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Control Design for a Perturbed System with an Ambiguous Nonlinearity

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Abstract—The object of this study is an n-dimensional system of ordinary differential equations with an ambiguous relay nonlinearity under a continuous periodic perturbation. We consider continuous periodic solutions of the system with the state-space trajectory consisting of two parts connected at relay switching points. We develop an algorithm for selecting the nonlinearity parameters under which there is a unique asymptotically orbitally stable periodic solution of the system with given oscillation properties, including a given period and two switching points per period.

Keywords: automatic control systems, canonical transformations, control design, relay nonlinearity, forced periodic oscillations, switching points, stable solutions

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1. INTRODUCTION

The theory of relay control systems has been actively developed for several decades [1–25]. Despite the accumulated experience and interesting scientific results [14], there are still open issues related to the existence and properties of solutions even for second-order differential equations [13]. They require a theoretical analysis of periodic and other oscillatory solutions to fully understand the dynamics of systems with relay characteristics. In relay feedback systems, solutions can have quite complex modes [14], e.g., sliding mode and multiple fast or slow switching [5]. For multidimensional systems, it is difficult to apply the methods developed for two-dimensional systems, so their study involves decomposition methods [10, 15]. Discontinuous control systems are widely used in the practice of automatic regulation. Methods of fitting, fixed points, and point mappings are still actively employed to investigate piecewise integrable (in particular, relay) systems [17, 19, 24]. Applications often require regulating the dynamics of an automatic system with relay feedback and translating different types of its motions into periodic oscillations, including control of the period and nature of the resulting oscillatory motions. Hysteresis nonlinearities (including relay ones) often arise in applications [6, 10, 14].

2. PROBLEM STATEMENT

In this paper, the object of control is an n-dimensional system of ordinary differential equations with a discontinuous nonlinearity and an external perturbing force on the right-hand side. The system is described in the Euclidean space by

$$\dot{Y} = AY + Bu(\sigma) + Kf(t), \quad \sigma = (C, Y). \tag{1}$$

Here the system matrix A and the vectors $B = (b_1, \ldots, b_n)^*$, $K = (k_1, \ldots, k_n)^*$ are real and do not depend on the time variable t. The symbol $*$ indicates transpose, and Y denotes the state vector of the system. The nonlinearity is the relay characteristic $u(\sigma)$ of a nonideal two-position relay element, which has a zone of hysteresis (ambiguity) with bypass in the plane (σ, u) counterclockwise with thresholds ℓ_1 and ℓ_2 and outputs m_1 and m_2. Note that ℓ_1, ℓ_2, m_1, and m_2 are real. For the sake of definiteness, we assume that $\ell_1 < \ell_2$ and $m_1 < m_2$. The real constant vector $C = (c_1, \ldots, c_n)^*$ defines the system feedback. The relay characteristic with hysteresis under consideration is widely used in automatic control systems, e.g., in autopilot models. The external perturbation is described by a function $f(t)$ from the class of continuous T-periodic functions.

Due to the complexity of studying relay (especially high-dimensional) feedback systems by analytical methods, approximate techniques with computer implementation are gradually gaining popularity. This paper investigates the nonlinear multidimensional system by analytical methods. Despite the cumbersome formulas, the results theoretically justify different ways of designing particular control systems. We propose a control design approach ensuring the existence of periodic oscillations with given properties in relay feedback systems. This approach, described in detail in [7], consists in the following: under definite relations of system parameters, namely, the elements of the matrix A and vectors B and C, the original system of the nth order is reduced by a nonsingular linear transformation to the canonical system of special form convenient for analytical study and parametric analysis. In the parameter space of system (1), the approach yields domains corresponding to periodic solutions with given properties.

We consider the solution of the system in the class of continuous T_B-periodic functions with two switching points lying on the hyperplanes $\sigma(t) = \ell_\eta$ $(\eta = 1, 2)$ and a period equal to or multiple of the period of the external perturbation. In other words, the solution has the period $T_B = kT$, where T is the period of the function $f(t)$ and k is some natural number. Such hyperplanes will be further referred to as switching hyperplanes. By a switching point we mean the system state in which σ reaches one of the threshold values and the nonlinear characteristic $u(\sigma)$ changes the value of the output. At the switching points, the parts of the trajectory of the representative point of the state-space solution are joined due to the linear systems

$$\dot{Y} = AY + Bm_\alpha + Kf(t), \qquad \alpha = 1, 2. \tag{2}$$

System (1) was previously studied in the author's papers [7–9, 15, 19–21]. In [7], a necessary condition was obtained for the existence of a periodic solution of system (1) with the requisite properties in the case of simple, real, and nonzero eigenvalues with at least one positive eigenvalue. In addition, formulas were derived for the switching points. In [8], a sufficient condition was established for the existence of the desired periodic solutions and the uniqueness of the fixed-period solution was proved in the case of the simple, real, and nonzero eigenvalues of the system matrix. Based on Pokrovskii's results [3], a theorem was formulated on the existence of a unique asymptotically stable solution with a period coinciding with that of the external perturbation. In [9], the model with the Hurwitzian system matrix was considered, the conditions for the existence of a unique periodic solution were obtained, and the solution was analyzed for the reachability of the switching hyperplanes and stability. In [15], the model with the system matrix containing a zero eigenvalue was studied. The papers [19–21] were devoted to examining nonperiodic and periodic oscillating solutions with periods commensurate with that of the external perturbation, $T_B = T/k$. In [19, 20], the existence of special harmonic oscillations was considered for different cases, namely, the system matrices with a positive eigenvalue and a multiple nonzero eigenvalue, respectively. In [20], the system matrix was reduced to the Jordan canonical form. The paper [21] considered the model with the symmetric system matrix containing a multiple nonzero eigenvalue, and its nonperiodic oscillatory solutions were studied. The system matrix was reduced to the diagonal form.

In this paper, we supplement and generalize the analysis results for the periodic solutions of system (1) with periods representing multiples of that of the function $f(t)$ in the case of simple nonzero eigenvalues of the system matrix (i.e., the results obtained in [7–9]). In contrast to [7–9], the conditions for the switching hyperplanes to be reached without touching and the conditions for the asymptotically orbital stability of the solution of the original system below are formulated as theorems generalizing the case of nonzero eigenvalues of the system matrix (Theorems 3 and 4, respectively). The main result of this study is a control design algorithm for system (1) that ensures the existence of a unique asymptotically orbitally stable periodic solution with a given period and two switching points per period.

