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & c_2 \end{pmatrix}, \quad \tau = \begin{pmatrix} \tau_1 & -\tau_1 \\ 0 & \tau_2 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \tau_1 & \frac{1}{2}(\tau_2 - \tau_1 - \tau_3) \\ \bullet & \tau_3 \end{pmatrix}. \tag{3.9}$$

In this case, the feasibility of the MI (2.10) is established by the frequency theorem [10, p. 54] (the classical KYP lemma), which gives the following criterion for the existence of a QLF for the switched system (3.1) with $N = 3$.

Theorem 3. *Consider system* (3.1) *with* $N = 3$ *and let the matrices* A_s *be defined by* (3.2) *and* A *be Hurwitz stable. Assume there exists a set of numbers* $\tau_j > 0$, $j = \overline{1,3}$, *such that* $\Gamma > 0$, *and for any* $\omega \in [-\infty, \infty]$ *the frequency inequality* (2.11) *holds, where the matrix* Γ *and the entries of the matrix* $W(i\omega)$ *are defined in* (3.9).

Then system (3.1) *possesses a QLF* (*the set of LMIs* (2.13) *is feasible, system* (3.1) *is stable*).

If system (3.1) *possesses a QLF* (*the set of LMIs* (2.13) *is feasible*), *then there exists such a set of numbers* $\tau_j > 0$, $j = \overline{1,3}$.

Clearly, the conditions of Theorem 3 are much simpler and better than those of Theorem 2 in [9].

4. AN ALTERNATIVE POINT OF VIEW ON THE CIRCLE CRITERION. REDUCTION OF THE DIMENSION OF LMI SYSTEMS

The circle criterion is obtained as a feasibility condition for the MICC (2.10), which is an LMI in the unknown L and τ_j, $j = 1, m$; it can be checked numerically via use of numerous software tools. Therefore, instead of checking the feasibility of the set of inequalities (2.13) having overall dimension $2^m n$, a single MICC (2.10) of dimension $n + m$ with m additional parameters can be considered. Clearly, one has to account for possible contraction (losses) of the domain of existence of the QLF, caused by the lossness of the S-lemma.

With the criterion in Theorem 2, the transition from the set (2.13) (having cumulative dimension $3n$ in the case $N = 3$) to the single MI (3.8) (of dimension $n + 2$ and having three extra parameters) can be performed with no losses in the domain of existence of a QLF.

The absolute stability of the Lurie system (2.3) for the case $m = 2$ is equivalent [9] to the stability of the switched system (3.1) (under arbitrary switching) with the matrices A_s defined by

$$A_1 = A, \quad A_2 = A + b_1 c_1^\top, \quad A_3 = A + b_2 c_2^\top, \quad A_4 = A + b_1 c_1^\top + b_2 c_2^\top, \quad b_s, c_s \in \mathbb{R}^n. \tag{4.1}$$

Theorem 2 can be applied to the corresponding set of LMIs (2.13) with $N = 4$. First, it should be applied to the three inequalities in (2.13), and the obtained inequality is then to be coupled with the fourth MI into one set of inequalities. The resulting set of the two MIs is equivalent to the initial set; it has overall dimension $2n + 2$ and depends on the three additional parameters. The initial set of LMIs (2.13) for $N = 4$ has dimension $4n$.

The theorem below presents yet another method of reducing the dimension of connected systems of LMIs; it combines Theorem 1 and the Schur complement.

Theorem 4. *Assume that the inequalities in* (2.1) *are LMIs in the unknown variable* ν; *i.e.,* $I_1 = I_1(\nu)$ *and* $I_2 = I_2(\nu)$, *and the matrix* Q *admits the representation* $Q(\nu) = I_2(\nu) - I_1(\nu) = p(\nu)q^\top + qp^\top(\nu)$, *where* $p = p(\nu)$ *depends linearly on* ν, *and* q *does not depend on* ν.

Then the set of MIs (2.1) *is equivalent to the single MI*

$$\begin{pmatrix} I_1(\nu) & p(\nu) + (\tau/2)q \\ (\bullet)^\top & -\tau \end{pmatrix} < 0,$$

which is an LMI in the variables (ν, τ).

Application of Theorem 4 to the set of LMIs (2.13) for $N = 4$ yields a set of two MIs which is equivalent to the original one, has overall dimension $2n + 2$, and depends on the two additional parameters. As compared to the application of Theorem 2, the benefit is minimal, just one extra parameter less.

We next compare the efficiency of Theorems 2 and 4 in the reduction of the dimension of sets of LMIs in the two following cases of (2.13): $N = 6$ and $N = 8$.

