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First- and Second-Order Necessary Optimality Conditions for a Control Problem Described by Nonlinear Fractional Difference Equations

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Abstract—This paper considers an optimal control problem for an object described by a system of nonlinear fractional difference equations. Such problems are a discrete analog of optimal control problems described by fractional ordinary differential equations. The first and second variations of a performance criterion are calculated using a modification of the increment method under the assumption that the control set is open. We establish a first-order necessary optimality condition (an analog of the Euler equation) and a general second-order necessary optimality condition. Adopting the representations of the solution of the linearized fractional difference equations from the general second-order optimality condition, we derive necessary optimality conditions in terms of the original problem parameters. Finally, with a special choice of an admissible variation of control, we formulate a pointwise necessary optimality condition for classical extremals.

Keywords: admissible control, optimal control, open set, fractional difference equation, fractional operator, fractional sum, analog of the Euler equation

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1. INTRODUCTION

Fractional calculus plays an important role in many fields of science and technology. As is known, fractional integro-differential calculus originates from the meaningful discussions of the derivative of order $\frac{1}{2}$ in the correspondence between G. de l'Hôpital and G. Leibniz; for example, see [1]. But the idea of using fractional difference has appeared rather recently. In this context, we mention, e.g., the papers [2, 5] and the books [3, 4].

Fractional calculus also finds application in optimal control problems described by fractional difference equations [6–8].

In view of theoretical and practical applications, it is of current interest to elaborate a qualitative theory of optimal control problems described by various fractional difference equations. Note that the theory of necessary optimality conditions for optimal control problems described by fractional difference equations is still underdeveloped.

In light of the foregoing, this paper is devoted to one optimal control problem described by a system of fractional difference equations [2, 3]. Assuming the openness of the control set, we establish an analog of the Euler equation [9, 10] and second-order necessary optimality conditions.

2. PRELIMINARIES AND AUXILIARY RESULTS

In this section, we recall some concepts and definitions that will be used in the considerations below.

The following standard definitions [3–6] underlie fractional calculus.

Let N denote the set of natural numbers together with zero. For $a \in Z$, we introduce the notations $N_a^+ = \{a, a+1, a+2, \ldots, \}$, $\sigma(t) = t+1$, and $\rho(t) = t-1$.

Definition 1. A fractional sum of order α is given by

$$\Delta^{-\alpha} u(n) = \sum_{j=0}^{n-1} \binom{j+\alpha-1}{j} u(n-j) = \sum_{j=0}^{n-1} \binom{n-j+\alpha-1}{n-j} u(j),$$

whereas a fractional operator of order α is given by

$$\Delta^{\alpha} u(n) = \sum_{j=0}^{n-1} \binom{j+\alpha-1}{j} \Delta u(n-j)$$

$$= \sum_{j=1}^{n} \binom{n-j-\alpha-1}{n-j} u(j) - \binom{n-\alpha-1}{n-1} u(0).$$

In these expressions, the binominal coefficient $\binom{a}{n}$ has the form

$$\binom{a}{n} = \begin{cases} \dfrac{\Gamma(a+1)}{\Gamma(a-n+1)\,\Gamma(n+1)}, & n > 0 \\ 1, & n = 0 \\ 0, & n < 0. \end{cases}$$

For $x, y \in R$, let $x^{(y)} = \frac{\Gamma(x+1)}{\Gamma(x+1-y)}$, where Γ stands for the gamma function. As is well known, it satisfies the identity

$$\Gamma(x+1) = x\Gamma(x).$$

Note that the fractional sum and operator of order α can be alternatively defined as follows.

Consider an arbitrary real number a and a number $b = k + a$, where $k \in N,\ k \geq 2$; for these numbers, let $T = \{a, a+1, \ldots, b\}$, $T^k = \{a, a+1, \ldots, b-1\}$, denoting by T the set of functions with the domain T.

Definition 2. For $f \in \mathrm{T}$, the left and right fractional sums of order $\alpha > 0$ are given by

$${}_a\Delta_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{(\alpha-1)} f(s),$$

$${}_t\Delta_b^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t+\alpha}^{b} (s-\sigma(t))^{(\alpha-1)} f(s),$$

respectively.