3. CONSTRUCTING THE SYSTEMS OF TRANSCENDENTAL EQUATIONS

We consider the solution of system (1) in the Cauchy form

$$Y(t) = e^{A(t-t_0)}Y(t_0) + \int\limits_{t_0}^{t} e^{-A(\tau-t)}\left(Bm_\alpha + Kf(\tau)\right)d\tau \quad (\alpha = 1, 2), \tag{3}$$

where t_0 denotes the initial time. The switching points Y^1 and Y^2 of the periodic solution of the system have the following property:

$$Y^\beta = Y(t_0, m_\alpha, t_0) = Y(t_0, m_\alpha, t_0 + T_B), \quad (C, Y^\beta) = \ell_\eta, \ \ \forall \alpha, \beta, \eta = 1, 2.$$

Let the representative point of the periodic solution of system (1) start moving at the point Y^1 on the hyperplane $\sigma = \ell_1$ at the time $t_0 = 0$ and reach the hyperplane $\sigma = \ell_2$ at the point Y^2 at the time $t = t_1$ (the first time to encounter $\sigma = \ell_2$, see Definition 1 in [19]) by virtue of system (2) under the condition $m_\alpha = m_1$. Then it returns to the hyperplane $\sigma = \ell_1$ at the point Y^1 at the time $t = T_B$ (the first time to encounter $\sigma = \ell_1$) by virtue of system (2) under the condition $m_\alpha = m_2$. Thus, by the motion sequence prescribed for the representative point of the solution of system (1), we have $Y(0) = Y(T_B) = Y^1$ and $Y(t_1) = Y^2$.

The external T-periodic perturbation is modeled by

$$f(t) = f_0 + f_1 \sin(\omega t + \varphi_1) + f_2 \sin(2\omega t + \varphi_2), \tag{4}$$

where f_0, f_1, f_2, φ_1, φ_2, and ω are real constants, $T = 2\pi/\omega$, and $\omega > 0$.

Suppose that system (1) has at least one periodic solution with the period T_B and the representative point of this solution moves along the trajectory in the prescribed sequence. We write the general system of transcendental equations with respect to the two switching times (the second time coincides with the period of the desired solution) and the corresponding switching points in the state space [7]:

$$\ell_1 = \left(C, Y^1\right), \quad \ell_2 = \left(C, Y^2\right), \tag{5}$$

where

$$Y^2 = e^{At_1}Y^1 + \int\limits_0^{t_1} e^{A(t_1-\tau)}(Bm_1 + Kf(\tau))d\tau,$$

$$Y^1 = e^{A(T_B-t_1)}Y^2 + \int\limits_{t_1}^{T_B} e^{A(T_B-\tau)}(Bm_2 + Kf(\tau))d\tau.$$

Note that the resulting system of four equations can be resolved in t_1, T_B, Y^1, and Y^2 using numerical methods, but the objective of this study is to address analytical methods. Therefore, to solve system (5) analytically, we reduce the original system to the canonical form.

Reversibility conditions. Treating the elements of the matrix A and vector B as the parameters of system (1), we impose the following conditions: 1) The matrix A has only simple eigenvalues. 2) The vectors $B, AB, A^2B, \ldots, A^{n-1}B$ are linearly independent.

These conditions guarantee the reversibility of the canonical transformation of the original system for analytical study. The original and canonical systems are interchangeable. Only real eigenvalues will be considered for the sake of simplicity.

We write the system of transcendental equations for different cases with nonzero eigenvalues of the system matrix. Let the matrix A have simple, real, and nonzero eigenvalues λ_i $(i = \overline{1,n})$. The general approach to transforming the original system and reducing the system of transcendental equations (5) to a simplified form with the separated switching times and switching points was considered in [7]. Here is its brief description. System (1) is reduced by a nonsingular transformation $Y = SX$ to the canonical form

$$\dot{X} = A_0X + B_0u(\sigma) + K_0f(t), \quad \sigma = (\Gamma, X), \tag{6}$$

where

$$A_0 = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \quad B_0 = \begin{pmatrix} 1 \\ \ldots \\ 1 \end{pmatrix}, \quad K_0 = \begin{pmatrix} k_0^1 \\ \ldots \\ k_0^n \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \gamma_1 \\ \ldots \\ \gamma_n \end{pmatrix}.$$

The elements γ_i $(i = \overline{1,n})$ are given by

$$\gamma_i = \frac{-1}{D'(\lambda_i)} \sum_{h=1}^{n} c_h N_h(\lambda_i), \tag{7}$$

where

$$D'(\lambda_i) = \left.\frac{dD(p)}{dp}\right|_{p=\lambda_i}, \quad D(p) = |A - pE|, \quad N_h(p) = \sum_{i=1}^{n} b_i D_{ih}(p).$$

These formulas have the following notations: $\lambda_i, i = \overline{1,n}$, are the roots of the characteristic equation $D(p) = 0$; E is an identity matrix of appropriate dimensions; $D_{ih}(p)$ is the algebraic complement of the element a_{ih} of the determinant $D(p)$ standing at the intersection of row i and column h; $b_i, i = \overline{1,n}$, are the elements of the vector B; c_h, $h = \overline{1,n}$, are the elements of the feedback vector C; finally, p is some real parameter.

The transformation matrix S is

$$S = -\begin{pmatrix} \dfrac{N_1(\lambda_1)}{D'(\lambda_1)} & \cdots & \dfrac{N_1(\lambda_n)}{D'(\lambda_n)} \\ \vdots & \ddots & \vdots \\ \dfrac{N_n(\lambda_1)}{D'(\lambda_1)} & \cdots & \dfrac{N_n(\lambda_n)}{D'(\lambda_n)} \end{pmatrix}. \tag{8}$$

We consider the elements of the feedback vector as the additional parameters of the original (hence, canonical) system. Let the parameters of the vector Γ be chosen so that all elements except one equal 0. The subscript at the nonzero element of the vector Γ will be denoted by s. In other words, $\gamma_s \neq 0$ and $\gamma_j = 0$, where $j = 1, \ldots, s-1, s+1, \ldots, n$. According to (7), we choose the parameters c_h $(h = \overline{1,n})$ from the system

$$\sum_{h=1}^{n} c_h N_h(\lambda_j) = 0, \qquad \sum_{h=1}^{n} c_h N_h(\lambda_s) \neq 0. \tag{9}$$

The assumption on the feedback vector parameters allows splitting the n-order system into first-order systems and simplifying the system of transcendental equations (5). The resulting first-order systems can be sequentially integrated and studied by analytical methods.

The function $\sigma(t)$ is determined from the system of differential equations

$$\sigma(t) = \gamma_s x_s, \quad \dot{x}_s = \lambda_s x_s + u(\sigma) + k_s^0 f(t); \tag{10}$$

the other variables x_j $(j \neq s)$ are determined from the first-order heterogeneous linear equations

$$\dot{x}_j = \lambda_j x_j + u(\sigma) + k_j^0 f(t), \quad j = 1, \ldots s-1, s+1, \ldots, n. \tag{11}$$

Note that under the above choice of the feedback parameter vector (all parameters are 0 except the one with subscript s), both switching hyperplanes in the coordinates x_i $(i = \overline{1,n})$ are oriented orthogonally to the axis x_s. Using (10), we can find the transition time of the representative point of the solution from one switching hyperplane to the other as a constant value independent of the initial position of this point in the original switching hyperplane. Furthermore, substituting the transition time into the solutions of equations (11) makes the time-dependent expressions constant. Thus, the solution of the systems of equations (10), (11) defines a point mapping of one switching hyperplane to the other.