In the case $N = 8$, let (2.13) determine the existence conditions of a QLF for the Lurie system (2.3) with $m = 3$; i.e., $N = 2^m$. Theorem 4 can be applied to the four pairs of MIs to obtain a set of the four inequalities of overall dimension $4n + 4$ and having four extra parameters. On the other hand, we can apply Theorem 2 to the two triplets of MIs, and Theorem 4 to the remaining two MIs, and arrive at the set of three inequalities of the overall dimension $3n + 5$ and having seven extra parameters. The original LMI set (2.13) for $N = 8$ is of dimension $8n$.

In the case $N = 6$, we consider a switched system with certain connectivity margin which allows us to apply Theorem 2 to the two triplets of this system, and Theorem 4 to the three pairs. For instance, it can be a system with switching of the prism type in which the matrices A_s are defined by the relations

$$\begin{aligned} A_1 &= A, & A_2 &= A + b_3 c_3^\top, \\ A_3 &= A + b_1 c_1^\top, & A_4 &= A + b_1 c_1^\top + b_3 c_3^\top, \\ A_5 &= A + b_2 c_2^\top, & A_6 &= A + b_2 c_2^\top + b_3 c_3^\top, \quad b_s, c_s \in \mathbb{R}^n. \end{aligned}$$

Then a twofold application of Theorem 2 leads to a system of two inequalities of cumulative dimension $2n + 4$ and with six additional parameters, whereas a threefold application of Theorem 4 leads to a system of three inequalities of cumulative dimension $3n + 3$ and with three additional parameters. The original set of LMIs (2.13) for $N = 6$ has dimension $6n$.

At the end of this section, we show how to apply Theorem 4 in order to reduce the dimensionality of the set (2.13) when checking the existence of a QLF for the Lurie system (2.3) with arbitrary finite m. On the one hand, each application of Theorem 4 reduces the number of MIs in the system by one; on the other hand, it increases by one the number of unknowns. From the set of LMIs (2.13) in $n(n+1)/2$ variables, which has overall dimension $2^m n$, we can move to an equivalent system of dimension $2^{m-1}(n+1)$ with respect to $n(n+1)/2 + 2^{m-1}$ variables.

Theorem 5. *For* $N = 2^m$, *the system of MIs* (2.13) *is equivalent to the system of MIs*

$$\begin{pmatrix} A_s^\top L + L A_s & L b_m + (\tau_s/2) c_m \\ (\bullet)^\top & -\tau_s \end{pmatrix} < 0, \quad s = \overline{1, 2^{m-1}}, \tag{4.2}$$

with 2^{m-1} *additional parameters* $\tau_s > 0$.

A proof of Theorem 5 is given in the Appendix.

5. IMPROVEMENT OF THE CIRCLE CRITERION FOR THE CASE OF TWO NONLINEARITIES

In Section 4 we mentioned the equivalence (see [9]) of the absolute stability of the Lurie system (2.3) for $m = 2$ and the switched system (3.1) with matrices A_s defined by relations (4.1). In this case, the set of LMIs (2.13) takes the form

$$I_s = A_s^\top L + LA_s < 0, \quad s = \overline{1,4}, \tag{5.1}$$

where the matrices A_s are defined in (4.1).

In [12], the existence of a QLF for discrete-time Lurie systems was discussed for the case $m = 2$, and criteria A and B were considered as improvements of the Tsypkin criterion. In [12] it was shown that, for some examples, criterion A produces a more accurate result than the Thypkin criterion, whereas other examples demonstrate the opposite relation. Criterion B either reproduces or improves the estimate of the stability domain obtained by the Tsypkin criterion. A continuous-time counterpart of the Tsypkin criterion is the circle criterion; therefore it is interesting to understand what are the outcomes of the approaches underlying criteria A and B in continuous case.

In order to pass on to the specific exposition, we briefly recall the diagram proposed in [12] for the derivation of an RMI [8] equivalent to the original system (5.1):

$$\begin{array}{ccccccccccccc}
I_2 & & \longleftrightarrow & & I_1 & & \longleftrightarrow & & I_3 & & \longleftrightarrow & & I_4 \\
 & & \downarrow_{\varepsilon_1} & & & & \downarrow_{\varepsilon_3} & & & & \downarrow_{\varepsilon_2} & & \\
 & & \widetilde{I}_1 & & \longleftrightarrow & & \widetilde{I}_3 & & \longleftrightarrow & & \widetilde{I}_2 & & \\
 & & & & \downarrow_{\varepsilon_4} & & & & \downarrow_{\varepsilon_5} & & & & \\
 & & & & \widetilde{\widetilde{I}}_1 & & \longleftrightarrow & & \widetilde{\widetilde{I}}_2 & & & & \\
 & & & & & & \downarrow_{\varepsilon_6} & & & & & & \\
 & & & & & & \widetilde{\widetilde{I}}\,. & & & & & &
\end{array}$$

The horizontal arrows in the diagram indicate pairs of inequalities which fall under the conditions of Theorem 1. The vertical arrows point to MIs resulting from the application of this theorem, and ε_ss denote the emerging new parameters.

