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Spacecraft Transfer Optimization with Releasing the Additional Fuel Tank and the Booster to the Earth Atmosphere

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Abstract—This paper considers the idea of reducing the debris of near-Earth space by releasing the spent parts of a spacecraft on orbits touching the conditional boundary of the Earth's atmosphere. We optimize the spacecraft transfer trajectory from a circular reference orbit of an artificial Earth satellite to a target elliptical orbit in a modified pulse problem statement. The convergence of Newton's method is improved by introducing a series of auxiliary coordinate systems at each point of applying an impulse action. The derivatives in the transversality conditions are calculated using a special numerical–analytical differentiation technique.

Keywords: spacecraft, space debris, additional fuel tank, booster, release to the atmosphere, pulse formulation, boundary value problem, Lagrange principle

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1. INTRODUCTION

Since the beginning of the space age, near-Earth space has accumulated a significant amount of space debris, i.e., objects of artificial origin and their fragments that are faulty, non-functional, and unable to serve any useful purposes. Moreover, they are dangerous factors affecting the operation of spacecraft. Every year the number of space debris objects grows constantly. These objects can be divided into large groups as follows: payloads, rocket stages, boosters, and tanks. The greatest accumulation of space debris is observed in low Earth orbits and the geostationary orbit (GEO) zone [1]. The collisions of large-size space debris objects with each other and the explosions of fuel residues in the tanks may appreciably increase the number of small-size space debris objects. The processes described may generate a chain reaction, the so-called Kessler effect [2].

Many publications were devoted to the compilation of catalogs of non-functional spacecraft and the state monitoring of near-Earth space; for example, see [3–7].

In [3], the current anthropogenic situation in near-Earth space was analyzed based on the available catalog of cosmic objects. In addition, the main measures to reduce space debris were listed, and an automated warning system of dangerous situations in near-Earth space was described.

The paper [4] considered the organizational, methodological, and technological problems of building an information monitoring system to reduce the debris of near-Earth space, including the description of the system databases.

The authors [5] presented mathematical support and software tools for examining the objects of artificial origin. The tools allow predicting the probability of collision for space debris objects with functional satellites and simulating the space debris formation process due to explosions and collisions.

In [6], an algorithm was developed to accelerate orbit construction for a non-cataloged space debris object.

Three generations of optoelectronic complexes to monitor near-Earth space were described in [7].

It is topical to elaborate measures on cleaning and avoiding or reducing space debris [8]. The ways of solving the space debris problem can be divided into two large groups: prevention and cleaning. Various projects to clean up near-Earth space are being developed: capturing a space debris fragment with nets [9–11] and harpoons [12–15] and using lasers [16–20]. The idea of a flyby of a large-size space debris object with its subsequent transfer to the disposal orbit was considered in [21, 22]. However, there exist no economically acceptable projects so far.

In the adopted international documents, one measure to prevent space debris formation is withdrawing space objects from the working orbits after the end of their active operation [23, 24]. The authors [25–30] studied the transfer problem of a finished spacecraft to the disposal orbit with a given lifetime. The ideas of moving spent spacecraft and their parts to the dense atmosphere were examined in [27–33].

The paper [25] considered the transfer problem of a finished spacecraft to the disposal orbit with a given lifetime. For a particular spacecraft, it is possible to choose the type of disposal orbit (circular or elliptical) and the time instant to start transition maneuver to this orbit.

The possibility of fulfilling the Inter-Agency Space Debris Coordination Committee space debris mitigation guidelines was analyzed in [26]. As noted, one of the easiest and most effective ways to prevent debris in mid-altitude orbits is minimizing the eccentricity of satellites after the end of their active operation. According to the statistical data, almost half of the satellites in 1999–2011 were transferred to the disposal orbit meeting the requirements of the Inter-Agency Space Debris Coordination Committee.

In [27], the transfer of the Gonets-M spacecraft to the disposal orbit and the dense atmosphere after the end of its active operation was considered. In addition, the characteristic velocity to execute the transition maneuver and the lifetime of the spacecraft in the disposal orbit were estimated therein.

Various ways to prevent the debris of near-Earth space were among the issues addressed in [28]. They include the reduction of exhaust products from solid-propellant engines, the passivization of spacecraft and launch vehicles, the decrease of fragmentation due to collisions, the transfer of spacecraft and launch vehicles to disposal orbits, the forced atmospheric entry of spacecraft and launch vehicles, and the reduction of the lifetime of space objects. It was proposed to use special jet engines or main engines for the atmospheric entry of spacecraft. According to the authors, surface geometry changes (e.g., inflatable balloons) can be used to enlarge the surface area and thereby enhance atmospheric braking in low orbits.

The paper [29] considered the transfer problem to the atmosphere and disposal orbits on an example of several satellites. The dependencies of the relative fuel mass on various parameters were constructed.

As noted in [30], the forced atmospheric entry of spacecraft using flight deceleration techniques is a promising approach. However, such maneuvers need an appropriate modification of traditional designs of spacecraft and launch vehicles.

In [31], besides various space debris removal methods, the authors suggested reducing the debris of near-Earth space by the transition of failed satellites to the upper layers of the Earth atmosphere using solar sails.

Super-small spacecraft, part of distributed spacecraft, can be transferred to the dense atmosphere; see [32].

A method to reduce the debris of near-Earth space [33] consists in maneuvering with the unused fuel residues to change the perigee altitude of the orbit for the fast controlled transition of detached spacecraft parts to the dense atmosphere. A corresponding optimization problem was posed in the cited publication. Based on the analysis of existing space launch vehicles, this method is practicable.

The paper [34] considered the problem of choosing orbits to transfer large-size space objects after the end of their active operation. The dependence of the ballistic lifetime of space objects on the altitudes of the exchange orbits was investigated. According to the presented results, as the minimum altitude of the orbit decreases, the lifetime of a space object in this orbit tends to zero. By assumption, an object terminates ballistic life when reaching an altitude less than 80 km above the Earth's surface.

In what follows, we optimize spacecraft transfer with releasing the spent stages to the atmosphere. This problem statement is original and has not been studied by other researchers within optimization approaches.

This paper considers the idea of reducing the debris of near-Earth space by releasing the spent parts of a spacecraft on orbits touching the conditional boundary of the Earth's atmosphere at the final ascent stage. Found in [35] for the apsidal pulse statement, the solution with extremely low overhead costs also turned out to be a solution of the problem without the prior apsidality assumption; for details, see [36, 37]. This result gives hope for the success of the problem hierarchy methodology, i.e., the sequential formalization and solution of a series of problems with the gradual clarification and complication of their statements, where the obtained solutions of simpler problems are used to solve the next (more difficult) ones as an initial approximation (directly or based on their parametric continuation). A good initial approximation and a good computational scheme of the shooting method yield one of the possible extremals in the problem. Note that the attempts to construct an arbitrary extremal fail without preliminary trajectory analysis.

In this work, we optimize the exchange trajectory of a spacecraft from a reference circular orbit of an artificial Earth satellite of given radius and inclination to a target elliptical orbit. This transfer is considered in the central Newtonian gravitational field, i.e., the Earth is treated as the point attraction center and all other gravitational forces are neglected. The problem under study has the pulse statement. As before, the sum of final ascent maneuver impulses of a spacecraft from a target orbit to the geostationary one is assumed limited by a given value. The characteristic velocity of orbit final ascent maneuvers is modeled within the simplified apsidal pulse scheme.

We consider the transfer trajectory of a spacecraft with orbit final ascent maneuvers of the maximum characteristic velocity of 1.5 km/s. As demonstrated by the previous studies, the characteristic velocities of orbit final ascent maneuvers below 1.47... km/s lead to another trajectory structure with significant overhead costs.