Solving the system of equations (10) with respect to the function $\sigma(t)$ with the initial and boundary conditions $\ell_1 = \sigma(\ell_1, 0, m_1, 0)$, $\ell_2 = \sigma(\ell_1, 0, m_1, t_1)$, and $\ell_1 = \sigma(\ell_2, t_1, m_2, T_B)$, we obtain a system of transcendental equations with respect to the switching times t_1 and T_B and formulas for finding the switching points X^1 and X^2. Under the condition $\lambda_s > 0$ (the case considered in [7]), the system of transcendental equations takes the form

$$\begin{gathered}
\ell_2 = \Bigg(\ell_1 + \frac{\gamma_s}{\lambda_s}(m_1 + k_s^0 f_0) + \frac{\gamma_s k_s^0 f_1}{\sqrt{\lambda_s^2 + \omega^2}} \sin(\varphi_1 + \delta_1) \\
+ \frac{\gamma_s k_s^0 f_2}{\sqrt{\lambda_s^2 + 4\omega^2}} \sin(\varphi_2 + \delta_2)\Bigg) e^{\lambda_s t_1} - \frac{\gamma_s}{\lambda_s}(m_1 + k_s^0 f_0) \\
- \frac{\gamma_s k_s^0 f_1}{\sqrt{\lambda_s^2 + \omega^2}} \sin(\omega t_1 + \varphi_1 + \delta_1) - \frac{\gamma_s k_s^0 f_2}{\sqrt{\lambda_s^2 + 4\omega^2}} \sin(2\omega t_1 + \varphi_2 + \delta_2),
\end{gathered}$$

$$\begin{gathered}
\ell_1 = \Bigg(\ell_2 + \frac{\gamma_s}{\lambda_s}(m_2 + k_s^0 f_0) + \frac{\gamma_s k_s^0 f_1}{\sqrt{\lambda_s^2 + \omega^2}} \sin(\omega t_1 + \varphi_1 + \delta_1) \\
+ \frac{\gamma_s k_s^0 f_2}{\sqrt{\lambda_s^2 + 4\omega^2}} \sin(2\omega t_1 + \varphi_2 + \delta_2)\Bigg) e^{\lambda_s (T_B - t_1)} - \frac{\gamma_s}{\lambda_s}(m_2 + k_s^0 f_0) \\
- \frac{\gamma_s k_s^0 f_1}{\sqrt{\lambda_s^2 + \omega^2}} \sin(\omega T_B + \varphi_1 + \delta_1) - \frac{\gamma_s k_s^0 f_2}{\sqrt{\lambda_s^2 + 4\omega^2}} \sin(2\omega T_B + \varphi_2 + \delta_2).
\end{gathered} \tag{12}$$

Under the condition $\lambda_s < 0$ (the case considered in [9]), it becomes

$$\ell_2 = \Bigg(\ell_1 + \frac{\gamma_s}{\lambda_s}(m_1 + k_s^0 f_0) - \frac{\gamma_s k_s^0 f_1}{\sqrt{\lambda_s^2+\omega^2}}\sin(\varphi_1+\delta_1) - \frac{\gamma_s k_s^0 f_2}{\sqrt{\lambda_s^2+4\omega^2}}\sin(\varphi_2+\delta_2)\Bigg)e^{\lambda_s t_1} - \frac{\gamma_s}{\lambda_s}(m_1 + k_s^0 f_0) + \frac{\gamma_s k_s^0 f_1}{\sqrt{\lambda_s^2+\omega^2}}\sin(\omega t_1+\varphi_1+\delta_1) + \frac{\gamma_s k_s^0 f_2}{\sqrt{\lambda_s^2+4\omega^2}}\sin(2\omega t_1+\varphi_2+\delta_2),$$

$$\ell_1 = \Bigg(\ell_2 + \frac{\gamma_s}{\lambda_s}(m_2 + k_s^0 f_0) - \frac{\gamma_s k_s^0 f_1\sin(\omega t_1+\varphi_1+\delta_1)}{\sqrt{\lambda_s^2+\omega^2}} - \frac{\gamma_s k_s^0 f_2\sin(2\omega t_1+\varphi_2+\delta_2)}{\sqrt{\lambda_s^2+4\omega^2}}\Bigg)e^{\lambda_s(T_B-t_1)} - \frac{\gamma_s}{\lambda_s}(m_2 + k_s^0 f_0) + \frac{\gamma_s k_s^0 f_1}{\sqrt{\lambda_s^2+\omega^2}}\sin(\omega T_B+\varphi_1+\delta_1) + \frac{\gamma_s k_s^0 f_2}{\sqrt{\lambda_s^2+4\omega^2}}\sin(2\omega T_B+\varphi_2+\delta_2). \tag{13}$$

Hereinafter, $\delta_1 = \arctan(\omega/\lambda_s)$ and $\delta_2 = \arctan(2\omega/\lambda_s)$.

The switching points $X^1 = (x_1^1, \ldots, x_n^1)^*$ and $X^2 = (x_1^2, \ldots, x_n^2)^*$ of the transformed system (6) belong to the switching hyperplanes $\sigma = \ell_\eta$ $(\eta = 1, 2)$. They are given by the following formulas: $x_s^1 = \ell_1/\gamma_s$, $x_s^2 = \ell_2/\gamma_s$,

$$\begin{gathered} x_j^1 = \left(1 - e^{\lambda_j T_B}\right)^{-1}\Bigg(e^{\lambda_j T_B}\Bigg[m_1\int_0^{t_1} e^{-\lambda_j\tau}d\tau + m_2\int_{t_1}^{T_B} e^{-\lambda_j\tau}d\tau + k_j^0 f_0\int_0^{T_B} e^{-\lambda_j\tau}d\tau \\ + k_j^0 f_1\int_0^{T_B} e^{-\lambda_j\tau}\sin(\omega\tau+\varphi_1)d\tau + k_j^0 f_2\int_0^{T_B} e^{-\lambda_j\tau}\sin(2\omega\tau+\varphi_2)d\tau\Bigg]\Bigg), \\ x_j^2 = \left(1 - e^{\lambda_j T_B}\right)^{-1} e^{\lambda_j t_1}\Bigg(\int_{t_1}^{T_B} e^{-\lambda_j(T_B-\tau)}\Big[m_2 + k_j^0\left(f_0 + f_1\sin(\omega t+\varphi_1) + f_2\sin(2\omega t+\varphi_2)\right)\Big]d\tau \\ + \int_0^{t_1} e^{-\lambda_j\tau}\Big[m_1 + k_j^0\left(f_0 + f_1\sin(\omega t+\varphi_1) + f_2\sin(2\omega t+\varphi_2)\right)\Big]d\tau\Bigg), \\ j = 1, \ldots, s-1, s+1, \ldots, n. \end{gathered} \tag{14}$$