In the diagram, to make the expressions for the MIs through $u_s^\pm$ the same for the continuous and discrete cases, we keep the expressions from (2.14) for p_j and q_j through Lb_j and c_j, but change the expressions from (2.15) for $u_s^\pm$ through p_j and q_j; i.e., now the vectors $u_s^\pm$ correspond to values other than those defined in (2.15) and used in Sections 2 and 3.

We present the expressions for the MIs in the diagram above in terms of p_s and q_s from (2.14) for $m = 2$; the notation $\varepsilon_s^\pm = 1 \pm 1/\varepsilon_s^2$ will be used.

First-level inequalities:

$$\widetilde{I}_1 = I_1 + \frac{\varepsilon_1^2}{2} u_1^+ (u_1^+)^\top = I_2 + \frac{\varepsilon_1^2}{2} u_1^- (u_1^-)^\top < 0, \quad u_1^\pm = p_1 \pm \frac{1}{\varepsilon_1^2} q_1, \tag{5.2}$$

$$\widetilde{I}_2 = I_3 + \frac{\varepsilon_2^2}{2} u_2^+ (u_2^+)^\top = I_4 + \frac{\varepsilon_2^2}{2} u_2^- (u_2^-)^\top < 0, \quad u_2^\pm = p_1 \pm \frac{1}{\varepsilon_2^2} q_1, \tag{5.3}$$

$$\widetilde{I}_3 = I_1 + \frac{\varepsilon_3^2}{2} u_3^+ (u_3^+)^\top = I_3 + \frac{\varepsilon_3^2}{2} u_3^- (u_3^-)^\top < 0, \quad u_3^\pm = p_2 \pm \frac{1}{\varepsilon_3^2} q_2, \tag{5.4}$$

where $\widetilde{I}_1 < 0 \;\cong\; I_1 < 0,\, I_2 < 0,\, \widetilde{I}_2 < 0 \;\cong\; I_3 < 0,\, I_4 < 0,\, \widetilde{I}_3 < 0 \;\cong\; I_1 < 0,\, I_3 < 0.$

Second-level inequalities:

$$\widetilde{\widetilde{I}}_1 = \widetilde{I}_1 + \frac{\varepsilon_4^2}{2} u_4^+ (u_4^+)^\top = \widetilde{I}_3 + \frac{\varepsilon_4^2}{2} u_4^- (u_4^-)^\top < 0, \quad u_4^\pm = \frac{\varepsilon_1 \varepsilon_4^\mp}{2} u_1^+ + \frac{\varepsilon_3 \varepsilon_4^\pm}{2} u_3^+, \tag{5.5}$$

$$\widetilde{\widetilde{I}}_2 = \widetilde{I}_2 + \frac{\varepsilon_5^2}{2} u_5^+ (u_5^+)^\top = \widetilde{I}_3 + \frac{\varepsilon_5^2}{2} u_5^- (u_5^-)^\top < 0, \quad u_5^\pm = \frac{\varepsilon_2 \varepsilon_5^\mp}{2} u_2^+ + \frac{\varepsilon_3 \varepsilon_5^\pm}{2} u_3^-, \tag{5.6}$$

where $\widetilde{\widetilde{I}}_1 < 0 \ \cong \ \widetilde{I}_1 < 0, \ \widetilde{I}_3 < 0, \ \widetilde{\widetilde{I}}_2 < 0 \ \cong \ \widetilde{I}_2 < 0, \ \widetilde{I}_3 < 0.$

The resulting MI:

$$\widetilde{\widetilde{I}} = \widetilde{\widetilde{I}}_1 + \frac{\varepsilon_6^2}{2} u_6^+ (u_6^+)^\top = \widetilde{\widetilde{I}}_2 + \frac{\varepsilon_6^2}{2} u_6^- (u_6^-)^\top < 0, \quad u_6^\pm = \frac{\varepsilon_4 \varepsilon_6^\mp}{2} u_4^- + \frac{\varepsilon_5 \varepsilon_6^\pm}{2} u_5^-, \tag{5.7}$$

where $\widetilde{\widetilde{I}} < 0 \ \cong \widetilde{\widetilde{I}}_1 < 0, \ \widetilde{\widetilde{I}}_2 < 0.$