In contrast to the spacecraft model with two stages and a satellite [36, 37], the spacecraft is supposed to consist of a booster, an auxiliary fuel tank (AFT), and a satellite. The first series of spacecraft transition maneuvers is executed using the AFT reserves. At the end of this series of maneuvers the spacecraft must reach the disposal orbit, i.e., the one touching the conditional boundary of the atmosphere (with a perigee altitude of 100 km). Then, there is a passive flight of duration 120 s to release the AFT. At the end of the passive flight section, the AFT remains in the disposal orbit, whereas the spacecraft is transferred by an impulse action to a "safe" orbit with a perigee altitude of 200 km. This impulse action, as well as the subsequent ones, is applied using the fuel reserves from the booster's main tank. Finally, after the second series of transition maneuvers, the spacecraft is delivered to the target orbit (one of the elliptical orbits such that the characteristic velocity of satellite final ascent maneuvers to the GEO is limited by a given value).

In this orbit, the satellite is undocked from the booster. Like the previous studies, the booster is assumed to be released from the apogee of the target orbit using the main tank fuel.

According to the parametric analysis in the earlier publications [35–37], a local minimum is achieved on the extremal constructed using the Lagrange principle.

We formalize the corresponding optimization problem and reduce it to a multipoint boundary value problem based on the Lagrange principle [38]. The boundary value problem of the Lagrange principle in the pulse statement is solved numerically by the shooting method. Bulky derivatives in the transversality conditions are computed using a special numerical–analytical differentiation technique.[1]

2. PROBLEM STATEMENT

The transfer is considered in the central Newtonian gravitational field in a vacuum in the rectangular Cartesian coordinate system related to the Earth's center. The axis z of this system is perpendicular to the equatorial plane and has is south-to-north direction; the axis x lies in the equatorial plane and is directed along the node line of the initial circular orbit from the descending node to the ascending one; the axis y completes the coordinate system to the right-hand triple. The passive motion of the spacecraft's center of mass is described by the system of differential equations

$$\begin{gathered} \dot{x}(t) = v_x(t), \quad \dot{y}(t) = v_y(t), \quad \dot{z}(t) = v_z(t), \\ \dot{v}_x(t) = -\frac{\mu x(t)}{r^3(t)}, \quad \dot{v}_y(t) = -\frac{\mu y(t)}{r^3(t)}, \quad \dot{v}_z(t) = -\frac{\mu z(t)}{r^3(t)}, \end{gathered} \tag{2.1}$$

with the following notations: $x(t)$, $y(t)$, and $z(t)$ are the coordinates of the spacecraft's center of mass;

$$r = \sqrt{x^2(t) + y^2(t) + z^2(t)}$$

is the distance between the spacecraft and the Earth's center; $v_x(t)$, $v_y(t)$, and $v_z(t)$ are the velocity vector components of the spacecraft's center of mass; finally, μ is the gravitational parameter of the Earth.

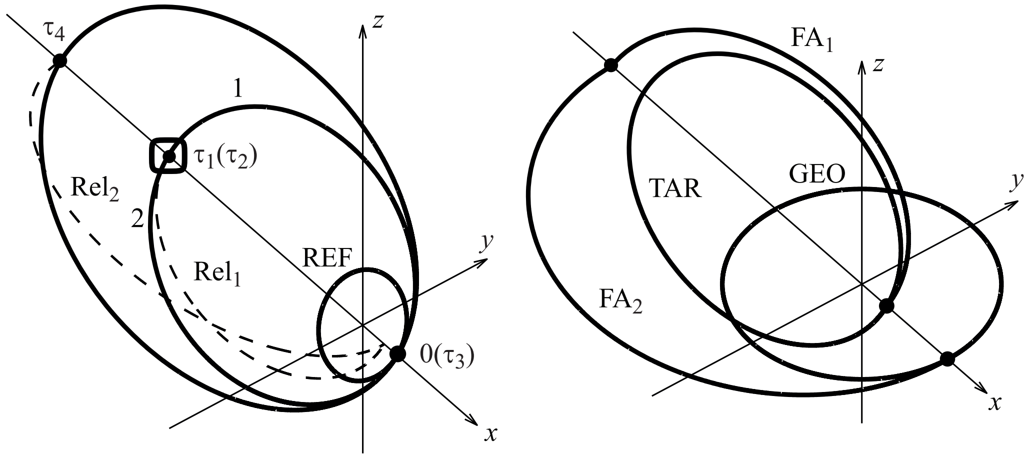
According to the previous studies, the desired spacecraft transfer trajectory from the reference orbit to the target orbit consists of four passive segments and five impulse actions; see the figure.

Let an impulse action be applied at a time instant τ. It occurs instantaneously and does not change the spacecraft coordinates:

$$\begin{gathered} x(\tau_+) - x(\tau_-) = 0, \quad y(\tau_+) - y(\tau_-) = 0, \quad z(\tau_+) - z(\tau_-) = 0, \\ \tau_+ - \tau_- = 0. \end{gathered} \tag{2.2}$$

From this point onwards, we employ the following notations: τ_- is the end of a current passive segment; $x(\tau_-)$, $y(\tau_-)$, and $z(\tau_-)$ are the state variables at this time instant (some left-continuous functions); τ_+ is the beginning of the next passive segment; $x(\tau_+)$, $y(\tau_+)$, and $z(\tau_+)$ are the state variables at this time instant (some right-continuous functions). In terms of the problem statement and application of the Lagrange principle [38], these are nonidentical segments and nonidentical state variables. According to the main theorem of [38], they differ and are indicated differently. The unified notation system of this paper serves to simplify the presentation: the subscripts are omitted.

[1] The numerical–analytical differentiation project is available at: http://mech.math.msu.su/∼iliagri/ext_value.htm.



Spacecraft transfer to target orbit (left) and final ascent maneuvering from target orbit to geostationary orbit (right): REF—reference orbit, 1—first exchange orbit (before highlighted domain), 2—safe orbit (after highlighted domain, the apogees of first and safe orbits are visually indistinguishable), TAR—target orbit, Rel_1—AFT release orbit, Rel_2—booster release orbit, FA_1—first final ascent maneuver orbit, FA_2—second final ascent maneuver orbit, and GEO—geostationary orbit.

At the initial time instant, the spacecraft before the first impulse action is in the circular reference orbit of a given inclination i_0 and radius R_0. Due to the chosen coordinate system, the ascending node has the longitude $\Omega_0 = 0$:

$$\begin{gathered} x^2(0_-) + y^2(0_-) + z^2(0_-) = R_0^2, \\ x(0_-)C_{0x} + y(0_-)C_{0y} + z(0_-)C_{0z} = 0, \\ v_x(0_-) + \frac{v_0}{R_0}\left(y(0_-)\cos i_0 + z(0_-)\sin i_0\right) = 0, \\ v_y(0_-) - \frac{v_0}{R_0}x(0_-)\cos i_0 = 0, \\ v_z(0_-) - \frac{v_0}{R_0}x(0_-)\sin i_0 = 0, \end{gathered} \tag{2.3}$$

where $C_0 = \sqrt{\mu R_0}$, $C_{0x} = 0$, $C_{0y} = -C_0 \sin i_0$, and $C_{0z} = C_0 \cos i_0$ are the magnitude and components of the kinetic momentum vector of the spacecraft relative to the Earth's center; $R_0 = R_{\text{Ear}} + h_0$ is the radius of the reference orbit; R_{Ear} is the Earth's radius; h_0 is the altitude of the reference orbit above the Earth's surface; finally, $v_0 = \sqrt{\frac{\mu}{R_0}}$ is the magnitude of the velocity vector in the reference orbit.