Definition 3. Let $0 < \alpha \leq 1$ and $\mu = 1 - \alpha$. For a function $f \in \mathrm{T}$, the left and right fractional operators of order α are given by

$${}_a\Delta_t^{\alpha} f(t) = \Delta\left({}_a\Delta_t^{-\mu} f(t)\right),$$

$${}_t\Delta_b^{\alpha} f(t) = -\Delta\left({}_t\Delta_b^{-\mu} f(t)\right),$$

respectively.

Here are some common properties of fractional sums and differences:

1. $\Delta^{\alpha}\Delta^{\beta}f(t) = \Delta^{\alpha+\beta}f(t)$;
2. $\Delta^{-\alpha}\Delta^{\alpha}f(t) = f(t) - f(0)$;
3. $\Delta^{\alpha}\Delta^{-\alpha}f(t) = f(t)$;
4. $\Delta^{\alpha}f(0) = 0$ and $\Delta^{\alpha}f(1) - f(0) = \Delta f(1)$.

The following result is true; for example, see [6].

Theorem 1 (on fractional summation by parts). *Let f and g be nonnegative real-valued functions with the domains T^k and T, respectively. If $0 < \alpha \le 1$ and $\mu = 1 - \alpha$, then*

$$\sum_{t=a}^{b-1} f(t)_a\Delta_t^{\alpha}g(t) = f(b-1)g(b) - f(a)g(a) + \sum_{t=a}^{b-2} {}_t\Delta_b^{\alpha}f(t)g^{\sigma}(t)$$
$$+ \frac{\mu}{\Gamma(\mu+1)}g(a)\left(\sum_{t=a}^{b-1}(t+\mu-\alpha)^{(\mu-1)}f(t) - \sum_{t=\sigma(a)}^{b-1}(t+\mu-\sigma(\alpha))^{(\mu-1)}f(t)\right).$$

Consider a system of linear inhomogeneous fractional difference equations

$$\Delta^{\alpha}y(t+1) = A(t)y(t) + g(t) \tag{1}$$

with initial conditions

$$y(t) = y_0. \tag{2}$$

Here, the notations are as follows: $y = (y_1, \ldots, y_n)'$ is an n-dimensional column vector; $g = (g_1, \ldots, g_n)'$ is a given n-dimensional vector; $y_0 = (y_{1_0}, \ldots, y_{n_0})'$ is a given constant column vector; t_0 and t_1 are given numbers; finally, $A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{pmatrix}$ is a given discrete matrix function of dimensions $n \times n$.

Problem (1)–(2) is a discrete analog of the Cauchy problem for a system of linear inhomogeneous fractional differential equations.

Theorem 2 [2]. *The solution $y(t)$ of the system of linear inhomogeneous fractional difference equations (1)–(2) can be represented as*

$$y(t) = y_0 \prod_{j=t_0}^{t-1}\left[1 + R_{\alpha}(t-1,j)A(j)\right]$$
$$+ \sum_{j=t_0}^{t-1} R_{\alpha}(t-1,j)f(j)\prod_{k=j+1}^{t-1}\left[1 + R_{\alpha}(t-1,k)A(k)\right].$$

In this formula,

$$R_{\alpha}(t,j) = \begin{pmatrix} t-j+\alpha-1 \\ t-j \end{pmatrix}.$$

3. OPTIMAL CONTROL: PROBLEM STATEMENT

Consider the following problem: minimize a terminal performance criterion

$$S(u) = \varphi\left(x\left(t_1\right)\right) \tag{3}$$

subject to constraints

$$u(t) \in U \subset R^r, \quad t \in T = \{t_0, t_0 + 1, \ldots, t_1 - 1\}, \tag{4}$$

$$\Delta^\alpha x\left(t+1\right) = f\left(t, x(t), u(t)\right), \quad t \in T, \tag{5}$$

$$x\left(t_0\right) = x_0. \tag{6}$$

Here, the notations are as follows: $x(t)$ is the n-dimensional state vector; $u(t)$ is the r-dimensional discrete control vector; U is a given non-empty, bounded, and open set; the numbers t_0 and t_1 and the constant vector x_0 are known; $f\left(t, x, u\right)$ is a given n-dimensional vector function that is jointly continuous together with its partial derivatives with respect to (x, u) up to the second order inclusive; $\varphi\left(x\right)$ is a given twice continuously differentiable scalar function; finally, $\Delta^\alpha x(t)$, $0 < \alpha \leq 1$, is a fractional operator of order α [11, 12].