The derivation of criterion A in [12] was based on the approach (approach A) which, being reformulated for the continuous case, consists of the following steps. First, two out of the six extra parameters in the RMI (5.7) are set to unity; specifically, $\varepsilon_4 = \varepsilon_5 = 1$. This assumption turns the inequality $\widetilde{\widetilde{I}}\,(\varepsilon_1, \varepsilon_2, \varepsilon_3, 1, 1, \varepsilon_6) \triangleq \widetilde{\widetilde{I}}_{(1)} < 0$ into a sufficient condition for the validity of the whole set of MIs (5.1). Second, the conditions of the validity of this MI $\widetilde{\widetilde{I}}_{(1)} < 0$ and the conditions of the circle criterion are compared.

Let us demonstrate the application of approach A in the continuous-time case. The expressions of the second-level MIs through $u_s^\pm$ are the same for discrete and continuous time; hence, we briefly repeat the derivations in [12]. By setting $\varepsilon_4 = \varepsilon_5 = 1$ in (5.5) and (5.6) (in that case, $\varepsilon_4^+ = \varepsilon_5^+ = 2$, $\varepsilon_4^- = \varepsilon_5^- = 0$), we obtain

$$u_4^+ = \varepsilon_3 u_3^+, \quad u_4^- = \varepsilon_1 u_1^+, \quad u_5^+ = \varepsilon_3 u_3^-, \quad u_5^- = \varepsilon_2 u_2^+,$$

and the corresponding second-level MIs take the form

$$\begin{aligned} \widetilde{\widetilde{I}}_{1(1)} &= \widetilde{I}_3 + \frac{\varepsilon_1^2}{2} u_1^+ (u_1^+)^\top = I_1 + \frac{\varepsilon_1^2}{2} u_1^+ (u_1^+)^\top + \frac{\varepsilon_3^2}{2} u_3^+ (u_3^+)^\top < 0, \\ \widetilde{\widetilde{I}}_{2(1)} &= \widetilde{I}_3 + \frac{\varepsilon_2^2}{2} u_2^+ (u_2^+)^\top = I_1 + \frac{\varepsilon_2^2}{2} u_2^+ (u_2^+)^\top + \frac{\varepsilon_3^2}{2} u_3^+ (u_3^+)^\top < 0. \end{aligned} \tag{5.8}$$

Accounting for the expressions in (5.2)–(5.4) for $u_s^\pm$ through p_j and q_j, we see that, for $m = 2$, the MICCK (2.16) has the form

$$I_{\text{cir}} = I_1 + \frac{\varepsilon_1^2}{2} u_1^+ (u_1^+)^\top (\varepsilon_1) + \frac{\varepsilon_3^2}{2} u_3^+ (u_3^+)^\top (\varepsilon_3) < 0, \tag{5.9}$$

where, to save space, the short-hand notation $u_s^+ (u_s^+)^\top (\varepsilon)$ for $u_s^+(\varepsilon) u_s^+(\varepsilon)^\top$ is adopted. It is obvious that the feasibility of any of the two MIs in (5.8) implies the feasibility of the MICC (5.9).

On the other hand, assume (from now on) that the MICC (5.9) is feasible for $\varepsilon_1 = \varepsilon_{1cir}$ and $\varepsilon_3 = \varepsilon_{2cir}$. Then the first inequality in (5.8) is feasible for $\varepsilon_1 = \varepsilon_{1cir}$ and $\varepsilon_3 = \varepsilon_{2cir}$, and the second one is feasible for $\varepsilon_2 = \varepsilon_{1cir}$ and $\varepsilon_3 = \varepsilon_{2cir}$. Therefore, the MI $\widetilde{\widetilde{I}}(\varepsilon_{1cir}, \varepsilon_{1cir}, \varepsilon_{2cir}, 1, 1, \varepsilon_6) < 0$ is feasible for some ε_6 which is defined when applying Theorem 1 to the set (5.8) of two inequalities. This result is formulated in the theorem below.

Theorem 6. *The feasibility of the MICC* (2.16) *for* $m = 2$ *is equivalent to the feasibility of the RMI* (5.7) *for* $\varepsilon_4 = \varepsilon_5 = 1$.

In other words, use of approach A in the continuous-time case leads to the same existence conditions of a QLF, as the conditions of the circle criterion.