The initial time instant $t = 0_-$ corresponds to the conditions before the impulse action in the circular orbit (and not to any passive segment). The values $x(0_-)$, $y(0_-)$, $z(0_-)$, $v_x(0_-)$, $v_y(0_-)$, and $v_z(0_-)$, representing the coordinates and components of the velocity vector in the circular orbit before the impulse action due to (2.2), are eliminated from consideration within the problem statement:

$$x^2(0_+) + y^2(0_+) + z^2(0_+) = R_0^2, \quad x(0_+)C_{0x} + y(0_+)C_{0y} + z(0_+)C_{0z} = 0. \tag{2.4}$$

(For details, see [38].) Note that the formalized problem statement includes just the two conditions (2.4) out of the five ones (2.3). The last three conditions of (2.3) are part of the initial impulse:

$$\begin{gathered}\Delta v_{0x} = v_x(0_+) + \frac{v_0}{R_0}\left(y(0_+)\cos i_0 + z(0_+)\sin i_0\right),\\ \Delta v_{0y} = v_y(0_+) - \frac{v_0}{R_0}x(0_+)\cos i_0,\\ \Delta v_{0z} = v_z(0_+) - \frac{v_0}{R_0}x(0_+)\sin i_0,\\ \Delta v_0 = \sqrt{\Delta v_{0x}^2 + \Delta v_{0y}^2 + \Delta v_{0z}^2}.\end{gathered} \tag{2.5}$$

The impulse action at the initial time instant $t = 0$ transfers the spacecraft to the first exchange orbit. At an a priori unknown time instant τ_1 (according to the previous studies, in a neighborhood of the apogee of the first exchange orbit), an impulse action is applied to transfer the spacecraft to the AFT release orbit, i.e., an elliptical orbit touching the conditional boundary of the atmosphere (with a perigee altitude of 100 km):

$$\begin{gathered}x(\tau_{1+}) - x(\tau_{1-}) = 0, \quad y(\tau_{1+}) - y(\tau_{1-}) = 0, \quad z(\tau_{1+}) - z(\tau_{1-}) = 0,\\ \tau_{1+} - \tau_{1-} = 0,\\ R_{\mathrm{p}}(x(\tau_{1+}), y(\tau_{1+}), z(\tau_{1+}), v_x(\tau_{1+}), v_y(\tau_{1+}), v_z(\tau_{1+})) = R_{\mathrm{Ear}} + 100 \text{ km},\\ \Delta v_1 = \sqrt{(v_x(\tau_{1+}) - v_x(\tau_{1-}))^2 + (v_y(\tau_{1+}) - v_y(\tau_{1-}))^2 + (v_z(\tau_{1+}) - v_z(\tau_{1-}))^2}.\end{gathered} \tag{2.6}$$

Depending on the coordinates and velocities of the spacecraft at the time instant of inserting into the AFT release orbit, the radius of the perigee $R_{\mathrm{p}}(x, y, z, v_x, v_y, v_z)$ is given by

$$R_{\mathrm{p}}(\cdot) = a(\cdot)(1 - e(\cdot)),$$

where $a(\cdot)$ and $e(\cdot)$ denote the semi-major axis and eccentricity, calculated as functions of the spacecraft coordinates and velocities by known formulas [39].

In the next passive segment of 120 s, the AFT is undocked from the spacecraft. At the time instant τ_2, an impulse action transfers the spacecraft to the safe orbit (with a perigee altitude of 200 km):

$$\begin{gathered}x(\tau_{2+}) - x(\tau_{2-}) = 0, \quad y(\tau_{2+}) - y(\tau_{2-}) = 0, \quad z(\tau_{2+}) - z(\tau_{2-}) = 0,\\ \tau_{2+} - \tau_{2-} = 0,\\ R_{\mathrm{p}}(x(\tau_{2+}), y(\tau_{2+}), z(\tau_{2+}), v_x(\tau_{2+}), v_y(\tau_{2+}), v_z(\tau_{2+})) = R_{\mathrm{Ear}} + 200 \text{ km},\\ \tau_{2-} - \tau_{1+} = 120 \text{ s},\\ \Delta v_2 = \sqrt{(v_x(\tau_{2+}) - v_x(\tau_{2-}))^2 + (v_y(\tau_{2+}) - v_y(\tau_{2-}))^2 + (v_z(\tau_{2+}) - v_z(\tau_{2-}))^2}.\end{gathered} \tag{2.7}$$

At an a priori unknown time instant τ_3 (according to the previous studies, in a neighborhood of the perigee of the safe orbit), an impulse action is applied to transfer the spacecraft to the target orbit:

$$\begin{gathered}x(\tau_{3+}) - x(\tau_{3-}) = 0, \quad y(\tau_{3+}) - y(\tau_{3-}) = 0, \quad z(\tau_{3+}) - z(\tau_{3-}) = 0,\\ \tau_{3+} - \tau_{3-} = 0,\\ \Delta v_3 = \sqrt{(v_x(\tau_{3+}) - v_x(\tau_{3-}))^2 + (v_y(\tau_{3+}) - v_y(\tau_{3-}))^2 + (v_z(\tau_{3+}) - v_z(\tau_{3-}))^2}.\end{gathered} \tag{2.8}$$

In the long passive segment of the target orbit, the satellite separates from the booster. Then the satellite's final ascent maneuver to the geostationary orbit is executed using the satellite's fuel reserves.

As demonstrated by the previous studies, the apogee of the target orbit is the booster's best transition point to the orbit touching the conditional boundary of the atmosphere. In this case, the release impulse is directed against the booster's velocity and the angle of inclination does not change:

$$R_{\rm ta} = r(\tau_{4-}) = \sqrt{x^2(\tau_{4-}) + y^2(\tau_{4-}) + z^2(\tau_{4-})},$$
$$V_{\rm ta} = v(\tau_{4-}) = \sqrt{v_x^2(\tau_{4-}) + v_y^2(\tau_{4-}) + v_z^2(\tau_{4-})},$$
$$V_{\rm Atm} = \sqrt{\frac{2\mu r_{\rm Atm}}{r(\tau_{4-})(r(\tau_{4-}) + r_{\rm Atm})}},$$
$$\Delta v_4 = V_{\rm ta} - V_{\rm Atm},$$

where τ_4 is the time instant of passing the apogee of the target orbit, $r_{\rm Atm} = R_{\rm Ear} + 100$ km, $V_{\rm Atm}$ is the velocity in the apogee of the orbit touching the conditional boundary of the atmosphere reached by the booster after the impulse action, $R_{\rm ta}$ is the radius of the apogee of the target orbit, and $V_{\rm ta}$ is the velocity in the apogee of the target orbit.

Note that the characteristic velocity of satellite's final ascent maneuvers from the target orbit to the GEO is considered within the simplified scheme and is executed using three impulse actions:

$$\Delta v_{\rm FA}(\cdot) = \Delta v_{\rm FA1}(\cdot) + \Delta v_{\rm FA2}(\cdot) + \Delta v_{\rm FA3}.$$

The three impulse actions of the final ascent maneuver are calculated by the apsidal formulas [39].

The first impulse action $\Delta v_{\rm FA1}(\cdot)$ is applied in the perigee of the target orbit; without changing the inclination, it increases the apogee to the maximum possible distance $R_{\max}$ between the spacecraft and the Earth:

$$\Delta v_{\rm FA1}(\cdot) = \sqrt{V_{\rm tp}^2 + V_{\rm 1p}^2 - 2V_{\rm tp}V_{\rm 1p}},$$
$$V_{\rm tp} = \sqrt{\frac{2\mu R_{\rm ta}}{R_{\rm tp}(R_{\rm ta} + R_{\rm tp})}}, \quad V_{\rm 1p} = \sqrt{\frac{2\mu R_{\max}}{R_{\rm tp}(R_{\max} + R_{\rm tp})}}, \tag{2.9}$$

where $R_{\rm tp}$ is the radius of the perigee of the target orbit and $V_{\rm tp}$ is the velocity in the perigee of the target orbit.