4. CONTROL DESIGN. SOLVABILITY CONDITIONS FOR TRANSCENDENTAL EQUATIONS

We consider the choice of the parameters ℓ_1, ℓ_2, m_1, m_2, and c_h $(h = \overline{1,n})$ of the nonlinear characteristic $u(\sigma)$ for the existence of stable periodic solutions of system (1), (4) with given oscillatory properties provided that the parameters of the matrix A and vector B ensure the reversible canonical transformation and all other coefficients of system (1), (4) are fixed. First, we design a control law for the system with the periodic perturbation so that the forced oscillations of system (1), (4) have a period T_B equal to or multiple of the period T of the function $f(t)$, i.e., $T_B = kT$, where $k \in \mathbb{N}$.

Then we study two subcases depending on the sign of the eigenvalue corresponding to the nonzero element of the feedback vector: its sign determines the kind of the system of transcendental equations to be investigated and solved by analytical methods. If two elements of the feedback vector are supposed nonzero, it is necessary to consider all combinations of the signs of the two corresponding eigenvalues, and the number of subcases will grow to four.

Case 1 (positive eigenvalue). We address the conditions ensuring the existence of a unique solution $t_1 \in (0, kT)$ of the system of equations (12) for a given natural number k.

Theorem 1 [7]. *Let the function $f(t)$ be given by* (4), *and let system* (1) *have a periodic solution with a period $T_B = kT$, where $k \in \mathbb{N}$, $T = 2\pi/\omega$, and $\omega > 0$. Assume that the eigenvalues of the matrix A are simple, real, and nonzero, with at least one of them being positive ($\lambda_s > 0$), and the element γ_s of the transformed feedback vector Γ is nonzero. Finally, let the following conditions hold:*

1) *the inequalities*

$$m_2 - m_1 e^{\lambda_s kT} + \lambda_s \left(1 - e^{\lambda_s kT}\right)\left(\ell_1/\gamma_s + k_s^0 L\right) > 0,$$
$$m_1 < -\lambda_s \left(\frac{\ell_1}{\gamma_s} + k_s^0 L\right) < m_2, \tag{15}$$

where

$$L = \frac{f_0}{\lambda_s} + \frac{f_1 \sin(\varphi_1 + \delta_1)}{\sqrt{\lambda_s^2 + \omega^2}} + \frac{f_2 \sin(\varphi_2 + \delta_2)}{\sqrt{\lambda_s^2 + 4\omega^2}}; \tag{16}$$
$$\delta_1 = \arctan(\omega/\lambda_s), \quad \delta_2 = \arctan(2\omega/\lambda_s);$$

2) *the inequality*

$$\begin{aligned}
&\left(\ell_1 + \frac{\gamma_s}{\lambda_s}(m_1 + k_s^0 f_0)\right)(e^{\lambda_s kT} H - 1) \\
&+\frac{\gamma_s k_s^0 f_1}{\sqrt{\lambda_s^2 + \omega^2}}\left(\sin(\varphi_1 + \delta_1) e^{\lambda_s kT} H - \sin\left(\frac{\omega}{\lambda_s}\ln H + \varphi_1 + \delta_1\right)\right) \\
&+\frac{\gamma_s k_s^0 f_2}{\sqrt{\lambda_s^2 + 4\omega^2}}\left(\sin(\varphi_2 + \delta_2) e^{\lambda_s kT} H - \sin\left(\frac{2\omega}{\lambda_s}\ln H + \varphi_2 + \delta_2\right)\right) > 0,
\end{aligned} \tag{17}$$

where

$$H = \frac{m_2 - m_1}{\lambda_s(1 - e^{\lambda_s kT})(\ell_1/\gamma_s + k_s^0 L) + m_2 - m_1 e^{\lambda_s kT}}; \tag{18}$$

3) *the equality*

$$\begin{aligned}
\ell_2 = {} & \ell_1 e^{\lambda_s kT} H + \frac{\gamma_s}{\lambda_s}\left(m_1 + k_s^0 f_0\right)\left(e^{\lambda_s kT} H - 1\right) \\
&+ \frac{\gamma_s k_s^0 f_1}{\sqrt{\lambda_s^2 + \omega^2}}\left(\sin(\varphi_1 + \delta_1) e^{\lambda_s kT} H - \sin\left(\frac{\omega}{\lambda_s}\ln H + \varphi_1 + \delta_1\right)\right) \\
&+ \frac{\gamma_s k_s^0 f_2}{\sqrt{\lambda_s^2 + 4\omega^2}}\left(\sin(\varphi_2 + \delta_2) e^{\lambda_s kT} H - \sin\left(\frac{2\omega}{\lambda_s}\ln H + \varphi_2 + \delta_2\right)\right).
\end{aligned} \tag{19}$$

Then system (12) *with the natural number k has a unique solution $t_1 \in (0, kT)$ given by*

$$t_1 = kT + \frac{1}{\lambda_s}\ln H. \tag{20}$$

Case 2 (negative eigenvalues). In this case, we apply the solvability conditions of system (13) from the paper [9].

Theorem 2 [9]. *Let the function $f(t)$ be given by* (4), *and let system* (1) *have a periodic solution with a period $T_B = kT$, where $k \in \mathbb{N}$, $T = 2\pi/\omega$, and $\omega > 0$. Assume that the eigenvalues of the matrix A are simple, real, and negative and one element of the transformed feedback vector Γ is nonzero* (*e.g.*, $\gamma_s \neq 0$). *Finally, let the following conditions hold:*

1) *the system of inequalities* (15), *where*

$$L = \frac{f_0}{\lambda_s} - \frac{f_1 \sin(\varphi_1 + \delta_1)}{\sqrt{\lambda_s^2 + \omega^2}} - \frac{f_2 \sin(\varphi_2 + \delta_2)}{\sqrt{\lambda_s^2 + 4\omega^2}}; \tag{21}$$