The derivation of criterion B in [12] was based on the approach (approach B) which, being reformulated for the continuous case, consists of the following steps. First, we we let $\varepsilon_4 = \varepsilon_5 = \varepsilon$ in the RMI (5.7); i.e., consider just five out of the six extra parameters. Second, we compare analytically the feasibility conditions of the MI $\stackrel{\approx}{I}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon, \varepsilon, \varepsilon_6) \triangleq \stackrel{\approx}{I}_{(\varepsilon)} < 0$ and the MICC (5.9).

Such a comparison has been performed (we omit the details). As a result, under the adopted coarsening assumptions, which make such a comparison possible, we failed to show analytically an improvement of the circle criterion within approach B. Nevertheless, the possibility of such an improvement remains. To test this, the domains of quadratic stability (QSD) obtained from the feasibility of the RMI (5.7) for $\varepsilon_4 = \varepsilon_5 = \varepsilon$ and from the MICC (5.9) are to be compared via numerical examples.

Note that the MIs (5.5) and (5.6) and hence, the RMI (5.7) are not LMIs in the considered parameters. We therefore pass on from the second-level inequalities (5.5) and (5.6) to the equivalent LMIs. By Theorem 2, the set of the three inequalities $I_s < 0$, $s = \overline{1,3}$, from the set (5.1) is equivalent to the single MI

$$\stackrel{\cong}{I}_1 = \begin{pmatrix} A^\top L + LA & Lb_1 + (\tau_1/2)c_1 & L(b_2 - b_1) - \dfrac{\tau_1}{2}c_1 + \dfrac{\tau_3}{2}c_2 \\ (\bullet)^\top & -\tau_1 & (\tau_1 - \tau_3 + \tau_4)/2 \\ (\bullet)^\top & \bullet & -\tau_4 \end{pmatrix} < 0, \tag{5.10}$$

which is an LMI in the variables L and τ_j, $j = 1, 3, 4$, and it is equivalent to the MI (5.5). Similarly, the set of the three inequalities $I_s < 0$, $s = \overline{2,4}$ in (5.1) is equivalent to the single MI

$$\stackrel{\cong}{I}_2 = \begin{pmatrix} A_3^\top L + LA_3 & Lb_1 + \dfrac{\tau_2}{2}c_1 & L(b_2 - b_1) - \dfrac{\tau_2}{2}c_1 - \dfrac{\tau_3}{2}c_2 \\ (\bullet)^\top & -\tau_2 & (\tau_2 - \tau_3 + \tau_5)/2 \\ (\bullet)^\top & \bullet & -\tau_5 \end{pmatrix} < 0, \tag{5.11}$$

which is an LMI in L and τ_j, $j = 2, 3, 5$, and it is equivalent to the MI (5.6). Hence, the feasibility of the set

$$\stackrel{\cong}{I}_1 < 0, \quad \stackrel{\cong}{I}_2 < 0, \tag{5.12}$$

of these two MIs is equivalent to the feasibility of the RMI (5.7).

In this context, of interest is the relation between the parameters ε_js, which enter the RMI (5.7), and the parameters τ_js, which enter the set (5.12). The ε_js appear in (5.7) according to the diagram and formulae (5.2)–(5.7). The τ_js appear in (5.12) according to Theorem 2. In the derivation of Theorem 2, the parameter ε_3 in (3.6) appears when applying the convolution theorem to the MI of dimension $n + 1$ (there is no counterpart of ε_3 in the diagram and in the RMI (5.7)). Hence, though relation (2.18) between the parameters ε_j, $j = \overline{1,3}$, from the diagram and (5.7) and the parameters τ_j, $j = \overline{1,3}$, from (5.12) does hold, there is no such a relation between the parameters τ_4 and τ_5 in (5.12) and the parameters ε_4 and ε_5 in the RMI (5.7). Therefore, the relation $\varepsilon_4 = \varepsilon_5 = \varepsilon$ does not imply $\tau_4 = \tau_5 = \tau$ and vice versa. Hence, the feasibility of (5.7) for $\varepsilon_4 = \varepsilon_5 = \varepsilon$ is not equivalent to the feasibility of (5.12) for $\tau_4 = \tau_5 = \tau$, whereas the feasibility of (5.7) is equivalent to the feasibility of (5.12).

A new version of approach B emerges. As a continuous-time counterpart of criterion B in [12], we consider the feasibility conditions of (5.12) for $\tau_4 = \tau_5 = \tau$; we refer to it as criterion C. We now compare the conditions of criterion C and the conditions of the circle criterion. To this end, we detect τ_4 and τ_5 implying the feasibility of the MIs in (5.12), provided that the conditions of the MICC hold.