The expressions (2.9) involve the formalized relation $\Delta v_{\rm FA1} = |V_{\rm 1p} - V_{\rm tp}|$ instead of the quite natural and clear one

$$\Delta v_{\rm FA1} = V_{\rm 1p} - V_{\rm tp}. \tag{2.10}$$

Of course, we have $V_{\rm 1p} > V_{\rm tp}$ in the optimal solution, and the absolute value (root of the square) becomes unnecessary in the formula for $\Delta v_{\rm FA1}$. However, it is desirable to consider the potential inequality $V_{\rm 1p} < V_{\rm tp}$ in intermediate calculations. Without the absolute value in such potential cases, the simplified formula (2.10) yields the unphysical result $\Delta v_{\rm FA1} < 0$. Nevertheless, it is not critical because the problem will not arise for the optimal solution.[2] The main reason for using the (more complex) absolute value-based formula is the deteriorated convergence of the iterative process compared to a simpler formula or "incompletely defined" conditions.

[2] More exactly, "extremal" since the first-order optimality conditions (the Lagrange principle) are verified without considering the second-order conditions and sufficient conditions.

The second impulse action $\Delta v_{\mathrm{FA2}}(\cdot)$ is applied in the apogee. It increases the perigee to the GEO radius R_{GEO} and changes the inclination to 0:

$$\Delta v_{\mathrm{FA2}}(\cdot) = \sqrt{V_{1\mathrm{a}}^2 + V_{2\mathrm{a}}^2 - 2V_{1\mathrm{a}}V_{2\mathrm{a}}\cos i_{\mathrm{t}}},$$
$$V_{1\mathrm{a}} = \sqrt{\frac{2\mu R_{\mathrm{tp}}}{R_{\max}(R_{\max}+R_{\mathrm{tp}})}},\ V_{2\mathrm{a}} = \sqrt{\frac{2\mu R_{\mathrm{GEO}}}{R_{\max}(R_{\max}+R_{\mathrm{GEO}})}}, \qquad (2.11)$$

where i_{t} denotes the inclination angle of the target orbit to the equatorial plane. At the time instant of passing the apogee, this value can be calculated by

$$\cos i_{\mathrm{t}} = \frac{\sqrt{v_x^2(\tau_{4-}) + v_y^2(\tau_{4-})}}{\sqrt{v_x^2(\tau_{4-}) + v_y^2(\tau_{4-}) + v_z^2(\tau_{4-})}}. \qquad (2.12)$$

The third impulse action Δv_{FA3} in the perigee decreases the apogee to the GEO radius without changing the inclination. Thus, it transfers the satellite to an a priori unknown point of the geostationary orbit:

$$\Delta v_{\mathrm{FA3}} = V_{2\mathrm{p}} - v_{\mathrm{GEO}},$$
$$V_{2\mathrm{p}} = \sqrt{\frac{2\mu R_{\max}}{R_{\mathrm{GEO}}(R_{\max}+R_{\mathrm{GEO}})}}, \quad v_{\mathrm{GEO}} = \sqrt{\frac{\mu}{R_{\mathrm{GEO}}}}. \qquad (2.13)$$

Note that the value Δv_{FA3} is actually constant. (It depends on the given value R_{GEO} and the given problem parameter $R_{\max}$.)

The values R_{tp}, R_{ta}, and i_{t} in these formulas are Kepler's integrals and can be calculated whenever the spacecraft moves in the target orbit. Their calculation at the time instant τ_4 of passing the apogee of the target orbit is technically more convenient. Thus, using (2.9)–(2.13), we represent $\Delta v_{\mathrm{FA}}(\cdot)$ as the relation

$$\Delta v_{\mathrm{FA}}(x(\tau_{4-}), y(\tau_{4-}), z(\tau_{4-}), v_x(\tau_{4-}), v_y(\tau_{4-}), v_z(\tau_{4-})).$$

As a result, the boundary conditions can be formalized as follows:

$$\begin{gathered}\Delta v_{\mathrm{FA}}(x(\tau_{4-}), y(\tau_{4-}), z(\tau_{4-}), v_x(\tau_{4-}), v_y(\tau_{4-}), v_z(\tau_{4-})) = \Delta v^*,\\ x(\tau_{4-})v_x(\tau_{4-}) + y(\tau_{4-})v_y(\tau_{4-}) + z(\tau_{4-})v_z(\tau_{4-}) = 0,\\ \lambda_3(x(\tau_{4-}), y(\tau_{4-}), z(\tau_{4-}), v_x(\tau_{4-}), v_y(\tau_{4-}), v_z(\tau_{4-}))\\ = v_x(\tau_{4-})(z(\tau_{4-})v_x(\tau_{4-}) - x(\tau_{4-})v_z(\tau_{4-}))\\ - v_y(\tau_{4-})(y(\tau_{4-})v_z(\tau_{4-}) - z(\tau_{4-})v_y(\tau_{4-})) - \frac{\mu z(\tau_{4-})}{r(\tau_{4-})} = 0,\end{gathered} \qquad (2.14)$$

where $\lambda_3(\cdot)$ is the z-component of the Laplace vector. The last relation at the apogee can be written as $z(\tau_{4-}) = 0$. However, we use the basic relation on the z-component of the Laplace vector in calculations for a convenient transition to the general case (the problem with limited thrust). The first relation in (2.14) restricts the characteristic velocity of the satellite final ascent maneuver from the target to geostationary orbit. According to the previous studies, this constraint is active; therefore, it is presented in the paper as an equality. We can also assume that only one main subcase is considered in the non-strict inequality case. (Under the complementary slackness conditions, the corresponding Lagrange multiplier exceeds 0.) The second orthogonality relation of the radius vector and velocity vector holds on the elliptical orbit at two points (perigee and apogee). They can be distinguished, e.g., as follows: before passing the apogee, the radial component of the velocity vector is positive; after passing the apogee, it becomes negative. Such strict inequalities will have no effect on the problem solution based on the Lagrange principle (will be passive) and are therefore omitted below. The third relation shows that the apogee is in the equatorial plane. On the one hand, this condition is not restrictive and eliminates the symmetry of the problem's rotation with respect to the spacecraft kinetic moment vector (C_{0x}, C_{0y}, C_{0z}) of the reference orbit. On the other hand, the condition that the apsidal line lies in the equatorial plane allows avoiding unnecessary final ascent maneuver costs. On the third hand, it seriously simplifies the final ascent maneuver formulas to the apsidal version.

The objective functional in this problem is the satellite's payload in the target orbit.

At the initial time instant, the spacecraft has the dimensionless mass $m(0) = 1$. At each time instant of applying the impulse action, the mass varies in accordance with Tsiolkovsky's formula:

$$m(\tau_+) = m(\tau_-)\exp\left(-\frac{\Delta v(\tau)}{c}\right),$$

where $c = P_{\text{spe}} g_{\text{Ear}}$ is the jet velocity, P_{spe} is the specific thrust, and g_{Ear} is the gravitational acceleration at the Earth surface.