A control function is called an admissible control if it satisfies the constraint (4).

Assume that for each given admissible control, the discrete analog of the Cauchy problem (problem (5)–(6)) has a unique solution.

An admissible control $u(t)$ minimizing the performance criterion (3) under the constraints (4)–(6) is called the optimal control, and the pair $(u(t), x(t))$ is called the optimal process.

4. THE INCREMENT FORMULA FOR THE PERFORMANCE CRITERION

Let $(u(t), x(t))$ and $(\overline{u}(t) = u(t) + \Delta u(t), \overline{x}(t) = x(t) + \Delta x(t))$ be fixed and arbitrary admissible processes, respectively.

We introduce the following notations:

$$\begin{aligned}
&H(t, x, u, \psi) = \psi'(t) f(t, x, u),\\
&H_x\left[t\right] \equiv H_x(t, x(t), u(t), \psi(t)),\\
&H_{xx}\left[t\right] \equiv H_{xx}(t, x(t), u(t), \psi(t)),\\
&H_u\left[t\right] \equiv H_u(t, x(t), u(t), \psi(t)),\\
&f_x\left[t\right] \equiv f_x(t, x(t), u(t)),\\
&f_u\left[t\right] \equiv f_u(t, x(t), u(t)).
\end{aligned}$$

Here, $\psi(t)$ is an unknown (as yet) n-dimensional column vector and $H\left(t, x, u, \psi\right)$ the Hamilton–Pontryagin function for the optimal control problem (3)–(6).

Applying a scheme similar to [11, 12], we obtain

$$\begin{aligned}
\Delta S(u) &= \varphi\left(x\left(t_1\right) + \Delta x\left(t_1\right)\right) - \varphi\left(x\left(t_1\right)\right)\\
&+\psi'\left(t_1 - 1\right) \Delta x\left(t_1\right) + \sum_{t=t_0}^{t_1-2} {}_t\Delta^\alpha{}_{\rho(t_1)} \psi\left(t-1\right) \Delta x(t)\\
&- \sum_{t=t_0}^{t_1-1} \left[H\left(t, \overline{x}(t), \overline{u}(t), \psi(t)\right) - H\left(t, x(t), u(t), \psi(t)\right)\right].
\end{aligned} \tag{7}$$

The representation (7) will serve for proving a first-order necessary optimality condition.

Under the assumptions above, using the Taylor expansion, the increment formula (7) for the performance criterion $S(u)$ with the admissible controls $\overline{u}(t)$ and $u(t)$ can be written as

$$\begin{aligned}\Delta S(u) &= \varphi_x(x(t_1))\Delta x(t_1) + \frac{1}{2}\Delta x'(t_1)\varphi_{xx}(x(t_1))\Delta x(t_1) \\ &+ \psi'(t_1-1)\Delta x(t_1) + \sum_{t=t_0}^{t_1-1}\psi'(t-1)\Delta x(t) \\ &- \sum_{t=t_0}^{t_1-2} {}_t\Delta^{\alpha}{}_{\rho(t_1)}\psi'(t-1)\Delta x(t) - \sum_{t=t_0}^{t_1-1}\left[H'_x[t]\Delta x(t) + H'_u[t]\Delta u(t)\right] \\ &- \frac{1}{2}\sum_{t=t_0}^{t_1-1}\left[\Delta x'(t)H_{xx}[t]\Delta x(t) + \Delta x'(t)H_{xu}[t]\Delta u(t)\right. \\ &\left. + 2\Delta u'(t)H_{ux}[t]\Delta x(t) + \Delta u'(t)H_{uu}[t]\Delta u(t)\right] \\ &+ o_1\left(\|\Delta x(t_1)\|^2\right) - \sum_{t=t_0}^{t_1-1} o_2[\|\Delta x(t)\| + \|\Delta u(t)\|]^2. \end{aligned} \tag{8}$$

Here, $\|\alpha\|$ is the norm of the vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ given by $\|\alpha\| = \sum_{i=1}^{n}|\alpha_i|$ and $o(\alpha)$ means the terms of a higher order of smallness than α, i.e., $o(\alpha) \to 0$ as $\alpha \to 0$.