Theorem 7. *Let the MICC* (5.9) *is feasible for* $\varepsilon_1 = \varepsilon_{1cir}$ *and* $\varepsilon_3 = \varepsilon_{2cir}$. *Then the set of the MIs* (5.12) *is feasible for*

$$\tau_1 = \tau_2 = 2/\varepsilon_{1cir}^2, \quad \tau_3 = 2/\varepsilon_{2cir}^2, \quad \tau_4 = \tau_5 = 2/\varepsilon_{1cir}^2 + 2/\varepsilon_{2cir}^2.$$

A proof of Theorem 7 is given in the Appendix.

From Theorem 7 it follows that the QSD obtained via criterion C is not worse than the one obtained via the circle criterion. However use of criterion C does not guarantee an improvement of the circle criterion. Still, such an improvement can be demonstrated via numerical examples. For the completeness of the exposition, we present such an example showing that the QSD obtained from criterion C is wider than that obtained from the circle criterion.

Example 1. Considered is the Lurie system of the form (2.3) with $n = 6$, where the matrix A is in the companion form; i.e., it is completely defined by the last row

$$\begin{gathered} A \sim [-10.0 \;\; -34.0 \;\; -49.0 \;\; -40.0 \;\; -20.0 \;\; -6.0\,], \\ \text{spectr}(A) = [-1.0 \;\; -1.0 \;\; -1.0 - i \;\; -1.0 + i \;\; -1.0 - 2i \;\; -1.0 + 2i\,], \\ b_1^\top = (0\;\,0\;\,0\;\,k_1\;\,0\;\,0),\;\; b_2^\top = (0\;\,0\;\,0\;\,0\;\,k_2\;\,0), \\ c_1^\top = (0\;\,0\;\,0\;\,0\;\,1\;\,1),\;\; c_2^\top = (0\;\,0\;\,0\;\,0\;\,1\;\,1). \end{gathered}$$

We evaluate the size of the QSDs obtained by means of the three algorithms. Algorithm NS consists in computing the QSD in accordance with necessary and sufficient conditions for the existence of a QLF by checking the feasibility of (5.1). Algorithm CC consists in evaluating the QSD via use of the circle criterion; i.e., by checking the MICC (2.10). Algorithm C consists in evaluating the QSD via use of criterion C; i.e., by checking the feasibility of (5.12) for $\tau_4 = \tau_5 = \tau$.

To find the size of a QSD, we consider a ray emanating from the origin. We then fix a directional vector $\overline{\alpha} = (\alpha_1, \alpha_2)$, $\alpha_s \geqslant 0$, along this ray and maximize the value of k such that the conditions of one or another criterion are satisfied for $(k_1, k_2) = k\overline{\alpha}$; five specific vectors were considered in the experiments.

The top row of Table presents the rays $\overline{\alpha}_i$, $i = \overline{1,5}$, along which the QSD is evaluated, and the left column shows the names of the corresponding algorithms used.

Table

Algorithm\Ray	$(1,1)$	$(1,2)$	$(1,3)$	$(2,1)$	$(3,1)$
NS	0.45684	0.32608	0.25301	0.28482	0.20674
CC	0.44831	0.31943	0.24813	0.28088	0.20453
C	0.45684	0.32608	0.25301	0.28482	0.20674

For this example, along all the considered directions $\overline{\alpha}_i$, the QSD obtained via criterion C is wider than the one obtained via the circle criterion; moreover, it coincides with the exact QSD, see Table.

The overall dimension of the set (5.12) is $2n + 4$, and for $\tau_4 = \tau_5 = \tau$, this set depends on five extra parameters. At the same time, each of the sets $\overset{\cong}{I}_1 < 0$, $I_4 < 0$, and $I_1 < 0$, $\overset{\cong}{I}_2 < 0$, is of the overall dimension $2n + 2$ and each of them depends on the three extra parameters. Moreover, each of these sets is equivalent to the original one (5.1) without losses in the domain of feasibility. On top of that, application of Theorem 5 to the set (5.1) leads to a set of two LMIs, which is equivalent to the original one, has cumulative dimension $2n+2$ and depends on the two extra parameters. Therefore, checking the condition of criterion C is exceptionally of theoretical interest. From the application point of view, the best way to check the feasibility of (5.1) numerically is to solve the system of the two LMIs obtained from Theorem 5.