Let u_1 and u_2 be the shares of the characteristic velocity of the spacecraft transition maneuver to the target orbit executed using the fuel reserves from the AFT and the booster's main tank, respectively. The dry mass of a tank is proportional to the mass of fuel held by it with a coefficient α; for details, see [40, p. 93]. Therefore, after the first series of transition maneuvers and AFT release, the spacecraft mass becomes

$$m_1 = (1+\alpha)\exp\left(-\frac{u_1}{c}\right) - \alpha.$$

Considering the booster's transition impulse to an orbit touching the conditional boundary of the atmosphere, the payload at the apogee of the target orbit [36] is given by

$$m_{\text{p}}(\cdot) = \left[(1+\alpha)\exp\left(-\frac{u_1}{c}\right) - \alpha\right]\left[\exp\left(-\frac{u_2}{c}\right) - \frac{\alpha\left(1-\exp\left(-\frac{u_2}{c}\right)\right)}{(1+\alpha)\exp\left(-\frac{\Delta v_4}{c}\right) - \alpha}\right].$$

For the spacecraft transfer under study, we have

$$u_1 = \Delta v(0) + \Delta v(\tau_1), \quad u_2 = \Delta v(\tau_2) + \Delta v(\tau_3).$$

Consequently, the payload is a complex function of the spacecraft coordinates and velocities after the impulse action at the initial time instant, the spacecraft velocities before and after the impulse actions at the time instants τ_1, τ_2, and τ_3, and the spacecraft coordinates and velocities at the time instant τ_4.

Thus, we have formalized the problem statement. It is reduced to a corresponding boundary value problem [38] based on the Lagrange principle.

3. THE LAGRANGE PRINCIPLE

The Lagrange function has the form

$$\Lambda = \sum_{i=0}^{4}\left(\int\limits_{\tau_{i+}}^{\tau_{(i+1)-}} L\,dt\right) + l, \quad (\tau_{0+} = 0_+),$$

with the Lagrangian

$$L = p_x(\dot{x} - v_x) + p_y(\dot{y} - v_y) + p_z(\dot{z} - v_z)$$
$$+ p_{vx}\left(\dot{v}_x + \frac{\mu x}{r^3}\right) + p_{vy}\left(\dot{v}_y + \frac{\mu y}{r^3}\right) + p_{vz}\left(\dot{v}_z + \frac{\mu z}{r^3}\right),$$

the Hamiltonian

$$H = p_x v_x + p_y v_y + p_z v_z + p_{vx}\left(-\frac{\mu x}{r^3}\right) + p_{vy}\left(-\frac{\mu y}{r^3}\right) + p_{vz}\left(-\frac{\mu z}{r^3}\right),$$

and the terminant

$$l = \sum_{i=1}^{3}\lambda_{xi}(x(\tau_{i+}) - x(\tau_{i-})) + \sum_{i=1}^{3}\lambda_{yi}(y(\tau_{i+}) - y(\tau_{i-}))$$
$$+ \sum_{i=1}^{3}\lambda_{zi}(z(\tau_{i+}) - z(\tau_{i-})) + \sum_{i=1}^{3}\lambda_{\tau i}(\tau_{i+} - \tau_{i-}) + \lambda_{\tau 12}(\tau_{2-} - \tau_{1+} - 120)$$
$$+ \lambda_{R0}(x(0_+)^2 + y(0_+)^2 + z(0_+)^2 - R_0^2) + \lambda_{C0}(x(0_+)C_{0x} + y(0_+)C_{0y} + z(0_+)C_{0z})$$
$$+ \lambda_{\mathrm{Rp1}}(R_{\mathrm{p}}(x(\tau_{1+}), y(\tau_{1+}), z(\tau_{1+}), v_x(\tau_{1+}), v_y(\tau_{1+}), v_z(\tau_{1+})) - R_{\mathrm{Ear}} - 100)$$
$$+ \lambda_{\mathrm{Rp2}}(R_{\mathrm{p}}(x(\tau_{2+}), y(\tau_{2+}), z(\tau_{2+}), v_x(\tau_{2+}), v_y(\tau_{2+}), v_z(\tau_{2+})) - R_{\mathrm{Ear}} - 200)$$
$$+ \lambda_{rv4}(x(\tau_{4-})v_x(\tau_{4-}) + y(\tau_{4-})v_y(\tau_{4-}) + z(\tau_{4-})v_z(\tau_{4-}))$$
$$+ \lambda_{zL}\lambda_3(x(\tau_{4-}), y(\tau_{4-}), z(\tau_{4-}), v_x(\tau_{4-}), v_y(\tau_{4-}), v_z(\tau_{4-}))$$
$$+ \lambda_{\mathrm{FA}}(\Delta v_{\mathrm{FA}}(x(\tau_{4-}), y(\tau_{4-}), z(\tau_{4-}), v_x(\tau_{4-}), v_y(\tau_{4-}), v_z(\tau_{4-})) - \Delta v^*) - \lambda_0 m_{\mathrm{p}}.$$

Here, λ_{xi}, λ_{yi}, λ_{zi}, $\lambda_{\tau i}$, $\lambda_{\tau 12}$, λ_{R0}, λ_{C0}, $\lambda_{\mathrm{Rp}k}$, λ_{rv4}, λ_{zL}, λ_{FA}, and λ_0 $(i = 1, 2, 3,\ k = 1, 2)$ are the numerical Lagrange multipliers; $p_x(\cdot)$, $p_y(\cdot)$, $p_z(\cdot)$, $p_{v_x}(\cdot)$, $p_{v_y}(\cdot)$, and $p_{v_z}(\cdot)$ are the conjugate variables (the functional Lagrange multipliers) on each of the four segments. The additional subscripts of the functions (according to the segment numbers), formally dictated by the theorem of [38], are omitted below to simplify the presentation.

The stationarity conditions in the state variables (the conjugate system of equations, the Euler–Lagrange equations) have the form

$$\begin{gathered}
\dot{p}_x = \frac{\mu}{r^3}\left[p_{vx} - \frac{3x}{r^2}(xp_{vx} + yp_{vy} + zp_{vz})\right], \\
\dot{p}_y = \frac{\mu}{r^3}\left[p_{vy} - \frac{3y}{r^2}(xp_{vx} + yp_{vy} + zp_{vz})\right], \\
\dot{p}_z = \frac{\mu}{r^3}\left[p_{vz} - \frac{3z}{r^2}(xp_{vx} + yp_{vy} + zp_{vz})\right], \\
\dot{p}_{vx} = -p_x, \quad \dot{p}_{vy} = -p_y, \quad \dot{p}_{vz} = -p_z.
\end{gathered} \tag{3.1}$$

Being cumbersome, the transversality conditions are formally written as

$$\begin{gathered}
p_\xi(\tau_{i+}) = \frac{\partial l}{\partial \xi(\tau_{i+})}, \quad p_\xi(\tau_{i-}) = -\frac{\partial l}{\partial \xi(\tau_{i-})}, \\
p_\xi(0_+) = \frac{\partial l}{\partial \xi(0_+)}, \quad p_\xi(\tau_{4-}) = -\frac{\partial l}{\partial \xi(\tau_{4-})}, \\
\xi = x, y, z, v_x, v_y, v_z, \quad i = 1, 2, 3.
\end{gathered} \tag{3.2}$$

The derivatives of the functions $R_{\mathrm{p}}(\cdot)$, $\Delta v_{\mathrm{FA}}(\cdot)$, and $m_{\mathrm{p}}(\cdot)$ in these conditions are calculated using a special numerical–analytical differentiation technique.

The stationarity conditions have the form

$$\begin{gathered} H(\tau_{1-}) = -\lambda_{\tau 1}, \quad H(\tau_{1+}) = -\lambda_{\tau 1} + \lambda_{\tau 12}, \quad H(\tau_{2-}) = -\lambda_{\tau 2} + \lambda_{\tau 12}, \\ H(\tau_{2+}) = -\lambda_{\tau 2}, \quad H(\tau_{3-}) = -\lambda_{\tau 3}, \quad H(\tau_{3+}) = -\lambda_{\tau 3}, \quad H(\tau_{4-}) = 0. \end{gathered} \tag{3.3}$$

There is no stationarity condition at the initial time instant. The stationarity conditions at the time instant τ_3 imply the continuity of the Hamiltonian: $H(\tau_{3+}) = H(\tau_{3-})$. From the stationarity conditions at the time instants τ_1 and τ_2 it follows that $H(\tau_{2+}) = H(\tau_{1-})$.