2) *the inequality*

$$\begin{aligned} &\left(\ell_1 + \frac{\gamma_s}{\lambda_s}(m_1 + k_s^0 f_0)\right)(e^{\lambda_s kT} H - 1) \\ &- \frac{\gamma_s k_s^0 f_1}{\sqrt{\lambda_s^2 + \omega^2}}\left(\sin(\varphi_1 + \delta_1) e^{\lambda_s kT} H - \sin\left(\frac{\omega}{\lambda_s} \ln H + \varphi_1 + \delta_1\right)\right) \\ &- \frac{\gamma_s k_s^0 f_2}{\sqrt{\lambda_s^2 + 4\omega^2}}\left(\sin(\varphi_2 + \delta_2) e^{\lambda_s kT} H - \sin\left(\frac{2\omega}{\lambda_s} \ln H + \varphi_2 + \delta_2\right)\right) > 0, \end{aligned} \tag{22}$$

where H is given by (18);

3) *the equality*

$$\begin{aligned} \ell_2 = \ell_1 e^{\lambda_s kT} H &+ \frac{\gamma_s}{\lambda_s}\left(m_1 + k_s^0 f_0\right)\left(e^{\lambda_s kT} H - 1\right) \\ &- \frac{\gamma_s k_s^0 f_1}{\sqrt{\lambda_s^2 + \omega^2}}\left(\sin(\varphi_1 + \delta_1) e^{\lambda_s kT} H - \sin\left(\frac{\omega}{\lambda_s} \ln H + \varphi_1 + \delta_1\right)\right) \\ &- \frac{\gamma_s k_s^0 f_2}{\sqrt{\lambda_s^2 + 4\omega^2}}\left(\sin(\varphi_2 + \delta_2) e^{\lambda_s kT} H - \sin\left(\frac{2\omega}{\lambda_s} \ln H + \varphi_2 + \delta_2\right)\right). \end{aligned} \tag{23}$$

Then system (13) *with the natural number k has a unique solution $t_1 \in (0, kT)$ given by formula* (20).

Let t_1 be the solution to the system of transcendental equations whose parameters satisfy the conditions above for a given number $k \in \mathbb{N}$. We consider the theorem on the existence of a unique periodic solution of the class under consideration in the case of nonzero eigenvalues of the system matrix, which was formulated and proved in [8]. Let us add the following condition to this theorem: t_1 is the smallest solution of the first equation of the transcendental system and kT is the smallest solution of the second equation of the system under a fixed time t_1 since t_1 and kT are the switching times (the first times to encounter the hyperplanes).

5. REACHABILITY CONDITIONS FOR SWITCHING HYPERPLANES

The quality of transients is related to two tasks: 1) maintaining the required mode, particularly with a given period; 2) preventing the sliding mode typical for relay systems.

Assume that the representative point of the desired solution of system (1) moves in the prescribed sequence. We study the relations between the system parameters under which the canonical transformation remains nonsingular, i.e., the reversibility conditions hold. Due to the nonsingular transformation, the analysis results of the canonical system will apply to the original system as well. We consider conditions under which the representative point of the solution of the canonical system will reach the switching hyperplanes (without touching to avoid the sliding mode).

Theorem 3. *Assume that either the conditions of Theorem 1 are satisfied with the negative eigenvalues λ_j $(j \neq s)$ or the conditions of Theorem 2 hold. Let system* (1) *be reduced to the canonical form* (6) *by a nonsingular transformation. In addition, suppose that:*

1) $n-1$ *elements* γ_j $(j \neq s)$ *of the vector* Γ *are zero;*

2) *for* $\gamma_s \lambda_s > 0$*,* $m_1 < m_2$*, and* $\ell_1 < \ell_2$*, we have the inequalities* $-\gamma_s m_2/\lambda_s < \ell_1$*,* $-\gamma_s m_1/\lambda_s > \ell_2$*;*

3) $\lambda_s \ell_1 + \gamma_s m_2 + \gamma_s k_s^0 f(t_\beta) \neq 0$ *and* $\lambda_s \ell_2 + \gamma_s m_1 + \gamma_s k_s^0 f(t_\beta) \neq 0$ $(\beta = 1, 2)$*, where* t_1 *is the first switching time and* $t_2 = 0$*;*

4) *the set* Q *is described by the system of inequalities*

$$\begin{cases} \|\bar{X}\| \leq \dfrac{1}{\min\limits_j |\lambda_j|} \left[\max\limits_{\alpha=1,\,2} |m_\alpha| \cdot \|\bar{B}_0\| + M \cdot \|\bar{K}_0\| \right], \; j = \overline{1,n}, j \neq s, \\ \ell_1 \leq x_s \gamma_s \leq \ell_2, \end{cases}$$

where the vectors $\bar{X}$*,* $\bar{B}_0$*, and* $\bar{K}_0$ *of dimension* $(n-1)$ *differ from their counterparts* X*,* B_0*, and* K_0*, respectively, in the canonical system by the excluded* s*th element; in addition, the constant* M *is determined from the inequality* $|f(t)| \leq |f_0| + |f_1| + |f_2| = M$*, valid for any* t*, in which* f_0*,* f_1*, and* f_2 *are the constant coefficients of the function* $f(t)$*.*

Then, starting its state-space motion at $X_0 \in Q$ *on one of the hyperplanes* $\sigma(t) = \ell_\eta$ $(\eta = 1, 2)$*, the representative point of the periodic solution of the canonical system reaches the other hyperplane without touching along the trajectory of system* (6)*.*

The proof of Theorem 3 is given in the Appendix.

In real systems, the initial conditions are set with some precision. Therefore, the natural question arises: How will small changes in the initial conditions affect the solution behavior at $t \to \infty$?

Theorem 4. *Assume that the canonical system* (6) *with* $\lambda_s \neq 0$ *has the* kT*-periodic solution of the class under consideration and the conditions of Theorem 3 are satisfied. Then the solution of system* (6) *is asymptotically stable and, due to the nonsingular transformation, the solution of the original system* (1) *is asymptotically orbitally stable.*

Proof of Theorem 4. The stability of the periodic solution was analyzed in the paper [9] after formulating Theorem 3.3. As established therein, there exists an asymptotically stable kT-periodic solution of system (6) and the solution of system (1) is asymptotically orbitally stable for $\lambda_j < 0$, $j \neq s$, regardless of the sign of λ_s. The conditions of the corollary of Theorem 3.3 are included in the statement of Theorem 3 above. Thus, Theorem 4 generalizes the results obtained in [9].

6. THE CONTROL DESIGN ALGORITHM

In relay systems with an external biharmonic perturbation of a period T, there may exist forced oscillations with a period multiple of it, i.e., kT, where $k \in \mathbb{N}$ and $k > 1$. Such oscillations are realized only in nonlinear systems and are called subharmonic of order k. Along with subharmonic oscillations, systems of this class may also contain forced harmonic oscillations with a period T. This phenomenon is known as the capture of the main frequency and its fraction [1].