Remark 1. Some of the conditions of the existence of a QLF presented above consist in checking a parameter-dependent LMI on feasibility. Due to linearity it is possible to set one additional parameter equal to unity, thus reducing the number of additional parameters by one. Among these conditions are the MICC (2.10), the LMI (3.8), and the systems of LMIs (4.2) and (5.12).

6. CONCLUSIONS

First, in this work, the circle criterion for a Lurie system with several nonlinearities is obtained without use of the S-lemma. Second, for a connected system with switching between three linear subsystems, the criterion for the existence of a quadratic Lyapunov function is obtained both in the form of the solvability conditions of a single LMI, and in the form of a frequency condition. Third, two theorems are proved which allow for an essential reduction in the dimensionality of the connected LMI system. Use of these theorems is demonstrated for the case of the Lurie system with $m=2$, $m=3$, and arbitrary finite m, and for systems with switching for $N=6$. Fourth, a comparison was performed for various approaches to the improvement of the circle criterion for the Lurie system with $m=2$.

FUNDING

This work was financially supported by the Program of fundamental scientific research on priority directions determined by the Presidium of the Russian Academy of Sciences, no. 7 “New developments in prospective areas of energy, mechanics, and robotics.”

APPENDIX

Proof of Theorem 5. Let $N=2^m$ and let the MIs in (2.13) be numerated in such a way that the first 2^{m-1} inequalities $I_s<0$, $s=\overline{1,2^{m-1}}$, coincide with the inequalities in (2.13) for $N=2^{m-1}$, and the rest 2^{m-1} inequalities $I_s<0$, $s=\overline{2^{m-1}+1,2^m}$, are numerated as follows:

$$I_{s+2^{m-1}} = I_s + \left(Lb_mc_m^\top + c_mb_m^\top L\right) < 0, \quad s=\overline{1,2^{m-1}}.$$

Then, Pyatnitskiy's theorem is applicable to the pairs of inequalities

$$I_s<0, \quad I_{s+2^{m-1}}<0, \quad s=\overline{1,2^{m-1}}. \tag{A.1}$$

As a result, the system of inequalities (2.13) is equivalent to the set of MIs

$$I_s + \frac{\varepsilon_s^2}{2}\left(Lb_m + \frac{1}{\varepsilon_s^2}c_m\right)\left(Lb_m + \frac{1}{\varepsilon_s^2}c_m\right)^\top < 0, \quad s=\overline{1,2^{m-1}}, \tag{A.2}$$

with 2^{m-1} additional parameters $\varepsilon_s>0$. Letting $\varepsilon_s=\varepsilon_m>0$, $s=\overline{1,2^{m-1}}$, in (A.2), we arrive at yet another proof, by induction, of the transition from (2.13) to the MICC (2.16). Application of Theorem 4 to every pair of inequalities (A.1) provides the equivalence of the set of MIs (2.13) to the set of MIs (4.2) with 2^{m-1} extra parameters $\tau_s \triangleq 2/\varepsilon_s^2$. Theorem 5 is proved.

Proof of Theorem 7. We show that the fulfilment of the MICC (5.9) implies the existence of τ_4 such that the MI (5.10) holds. Similarly to [12], to determine the conditions of negative definiteness of a parameter-dependent matrix $I_b(\nu)$, given the negative definiteness of the matrix $I_a(\nu)$, we use the following obvious sufficient condition: If $I_a(\nu) < 0$ and $I_b(\nu) \leqslant I_a(\nu)$, then $I_b(\nu) < 0$.

To simplify derivations, we return to the notation used in the proof of Theorem 2 and transform the MI (5.10) (see Lemma A4 [13, p. 253]) as follows:

$$\overset{\cong}{I}_1 = \begin{pmatrix} I_1 & u_1^+ & u_3^+ - u_1^+ \\ (\bullet)^\top & -\tau_1 & \delta_1 \\ (\bullet)^\top & \bullet & -\tau_4 \end{pmatrix} < 0 \cong \widehat{I}_1 = \begin{pmatrix} I_1 & u_3^+ - u_1^+ \\ (\bullet)^\top & -\tau_4 \end{pmatrix} + \frac{1}{\tau_1}\begin{pmatrix} u_1^+ \\ \delta_1 \end{pmatrix}\begin{pmatrix} u_1^+ \\ \delta_1 \end{pmatrix}^\top$$