Indeed, the right-hand side of the system of differential equations does not explicitly depend on time. Hence, the function $H(t)$ is constant on the solution of this system, i.e., $H(\tau_{2-}) = H(\tau_{1+})$. Due to the stationarity conditions, we have $H(\tau_{1+}) = H(\tau_{1-}) + \lambda_{\tau 12}$ and $H(\tau_{2-}) = H(\tau_{2+}) + \lambda_{\tau 12}$. Subtracting one equality from the other yields $H(\tau_{1+}) - H(\tau_{2-}) = H(\tau_{1-}) - H(\tau_{2+})$. Therefore, $H(\tau_{1-}) - H(\tau_{2+}) = 0$, i.e., $H(\tau_{2+}) = H(\tau_{1-})$, as required.

The normalization condition is $\lambda_0 = 1$.

Since the trajectory has a complex structure, the impossibility of the abnormal case $\lambda_0 = 0$ should be studied separately. This subject goes beyond the scope of the paper.

The unknowns in the boundary value problem are as follows: 48 arbitrary integration constants in the system of differential equations (2.1), (3.1) (12 unknowns for each of the four segments), the time instants $\tau_{1\pm}$, $\tau_{2\pm}$, $\tau_{3\pm}$, and τ_{4-}, and 20 numerical Lagrange multipliers. The total number is 75 unknowns. They have to be determined using 19 conditions on the spacecraft coordinates and the time instants (2.2), (2.4), (2.6), (2.7), and (2.14), 48 transversality conditions (3.2), and 7 stationarity conditions (3.3). The total number is 75 conditions, i.e., the number of unknowns in the boundary value problem coincides with the number of conditions imposed on them.

4. NUMERICAL SOLUTION

With the shooting method, the boundary value problem of the Lagrange principle is reduced to a system of nonlinear equations. This system is then solved numerically by the modified Newton–Isayev–Sonin–Fedorenko method [41, 42, Ch. 1].

The choice of the computational scheme of the shooting method significantly affects the convergence rate of Newton's method. It is reasonable to adjust the shooting parameters that determine the impulse action in a special basis, the so-called modified orbital basis. (Note that the new coordinate system has the same origin as the initial one.)

The basis vectors $\vec{e}_r(\tau)$, $\vec{e}_{v\mathrm{tr}}(\tau)$, and $\vec{e}_c(\tau)$ of the local coordinate system at the time instants of applying the impulse actions are defined as follows. The vector $\vec{e}_r(\tau)$ is directed along the radius vector of the spacecraft at the time instant of the impulse action; the direction of $\vec{e}_{v\mathrm{tr}}(\tau)$ coincides with that of the transversal velocity component; the vector $\vec{e}_c(\tau)$ completes the system to the right-hand triple. Then

$$\begin{gathered} \vec{e}_r(\tau) = \frac{\vec{r}(\tau)}{|\vec{r}(\tau)|}, \quad \vec{e}_{v\mathrm{tr}}(\tau) = \frac{\vec{v}_{\mathrm{tr}}(\tau)}{|\vec{v}_{\mathrm{tr}}(\tau)|}, \\ \vec{e}_c(\tau) = \frac{\vec{C}(\tau)}{|\vec{C}(\tau)|}, \quad \vec{e}_v(\tau) = \frac{\vec{v}(\tau)}{|\vec{v}(\tau)|}, \\ \vec{C}(\tau) = [\vec{e}_r(\tau), \vec{e}_v(\tau)], \quad \vec{v}_{\mathrm{tr}}(\tau) = [\vec{e}_c(\tau), \vec{e}_r(\tau)]. \end{gathered} \tag{4.1}$$

The spacecraft velocities and, hence, the components of the impulse vector $\Delta v_r(\tau)$, $\Delta v_{\mathrm{tr}}(\tau)$, and $\Delta v_c(\tau)$ at each time instant τ of applying the impulse action are specified in the system coordinates associated with the spacecraft:

$$\Delta v_r(\tau) = \Delta v(\tau)\cos\psi(\tau)\cos\theta(\tau), \quad \Delta v_{\rm tr}(\tau) = \Delta v(\tau)\sin\psi(\tau)\cos\theta(\tau),$$
$$\Delta v_c(\tau) = \Delta v(\tau)\sin\theta(\tau), \tag{4.2}$$

where $\theta(\tau)$ denotes the pitch angle (the angle between the impulse vector and the orbital plane) at the time instant τ, $\psi(\tau)$ is the yaw angle (counted in the orbital plane from the radius vector in direction of the velocity vector) at the time instant τ, and $\Delta v(\tau)$ is the impulse magnitude (its value in the basic coordinate system coincides with that in the local one).

The components of the impulse vector in the initial coordinate system, $\Delta v_x(\tau)$, $\Delta v_y(\tau)$, and $\Delta v_z(\tau)$, are obtained by the basis transition formulas in matrix representation:

$$\begin{pmatrix}\Delta v_x(\tau)\\ \Delta v_y(\tau)\\ \Delta v_z(\tau)\end{pmatrix} = \begin{pmatrix} e_{rx}(\tau) & e_{vx{\rm tr}}(\tau) & e_{cx}(\tau)\\ e_{ry}(\tau) & e_{vy{\rm tr}}(\tau) & e_{cy}(\tau)\\ e_{rz}(\tau) & e_{vz{\rm tr}}(\tau) & e_{cz}(\tau)\end{pmatrix}\begin{pmatrix}\Delta v_r(\tau)\\ \Delta v_{\rm tr}(\tau)\\ \Delta v_c(\tau)\end{pmatrix}. \tag{4.3}$$

These formulas have the following notations: $e_{rx}(\tau)$, $e_{ry}(\tau)$, and $e_{rz}(\tau)$ are the coordinates of the vector $\vec{e}_r(\tau)$; $e_{vx{\rm tr}}(\tau)$, $e_{vy{\rm tr}}(\tau)$, and $e_{vz{\rm tr}}(\tau)$ are the coordinates of the vector $\vec{e}_{v{\rm tr}}(\tau)$; finally, $e_{cx}(\tau)$, $e_{cy}(\tau)$, and $e_{cz}(\tau)$ are the coordinates of the vector $\vec{e}_c(\tau)$ in the basic coordinate system at the time instant τ.