In this paper, we develop an algorithm to localize the domains corresponding to kT-periodic solutions with given properties in the parameter space of system (1) and identify the domains not matching the desired solutions. Note that the natural number k is specified depending on which forced oscillation (harmonic or subharmonic) to study. The algorithm chooses the values of the parameters ℓ_1, ℓ_2, m_1, m_2, and c_h $(h = \overline{1,n})$ of the relay characteristic $u(\sigma)$ under which there is an asymptotically orbitally stable kT-periodic solution of system (1) with two switching points per period on the hyperplanes $\sigma = \ell_\eta$ $(\eta = 1, 2)$. We emphasize that all other parameters of system (1) are fixed and satisfy the reversibility conditions of system (1).

The algorithm includes the following steps.

1. Calculate the period of the function $f(t)$ of the form (4) by the formula $T = 2\pi/\omega$. Specify any $k \in \mathbb{N}$ to choose the period $T_B = kT$ of the desired solution.

2. Construct the characteristic equation $D(p) = |A - pE| = 0$.

3. Find the eigenvalues λ_i $\left(i = \overline{1, n}\right)$ of the matrix A. According to the assumptions, they are simple, nonzero, and real. The algorithm is applicable in two cases: 1) one eigenvalue is positive, whereas the rest are negative; 2) all eigenvalues are negative.

3.1. If there is a positive number among λ_i, we denote it by λ_s. If all λ_i are negative, then let $\lambda_s = \max\limits_i \lambda_i$. Now, $\lambda_s > 0$ or $\lambda_s < 0$, and the rest $(n-1)$ eigenvalues λ_j are negative.

3.1.1. Construct the nonsingular transformation matrix S by formula (8).

3.1.2. Find the inverse matrix S^{-1}.

3.1.3. Calculate the vector K_0 by formula $K_0 = S^{-1}K$, where K is the real vector at the function $f(t)$ in system (1).

3.1.4. Let $\gamma_s = \operatorname{sgn}(\lambda_s)$; in this case, the reachability conditions for the switching hyperplanes hold.

3.1.5. Let $\sum\limits_{h=1}^{n} c_h N_h(\lambda_s) = -D'(\lambda_s)\gamma_s$ based on (7) and (9). Then system (9) becomes

$$\sum_{h=1}^{n} c_h N_h(\lambda_j) = 0, \qquad \sum_{h=1}^{n} c_h N_h(\lambda_s) = -D'(\lambda_s)\gamma_s.$$

Solve the resulting heterogeneous system of n linear algebraic equations and find the solution c_h $\left(h = \overline{1, n}\right)$. Heterogeneity follows from $D'(\lambda_s)\gamma_s \neq 0$.

3.1.6. Calculate the values $\delta_1 = \arctan(\omega/\lambda_s)$ and $\delta_2 = \arctan(2\omega/\lambda_s)$.

3.1.7. Let $\ell_1 = 10\operatorname{sgn}(\lambda_s)$.

3.1.8. If the eigenvalue $\lambda_s > 0$, proceed to Step 3.1.9; if $\lambda_s < 0$, go to Step 3.2.9.

3.1.9. Calculate the value L by formula (16).

3.1.10. Denote by P the expression in the second inequality of system (15), i.e., $P = -\lambda_s\left(\ell_1/\gamma_s + k_s^0 L\right)$, and calculate P. Next, specify any real number m_1 such that $m_1 < P$ according to (15). Then specify any real number m_2 such that $m_2 > P$ according to (15) and $m_2 > -\lambda_s\ell_1/\gamma_s$ according to condition 2) of Theorem 3. Verify condition 3) of Theorem 3 for $t_\beta = 0$, i.e., $\lambda_s\ell_1 + \gamma_s m_2 + \gamma_s k_s^0 f(0) \neq 0$. If this inequality holds, proceed to Step 3.2.11; otherwise, the conditions for the switching hyperplanes to be reached without touching are violated, go to Step 8.

3.1.11. Calculate the value H by formula (18).

3.1.12. Verify condition (17): if it holds, proceed to Step 3.1.13; otherwise, one condition of Theorem 1 is violated, go to Step 8.

3.1.13. Find the parameter ℓ_2 by formula (19). If it satisfies conditions 2) and 3) of Theorem 3 for $t_\beta = 0$, i.e., $m_1 < -\lambda_s\ell_2/\gamma_s$ and $\lambda_s\ell_2 + \gamma_s m_1 + \gamma_s k_s^0 f(0) \neq 0$, proceed to Step 3.1.14; otherwise, go to Step 8.

3.1.14. Calculate the value t_1 by formula (20). If t_1 and kT are the smallest solutions of the first and second equations of the transcendental system, respectively, and condition 3) of Theorem 3 holds for $t_\beta = t_1$, go to Step 4; otherwise, go to Step 8.

3.2.9. Calculate the value L by formula (21).

3.2.10. Denote by P the expression in the second inequality of system (15) and calculate P. Next, specify any real number m_1 such that $m_1 < P$ according to (15). Then specify any real number m_2 such that $m_2 > P$ according to (15) and $m_2 > -\lambda_s\ell_1/\gamma_s$ according to condition 2) of

Theorem 3. Verify condition 3) of Theorem 3 for $t_\beta = 0$, i.e., $\lambda_s \ell_1 + \gamma_s m_2 + \gamma_s k_s^0 f(0) \neq 0$. If this inequality holds, proceed to Step 3.2.11; otherwise, go to Step 8.

3.2.11. Calculate the value H by formula (18).

3.2.12. Verify condition (22): if it holds, proceed to Step 3.2.13; otherwise, one condition of Theorem 2 is violated, go to Step 8.

3.2.13. Calculate the value ℓ_2 by formula (23). If it satisfies conditions 2) and 3) of Theorem 3 for $t_\beta = 0$, proceed to Step 3.2.14; otherwise, go to Step 8.

3.2.14. Calculate the value t_1 by formula (20). If condition 3) of Theorem 3 holds for $t_\beta = t_1$, proceed to Step 4; otherwise, pass to Step 8.

4. Find the coordinates of the switching points X^1 and X^2 of the canonical system (6): $x_s^1 = \ell_1/\gamma_s$, $x_s^2 = \ell_2/\gamma_s$, and the rest by formulas (14). These points belong to the trajectory of an asymptotically stable kT-periodic solution of system (6).

5. Calculate the switching points Y^1 and Y^2 by the inverse transformation $Y^\beta = SX^\beta$ $(\beta = 1, 2)$.

6. Construct the trajectory of the desired asymptotically orbitally stable periodic solution of system (1) with the initial value Y^1 by formula (3) or the trajectory of the asymptotically stable periodic solution of system (6) with the initial value X^1 by the formula $X(t) = S^{-1}Y(t)$.