$$= \begin{pmatrix} I_1 & u_3^+ - u_1^+ \\ (\bullet)^\top & -\tau_4 \end{pmatrix} + \frac{1}{\tau_1}\begin{pmatrix} u_1^+(u_1^+)^\top & \delta_1 u_1^+ \\ (\bullet)^\top & \delta_1^2 \end{pmatrix} = \begin{pmatrix} I_1 + \dfrac{1}{\tau_1}u_1^+(u_1^+)^\top & \dfrac{\delta_1 - \tau_1}{\tau_1}u_1^+ + u_3^+ \\ (\bullet)^\top & \dfrac{\delta_1^2}{\tau_1} - \tau_4 \end{pmatrix} < 0,$$

where, for brevity, the notation $\delta_1 \triangleq (\tau_1 - \tau_3 + \tau_4)/2$ is introduced and the arguments of the vectors $u_j^\pm = u_j^\pm(\tau_j)$ are omitted. Adopting yet another simplifying notations $\alpha_1 \triangleq (\delta_1 - \tau_1)/\tau_1$ and $\beta_1 \triangleq \tau_1/(\tau_1\tau_4 - \delta_1^2)$, we arrive at

$$\widehat{I}_1 < 0 \cong \widehat{\widehat{I}}_1 = I_1 + \frac{1}{\tau_1}u_1^+(u_1^+)^\top + \beta_1\left(\alpha_1 u_1^+ + u_3^+\right)\left(\alpha_1 u_1^+ + u_3^+\right)^\top < 0 \tag{A.3}$$

via use of the Schur complement.

For $\tau_1 = 2/\varepsilon_{1cir}^2$ and $\tau_3 = 2/\varepsilon_{2cir}^2$, the difference between the quadratic forms associated with the matrices I_{cir} in (5.9) and $\widehat{\widehat{I}}_1$ in (A.3) is the difference of squares

$$\widehat{\widehat{I}}_1 - I_{cir} \triangleq \Delta_1 = \beta_1\left(\alpha_1 u_1^+ + u_3^+\right)\left(\alpha_1 u_1^+ + u_3^+\right)^\top - \frac{1}{\tau_3}u_3^+(u_3^+)^\top.$$

Inequality $\Delta_1 \leqslant 0$ for the difference of squares is valid if the corresponding (squared) linear forms are proportional; i.e., $\alpha_1 u_1^+ + u_3^+ = \lambda_1 u_3^+$, which is only possible if $\alpha_1 = 0$ or $\tau_4 = \tau_1 + \tau_3$. In that case, $\Delta_1 = 0$.

We next determine the values of τ_5 which guarantee the feasibility of (5.11) provided that the MICC (5.9) is feasible. To this end, we perform the manipulations with the MI (5.11) similar to those performed above with the MI (5.10). As a result, we obtain

$$\overset{\cong}{I}_2 < 0 \cong \widehat{\widehat{I}}_2 = I_3 + \frac{1}{\tau_2}u_2^+(u_2^+)^\top + \beta_2\left(\alpha_2 u_2^+ + u_3^-\right)\left(\alpha_2 u_2^+ + u_3^-\right)^\top < 0, \tag{A.4}$$

where $\delta_2 \triangleq (\tau_2 - \tau_3 + \tau_5)/2$, $\alpha_2 \triangleq (\delta_2 - \tau_2)/\tau_2$, and $\beta_2 \triangleq \tau_2/(\tau_2\tau_5 - \delta_2^2)$. With account for $u_2^+(\tau) = u_1^+(\tau)$ and using the relation $I_1 + \frac{1}{\tau_3}u_3^+(u_3^+)^\top(\tau_3) = I_3 + \frac{1}{\tau_3}u_3^-(u_3^-)^\top(\tau_3)$, we arrive at the conclusion that, for $\tau_2 = 2/\varepsilon_{1cir}^2$ and $\tau_3 = 2/\varepsilon_{2cir}^2$, the difference between the quadratic forms associated with the matrices I_{cir} in (5.9) and $\widehat{\widehat{I}}_2$ in (A.4) is nothing but the difference of squares

$$\widehat{\widehat{I}}_2 - I_{cir} \triangleq \Delta_2 = \beta_2\left(\alpha_2 u_2^+ + u_3^-\right)\left(\alpha_2 u_2^+ + u_3^-\right)^\top - \frac{1}{\tau_3}u_3^-(u_3^-)^\top.$$

The inequality $\Delta_2 \leqslant 0$ for the difference of squares is valid if the corresponding linear forms are proportional; i.e.,

$$\alpha_2 u_2^+ + u_3^- = \lambda_2 u_3^-;$$

this takes place only if $\alpha_2 = 0$ or $\tau_5 = \tau_2 + \tau_3$. In that case $\Delta_2 = 0$. Theorem 7 is proved.

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This paper was recommended for publication by P.V. Pakshin, a member of the Editorial Board