The values $v_x(\tau_-)$, $v_y(\tau_-)$, and $v_z(\tau_-)$, representing the components of the spacecraft velocity vector in the initial coordinate system before the impulse action at the time instant τ, are determined by solving the Cauchy problem. (At the initial time instant, when the spacecraft flies in the reference circular orbit, they are calculated by the well-known formulas from the handbook [39].) The values $v_x(\tau_+)$, $v_y(\tau_+)$, and $v_z(\tau_+)$, representing the components of the spacecraft velocity vector in the basic coordinate system after the impulse action at the time instant τ, are given by

$$\begin{aligned} v_x(\tau_+) &= \Delta v_x(\tau) + v_x(\tau_-),\\ v_y(\tau_+) &= \Delta v_y(\tau) + v_y(\tau_-),\\ v_z(\tau_+) &= \Delta v_z(\tau) + v_z(\tau_-). \end{aligned} \tag{4.4}$$

The shooting parameter vector at the initial time instant includes φ_0 (the angular position of the spacecraft in the reference circular orbit), Δv_0 (the impulse value), ψ_0 and θ_0 (the two angles specifying the direction of the first impulse), and $p_x(0_+)$, $p_y(0_+)$, $p_z(0_+)$, $p_{vx}(0_+)$, $p_{vy}(0_+)$, and $p_{vz}(0_+)$ (the six values of the conjugate variables). The angular position φ_0 allows determining the spacecraft coordinates and velocity in the reference orbit before the impulse action according to the formulas from [39]. At the time instants τ_k $(k = 1, 2, 3)$ of the impulse actions, which transfer the spacecraft to the AFT release orbit, the safe orbit, and the target orbit, respectively, the shooting parameter vectors include the impulse values Δv_k, the two angles ψ_k and θ_k specifying the impulse direction, and the six values $p_x(\tau_{k+})$, $p_y(\tau_{k+})$, $p_z(\tau_{k+})$, $p_{vx}(\tau_{k+})$, $p_{vy}(\tau_{k+})$, and $p_{vz}(\tau_{k+})$ of the conjugate variables after the impulse actions. The spacecraft velocities after the impulse action are calculated by formulas (4.1)–(4.4). The shooting parameter vector also contains the durations $\Delta\tau_1$, $\Delta\tau_3$, and $\Delta\tau_4$ of the passive segments ($\Delta\tau_2$=120 s is the problem parameter) and the numerical Lagrange multipliers λ_{R0}, λ_{C0}, $\lambda_{R{\rm p}1}$, $\lambda_{R{\rm p}2}$, λ_{rv4}, $\lambda_{\rm FA}$, and λ_{zL}. Due to the continuity conditions, the spacecraft coordinates on a current integration segment are taken equal to those on the previous segment and are not included in the shooting parameter vector. In addition, the corresponding continuity conditions are not included in the residual vector.

The residual vector contains the following elements: the conditions on the radius of the perigee when the spacecraft is inserted into the AFT release and safe orbits (at the time instants τ_1 and τ_2, respectively); the constraint on the final ascent maneuver impulse at the end of the transfer; the conditions under which the spacecraft is in the apogee of the target orbit at the time instant τ_4 (the last impulse action to transfer the booster to the orbit touching the conditional boundary of the atmosphere); the zero value of the z-component of the Laplace vector at the time instant τ_4; 12 transversality conditions at the initial and terminal time instants; 9 implications of the transversality conditions on the spacecraft coordinates, i.e., the jumps in the conjugate variables due to the limited radius of the perigee at the time instants τ_1 and τ_2, and the continuity conditions of the conjugate variables at the time instant τ_3; 18 transversality conditions on the spacecraft velocities before and after the impulse action at the time instants τ_1, τ_2, and τ_3; the implication of the stationarity conditions at the time instants τ_1 and τ_2; the continuity of the Hamiltonian at the time instant τ_3; the zero value of the Hamiltonian at the time instant τ_4.

5. RESULTS

Let us present the extremal under the following problem parameters:

$$\mu = 398\,601.19 \text{ km}^3/\text{s}^2,\ R_{\text{Ear}} = 6378.25 \text{ km},\ h_0 = 200 \text{ km},$$
$$R_{\max} = 280\,000 \text{ km},\ R_{\text{GEO}} = 42\,164 \text{ km},\ P_{\text{spe}} = 350 \text{ s},\ g_{\text{Ear}} = 9.80665 \text{ m/s}^2,$$
$$i_0 = 0.9 \text{ rad},\ \alpha = 0.08,\ \Delta v^* = 1.5 \text{ km/s},\ \Delta\tau_2 = 120 \text{ s}.$$

The main units of measurement used in the calculations are 1000 km and 1 h. In the case of transition to other units, the conjugate variables must be recalculated using the appropriate formulas. For numerical integration, we employed the 8(7)th order Dorman–Prince method. All values (the shooting parameters, the numerical Lagrange multipliers, and the state and conjugate variables) at the initial and terminal points of the passive segments are given with an accuracy necessary for repeated calculations. Their refinement may require one or two iterations of Newton's method. The matching of the state and conjugate variables at the initial and terminal points of the passive segments can be verified by numerical integration. The transversality and stationarity conditions can be verified using the numerical–analytical differentiation technique.

At the initial time instant, in the reference orbit at the point corresponding to the angular position $\varphi_0 = 0.000307492$ rad of the spacecraft, the impulse action $\Delta v_0 = 1.790280$ km/s is applied in the direction specified by the two angles $\psi_0 = 1.570796511$ rad and $\theta_0 = -0.031065593$ rad. The coordinates and velocities of the spacecraft and the conjugate variables after the first impulse action are as follows:

$$x(0_+) = 6578.250 \text{ km},\ y(0_+) = 1.257 \text{ km},\ z(0_+) = 1.584 \text{ km},$$
$$v_x(0_+) = -0.002944 \text{ km/s},\ v_y(0_+) = 5.994615 \text{ km/s},\ v_z(0_+) = 7.464706 \text{ km/s},$$
$$p_x(0_+) = 0.089295766,\ p_y(0_+) = 1.813741405 \times 10^{-5},\ p_z(0_+) = 2.062570174 \times 10^{-5},$$
$$p_{vx}(0_+) = -1.047893452 \times 10^{-5},\ p_{vy}(0_+) = 0.022000079,\ p_{vz}(0_+) = 0.026020930.$$

The duration of the first passive segment is $\Delta\tau_1 = 7778.265$ s. The coordinates and velocities of the spacecraft and the conjugate variables before the impulse action transferring the spacecraft to the AFT release orbit are as follows:

$$x(\tau_{1-}) = -20\,417.506 \text{ km},\ y(\tau_{1-}) = 44.699 \text{ km},\ z(\tau_{1-}) = 55.603 \text{ km},$$
$$v_x(\tau_{1-}) = -0.023113 \text{ km/s},\ v_y(\tau_{1-}) = -1.931335 \text{ km/s},\ v_z(\tau_{1-}) = -2.404967 \text{ km/s},$$
$$p_x(\tau_{1-}) = 0.009217659,\ p_y(\tau_{1-}) = -7.999319341 \times 10^{-5},\ p_z(\tau_{1-}) = -0.000116968,$$
$$p_{vx}(\tau_{1-}) = 5.397932380 \times 10^{-5},\ p_{vy}(\tau_{1-}) = 0.019187690,\ p_{vz}(\tau_{1-}) = 0.028158871.$$

At the time instant τ_1 in the neighborhood of the apogee of the first exchange orbit, the impulse action $\Delta v_1 = 0.017905$ km/s is applied in the direction specified by the two angles $\psi_1 = -1.568891340$ rad and $\theta_1 = 0.078465462$ rad. The coordinates and velocities of the spacecraft and the conjugate variables after the impulse action transferring the spacecraft to the AFT release orbit are as follows:

$$x(\tau_{1+}) = -20\,417.506 \text{ km},\ y(\tau_{1+}) = 44.699 \text{ km},\ z(\tau_{1+}) = 55.603 \text{ km},$$
$$v_x(\tau_{1+}) = -0.023085 \text{ km/s},\ v_y(\tau_{1+}) = -1.921253 \text{ km/s},\ v_z(\tau_{1+}) = -2.390170 \text{ km/s},$$
$$p_x(\tau_{1+}) = -0.022777483,\ p_y(\tau_{1+}) = 9.226665723 \times 10^{-5},\ p_z(\tau_{1+}) = 9.732570693 \times 10^{-5},$$
$$p_{vx}(\tau_{1+}) = -0.000150718,\ p_{vy}(\tau_{1+}) = -0.022960891,\ p_{vz}(\tau_{1+}) = -0.024276861.$$