7. Complete the algorithm. The resulting parameters c_h $(h = \overline{1, n})$, ℓ_1, ℓ_2, m_1, and m_2 satisfy the conditions of Theorem 1 in [8] as well as those of Theorems 3 and 4 in this paper. Under these parameters, the system has a unique asymptotically orbitally stable kT-periodic solution with the switching points Y^1 and Y^2 per period, and the representative point of the solution reaches the hyperplanes $\sigma(t) = \ell_\eta$ $(\eta = 1, 2)$ at the switching points without touching. Thus, the solution trajectory is constructed.

8. Complete the algorithm. If at least one condition of Theorem 1 or Theorem 2 is violated, the necessary conditions for the existence of a kT-periodic solution with two switching points per period belonging to the switching hyperplanes $\sigma(t) = \ell_\eta$ $(\eta = 1, 2)$ will also fail. If conditions 2) and 3) of Theorem 3 are violated, the conditions for the switching hyperplanes to be reached without touching will also fail. (In other words, the solution can pass to an equilibrium (if any) or the sliding mode.) Hence, the selected parameter values c_h $(h = \overline{1, n})$, ℓ_1, m_1, and m_2 of the relay characteristic $u(\sigma)$ do not ensure that system (1) has an orbitally stable kT-periodic solution with two switching points per period lying on the switching hyperplanes. In this case, other parameter values should be chosen.

7. AN EXAMPLE OF ALGORITHM IMPLEMENTATION

Let $n = 3$, $A = \begin{pmatrix} -12 & -11 & -37 \\ 4.6 & 3.6 & 15.4 \\ 1.8 & 1.8 & 5.2 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and $K = \begin{pmatrix} 12.5 \\ -4.5 \\ -2.5 \end{pmatrix}$. The vectors B, AB, and A^2B are linearly independent since

$$\begin{vmatrix} 1 & -12 & 26.8 \\ 0 & 4.6 & -10.92 \\ 0 & 1.8 & -3.96 \end{vmatrix} = 1.44 \neq 0.$$

Assume that the external perturbation is described by the T-periodic function

$$f(t) = 1 + 2\sin(t + \pi/3) + 5\sin(2t).$$

1. We determine the period $T = 2\pi$ and specify $k = 2$.

2. We write the characteristic equation

$$D(p) = \begin{vmatrix} -12-p & -11 & -37 \\ 4.6 & 3.6-p & 15.4 \\ 1.8 & 1.8 & 5.2-p \end{vmatrix} = 0.$$

3. We find the roots of the characteristic equation, i.e., the eigenvalues of the matrix A: $\lambda_1 = -0.2$, $\lambda_2 = -1$, and $\lambda_3 = -2$.

3.1. All eigenvalues are negative. We choose $\lambda_s = \lambda_1$ since λ_1 is the largest eigenvalue.

3.1.1. We construct the nonsingular transformation matrix $S = \begin{pmatrix} -5 & -1 & 7 \\ 2 & 1 & -3 \\ 1 & 0 & -1 \end{pmatrix}$.

3.1.2. We find the inverse matrix $S^{-1} = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$.

3.1.3. We find the vector $K_0 = \begin{pmatrix} -2 \\ 1 \\ 0.5 \end{pmatrix}$.

3.1.4. We specify $\gamma_s = -1$; in this case, the condition $\gamma_s \lambda_s > 0$ holds.

3.1.5. We calculate $D'(\lambda_s)\gamma_s = -1.44$. The inhomogeneous system of linear algebraic equations yields the following values of the feedback vector parameters: $c_1 = -1$, $c_2 = -1$, and $c_3 = -4$.

3.1.6. We determine $\delta_1 \approx -1.373401$ and $\delta_2 \approx -1.471128$ (here and further the calculations are performed with an accuracy of 10^{-6}).

3.1.7. We specify $\ell_1 = -10$.

3.1.8. Because $\lambda_s < 0$, we proceed to Step 3.2.9.

3.2.9. We determine $L \approx -1.896301$.

3.2.10. We calculate the expression in the second inequality of system (15), $P \approx 2.7585203$. Next, we specify any real number m_1 such that $m_1 < P$, choosing $m_1 = -9$. Then we calculate $-\lambda_s \ell_1/\gamma_s = 2$ and specify any real number m_2 such that $m_2 > P$ and $m_2 > 2$, choosing $m_2 = 7.54$. We verify condition 3) of Theorem 3 for $t_\beta = 0$, namely, $\lambda_s \ell_1 + \gamma_s m_2 + \gamma_s k_s^0 f(0) \approx -0.07478 \neq 0$. Hence, we proceed to Step 3.2.11.

3.2.11. We determine $H \approx 2.884940$.

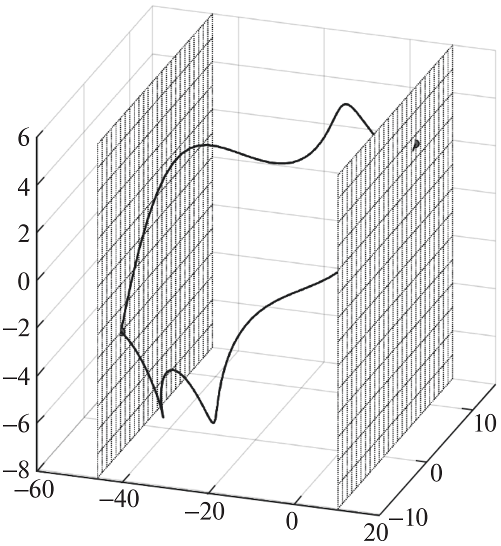
3.2.12. The parameters ℓ_1, m_1, and m_2 satisfy inequality (22): $43.570427 > 0$. Therefore, we proceed to Step 3.2.13.

3.2.13. We find $\ell_2 \approx 46.050182$. Next, we verify the condition for reachability without touching for $t_\beta = 0$: $5.254065 \neq 0$ is true. As a result, we proceed to Step 3.2.14.

3.2.14. We determine the first switching time $t_1 \approx 7.268850$ (t_1 and $2T$ are the smallest solutions of the first and second equations of system (13), respectively.) Then we verify condition 3) of Theorem 3 for $t_\beta = t_1$: the first and second inequalities hold, $9.250182 \neq 0$ and $14.579030 \neq 0$. Hence, we proceed to Step 4.

4. We find the switching points X^1 and X^2 of the canonical system:

$$X^1 \approx \begin{pmatrix} 10 \\ 6.822206 \\ 3.640645 \end{pmatrix}, \quad X^2 \approx \begin{pmatrix} -46.050182 \\ -4.946928 \\ -2.983556 \end{pmatrix}.$$



The solution with period 4π and switching points X^1 and X^2.