The duration of the second passive segment is $\Delta\tau_2 = 120$ s (the problem parameter). The coordinates and velocities of the spacecraft and the conjugate variables before the impulse action transferring the spacecraft to the safe orbit are as follows:

$$x(\tau_{2-}) = -20\,413.392 \text{ km},\ y(\tau_{2-}) = -185.840 \text{ km},\ z(\tau_{2-}) = -231.204 \text{ km},$$
$$v_x(\tau_{2-}) = 0.091656 \text{ km/s},\ v_y(\tau_{2-}) = -1.920856 \text{ km/s},\ v_z(\tau_{2-}) = -2.389677 \text{ km/s},$$
$$p_x(\tau_{2-}) = -0.022775591,\ p_y(\tau_{2-}) = -0.000372271,\ p_z(\tau_{2-}) = -0.000393840,$$
$$p_{vx}(\tau_{2-}) = 0.000608516,\ p_{vy}(\tau_{2-}) = -0.022956224,\ p_{vz}(\tau_{2-}) = -0.024271920.$$

At the time instant τ_2 the impulse action $\Delta v_2 = 0.017910$ km/s is applied in the direction specified by the two angles $\psi_2 = 1.574537652$ rad and $\theta_2 = 0.080471408$ rad. The coordinates and velocities of the spacecraft and the conjugate variables after the impulse action transferring the spacecraft to the safe orbit are as follows:

$$x(\tau_{2+}) = -20\,413.392 \text{ km},\ y(\tau_{2+}) = -185.840 \text{ km},\ z(\tau_{2+}) = -231.204 \text{ km},$$
$$v_x(\tau_{2+}) = 0.091982 \text{ km/s},\ v_y(\tau_{2+}) = -1.933160 \text{ km/s},\ v_z(\tau_{2+}) = -2.402686 \text{ km/s},$$
$$p_x(\tau_{2+}) = -0.001946427,\ p_y(\tau_{2+}) = 7.114712350 \times 10^{-5},\ p_z(\tau_{2+}) = 0.000157508,$$
$$p_{vx}(\tau_{2+}) = 5.197160474 \times 10^{-5},\ p_{vy}(\tau_{2+}) = 0.004386100,\ p_{vz}(\tau_{2+}) = 0.009711318.$$

The duration of the third passive segment is $\Delta\tau_3 = 7707.227$ s. The coordinates and velocities of the spacecraft and the conjugate variables before the impulse action transferring the spacecraft to the target orbit are as follows:

$$x(\tau_{3-}) = 6578.250 \text{ km},\ y(\tau_{3-}) = -0.053 \text{ km},\ z(\tau_{3-}) = 0.007 \text{ km},$$
$$v_x(\tau_{3-}) = 4.743611 \times 10^{-5} \text{ km/s},\ v_y(\tau_{3-}) = 6.001513 \text{ km/s},\ v_z(\tau_{3-}) = 7.459164 \text{ km/s},$$
$$p_x(\tau_{3-}) = 0.102240000,\ p_y(\tau_{3-}) = -8.659081488 \times 10^{-7},\ p_z(\tau_{3-}) = 9.583528338 \times 10^{-8},$$
$$p_{vx}(\tau_{3-}) = 1.591310975 \times 10^{-7},\ p_{vy}(\tau_{3-}) = 0.021609646,\ p_{vz}(\tau_{3-}) = 0.025485443.$$

At the time instant τ_3 in the neighborhood of the perigee of the safe orbit, the impulse action $\Delta v_3 = 1.278611$ km/s is applied in the direction specified by the two angles $\psi_3 = 1.570795995$ rad and $\theta_3 = -0.025757010$ rad. The coordinates and velocities of the spacecraft and the conjugate variables after the impulse action transferring the spacecraft to the target orbit are as follows:

$$x(\tau_{3+}) = 6578.250 \text{ km},\ y(\tau_{3+}) = -0.053 \text{ km},\ z(\tau_{3+}) = 0.007 \text{ km},$$
$$v_x(\tau_{3+}) = 5.352540 \times 10^{-5} \text{ km/s},\ v_y(\tau_{3+}) = 6.828426 \text{ km/s},\ v_z(\tau_{3+}) = 8.434388 \text{ km/s},$$
$$p_x(\tau_{3+}) = 0.102240000,\ p_y(\tau_{3+}) = -8.659081482 \times 10^{-7},\ p_z(\tau_{3+}) = 9.583528330 \times 10^{-8},$$
$$p_{vx}(\tau_{3+}) = 1.591310971 \times 10^{-7},\ p_{vy}(\tau_{3+}) = 0.021609646,\ p_{vz}(\tau_{3+}) = 0.025485443.$$

The duration of the fourth passive segment is $\Delta\tau_4 = 197\,878.402$ s. The satellite is undocked from the booster in the target orbit. The last braking impulse is applied at the point corresponding to the apogee of the target orbit. The coordinates and velocities of the spacecraft and the conjugate variables before the impulse action transferring the booster to the orbit touching the conditional boundary of the atmosphere are as follows:

$$x(\tau_{4-}) = -226\,432.098 \text{ km},\ y(\tau_{4-}) = 2.026 \text{ km},\ z(\tau_{4-}) = -6.115 \times 10^{-10} \text{ km},$$
$$v_x(\tau_{4-}) = -1.775160 \times 10^{-6} \text{ km/s},\ v_y(\tau_{4-}) = -0.198378 \text{ km/s},\ v_z(\tau_{4-}) = -0.245034 \text{ km/s},$$
$$p_x(\tau_{4-}) = -9.581390462 \times 10^{-5},\ p_y(\tau_{4-}) = 8.230843826 \times 10^{-10},\ p_z(\tau_{4-}) = -1.214480715 \times 10^{-9},$$
$$p_{vx}(\tau_{4-}) = -2.090933500 \times 10^{-7},\ p_{vy}(\tau_{4-}) = -0.022151189,\ p_{vz}(\tau_{4-}) = 0.014169354.$$

The numerical Lagrange multipliers are as follows:

$$\lambda_{R0} = 0.017815018,\ \lambda_{C0} = -7.520663278 \times 10^{-9},\ \lambda_{R\text{p}1} = 0.043517462,$$
$$\lambda_{R\text{p}2} = -0.027848883,\ \lambda_{rv4} = 1.997985665 \times 10^{-15},\ \lambda_{\text{FA}} = 0.033387259,$$
$$\lambda_{zL} = -5.445119694 \times 10^{-11},\ \lambda_{x1} = 0.009217659,\ \lambda_{y1} = -7.999319341 \times 10^{-5},$$
$$\lambda_{z1} = -0.000116968,\ \lambda_{x2} = -0.022775591,\ \lambda_{y2} = -0.000372271,$$
$$\lambda_{z2} = -0.000393840,\ \lambda_{x3} = 0.102240000,\ \lambda_{y3} = -8.659081488 \times 10^{-7},$$
$$\lambda_{z3} = 9.583528338 \times 10^{-8},\ \lambda_{\tau 1} = 0,\ \lambda_{\tau 2} = 0,\ \lambda_{\tau 3} = 0,\ \lambda_{\tau 12} = 0.$$

6. CONCLUSIONS

The methodology adopted in this paper—the simultaneous selection of a computational scheme of the shooting method and a good initial approximation of the corresponding shooting parameters based on the problem solved previously in a simpler statement—has turned out to be effective: the extremals have been successfully constructed. The problem presented in this paper and its solution, being another step in the problem hierarchy methodology, have served for optimizing the spacecraft transfer with a large limited thrust (not in the pulse statement). In future work, this methodology will be used to construct an extremal in the similar problem with perturbations due to the noncentral nature of the gravitational field of the Earth, the resistance of the atmosphere, and the attraction of other celestial bodies.

Thus, we have found an extremal without verifying its optimality based on higher-order conditions or the global optimality theorem. Additional research is required here.

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