5. We find the switching points Y^1 and Y^2 of the original system:

$$Y^1 \approx \begin{pmatrix} -31.337690 \\ 15.900271 \\ 6.359355 \end{pmatrix}, \quad Y^2 \approx \begin{pmatrix} 214.312942 \\ -88.096622 \\ -43.066625 \end{pmatrix}.$$

6. We construct the trajectory of the canonical system solution. The figure shows the graph of the 4π-periodic solution in the state space (x_1, x_2, x_3) of the system with the initial point X^1. The switching points are marked on the corresponding switching hyperplanes (highlighted by shading) being oriented orthogonally to the axis x_1 since $\gamma_1 \neq 0$ and $-46.050182 \leq x_1 \leq 10$.

7. We complete the algorithm with the following conclusion: for $\ell_1 = -10$, $\ell_2 \approx 46.050182$, $m_1 = -9$, $m_2 = 7.54$, and $\gamma_1 = -1$ in the transformed system (for $c_1 = -1$, $c_2 = -1$, and $c_3 = -4$ in the original system), there exists a unique asymptotically stable (asymptotically orbitally stable, respectively) 4π-periodic solution with two switching points per period.

8. CONCLUSIONS

Based on the results obtained in [7–9] and this paper, we have developed an algorithm for selecting domains in the system parameter space that correspond to the desired solution and finding the switching points of this solution in the state space. The algorithm has been described using the theorems established by rigorous analytical considerations with equivalent transformations and properties of the logarithmic function. The resulting system of conditions on the parameters appears to be consistent and has a solution in the form of a non-empty set. The effectiveness of the algorithm has been confirmed by a numerical example: calculating the parameters of the nonlinear characteristic and feedback vector under which there exists an asymptotically orbitally stable 4π-periodic solution (a subharmonic forced oscillation of the 2nd order) with two switching points per period. The algorithm has been implemented for a three-dimensional system using standard MATLAB functions. This algorithm can serve as a step-by-step instruction for writing program code for the completely automatic selection of control parameters and construction of the solution trajectory.

APPENDIX

Proof of Theorem 3. Starting its motion from a point on the hyperplane, the representative point of the canonical system solution moves in the prescribed sequence between the two switching hyperplanes along the coordinate axis x_s under condition 1) of Theorem 3. Moreover, the hyperplanes in the state space are orthogonal to the axis x_s.

Under condition 1) of Theorem 3, we write the canonical system (6) as the two systems

$$\dot{\bar{X}} = \bar{A}_0\bar{X} + \bar{B}_0 m_\alpha + \bar{K}_0 f(t), \quad \begin{cases} \sigma(t) = \gamma_s x_s \\ \dot{x}_s = \lambda_s x_s + m_\alpha + k_s^0 f(t), \end{cases}$$

where $\bar{A}_0$ is a diagonal matrix with the eigenvalues λ_j $(j \neq s)$ placed on the diagonal (the other elements are 0), $\bar{X} = (x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_n)^*$, $\bar{B}_0 = (b_1^0, \ldots, b_{s-1}^0, b_{s+1}^0, \ldots, b_n^0)^*$, $\bar{K}_0 = (k_1^0, \ldots, k_{s-1}^0, k_{s+1}^0, \ldots, k_n^0)^*$, $\lambda_j < 0$, and $b_j^0 = 1$, where $j = 1, \ldots, s-1, s+1, \ldots, n$.

Provided that the real eigenvalues λ_j $(j \neq s)$ are negative, we can apply Lyapunov functions in the state space of system (6) to separate on the switching hyperplanes a bounded, closed, and convex set that is mapped into itself due to the canonical system solution. The trivial solution of the system $\dot{\bar{X}} = \bar{A}_0\bar{X}$ is asymptotically stable, so there exists a positive definite quadratic form $V(\bar{X}) = \bar{X}^* V \bar{X}$. The equation $V(\bar{X}) = C_\nu$ with constants C_ν $(\nu \in \mathbb{N})$ describes cylindrical surfaces in the n-dimensional state space of the canonical system.

The attraction domain $V(\bar{X}) \le \min\limits_{\nu} C_\nu$ will intersect the switching hyperplanes $(\Gamma, X) = \ell_\eta$ $(\eta = 1, 2)$ under the conditions

$$-(\Gamma, A_0^{-1} B_0 m_2) < \ell_1, \quad -(\Gamma, A_0^{-1} B_0 m_1) > \ell_2. \tag{A.1}$$

They have the following interpretation: in the case of no external perturbation, the virtual stability points $X^{(\alpha)} = -A_0^{-1} B_0 m_\alpha$ $(\alpha = 1, 2)$ of the canonical system are located outside the ambiguity domain of $u(\sigma)$.

Selecting the elements of the vector Γ based on condition 1) of Theorem 3 simplifies inequalities (A.1) with $\lambda_s \neq 0$ to

$$-\gamma_s m_2/\lambda_s < \ell_1, \quad -\gamma_s m_1/\lambda_s > \ell_2. \tag{A.2}$$

Obviously, the system of inequalities (A.2) holds if $\gamma_s \lambda_s > 0$ for $m_1 < m_2$ and $\ell_1 < \ell_2$. Note that we have $\gamma_s > 0$ for $\lambda_s > 0$ and $\gamma_s < 0$ for $\lambda_s < 0$. Thus, condition 2) of Theorem 3 must be imposed for the attraction domain to intersect the switching hyperplanes.

The intersection of the set described by the inequality $V(\bar{X}) \le \min\limits_{\nu} C_\nu$ with the switching hyperplanes gives a compact and convex set Q, which is defined according to condition 4) of Theorem 3. If the initial points X_0 are taken from the domain bounded by the surface $V(\bar{X}) = \min\limits_{\nu} C_\nu$, the trajectory of the representative point of the solution due to the canonical system will remain in this domain of the state space. Therefore, starting its motion at $X_0 \in Q$ on one of the hyperplanes $\sigma(t) = \ell_\eta$ $(\eta = 1, 2)$, the representative point of the solution of system (6) will reach the other hyperplane in a finite time.

Next, let us establish conditions of no sliding mode occurrence. We write the condition under which the representative point of the solution of system (6) will reach the hyperplane without touching at the switching points $X = X^\beta$ at the corresponding times t_β $(\beta = 1, 2)$, where t_1 and $t_2 = kT$ are the first and second switching times, respectively. The existence of a unique time t_1 for a given number $k \in \mathbb{N}$ is ensured by the conditions of Theorem 1 or Theorem 2, depending on the sign of the eigenvalue λ_s. Thus, we have the inequality $(\Gamma, \dot{X}) \neq 0$. Considering condition 1) of Theorem 3, this inequality for $\lambda_s \neq 0$ is rewritten as condition 3) of Theorem 3, which ensures the reachability of the switching hyperplanes without touching. Note that the function $f(t)$ is T-periodic: its value at the second switching time kT coincides with that at zero. Therefore, we use $t_2 = 0$ in condition 3) of Theorem 3 for simplicity. The proof of Theorem 3 is complete.

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