Assume now that $\psi(t)$ is the solution of the following system of linear fractional difference equations:

$$\begin{cases} {}_t\Delta^{\alpha}{}_{\rho(t_1)}\psi'(t-1) = H_x[t], \quad t = t_1-1, t_1-2, \ldots, t_0 \\ \psi(t_1-1) = -\varphi_x(x(t_1)). \end{cases} \tag{9}$$

System (9) is said to be conjugate for problem (3)–(6). Due to (9), the increment formula (8) turns into

$$\begin{aligned}\Delta S(u) &= \frac{1}{2}\Delta x'(t_1)\varphi_{xx}(x(t_1))\Delta x(t_1) - \sum_{t=t_0}^{t_1-1} H'_u[t]\Delta u(t) \\ &- \frac{1}{2}\sum_{t=t_0}^{t_1-1}\left[\Delta x'(t)H_{xx}[t]\Delta x(t) + \Delta x'(t)H_{xu}[t]\Delta u(t)\right. \\ &\left. + 2\Delta u'(t)H_{ux}[t]\Delta x(t) + \Delta u'(t)H_{uu}[t]\Delta u(t)\right] \\ &+ o_1\left(\|\Delta x(t_1)\|^2\right) - \sum_{t=t_0}^{t_1-1} o_2[\|\Delta x(t)\| + \|\Delta u(t)\|]^2. \end{aligned} \tag{10}$$

Since the set U is open, we may define a special increment of the admissible control $u(t)$ as

$$\Delta u_{\varepsilon}(t) = \varepsilon\delta u(t). \tag{11}$$

Here, ε is a sufficiently small number by absolute value and $\delta u(t)$ is an arbitrary r-dimensional vector function with the domain R^r.

We denote by $\Delta x_\varepsilon(t)$ a special increment of the admissible trajectory $x(t)$ that corresponds to the special increment (11) of the control $u(t)$.

According to [12],

$$\|\Delta x(t)\| \le L_1 \prod_{j=t_0}^{t-1} (1 + A_\alpha(t,j)\,\|\Delta u(j)\|), \quad t \in T \cup t_1, \quad L_1 = \text{const} > 0.$$

Due to this upper bound,

$$\|\Delta x_\varepsilon(t)\| \le L_2\varepsilon, \quad t \in T \cup t_1, \quad L_2 = \text{const} > 0. \tag{12}$$

In view of formulas (11) and (12), we can employ, e.g., the scheme from [9, 13] to establish the following result: the special increment $\Delta x_\varepsilon(t)$ of the trajectory $x(t)$ has the expansion

$$\Delta x_\varepsilon(t) = \varepsilon\delta x(t) + o\left(\varepsilon; t\right), \tag{13}$$

where $\delta x(t)$ is the n-dimensional vector function satisfying the variational equation

$$\Delta^\alpha \delta x\left(t+1\right) = f_x\left[t\right]\delta x(t) + f_u\left[t\right]\delta u(t) \tag{14}$$

with the initial condition

$$\delta x\left(t_0\right) = 0. \tag{15}$$

Considering the expressions (11)–(15), from the increment formula (10) we obtain

$$\Delta S_\varepsilon(u) = S(u+\varepsilon\delta u) - S(u)$$

$$= -\sum_{t=t_0}^{t_1-1} H_u'\left[t\right]\varepsilon\delta u(t) + \frac{1}{2}(\varepsilon\delta x(t_1) + o(\varepsilon;t_1))'\varphi_{xx}(x(t_1))(\varepsilon\delta x(t_1) + o(\varepsilon;t_1))$$

$$-\frac{1}{2}\sum_{t=t_0}^{t_1-1}\Big[(\varepsilon\delta x(t) + o(\varepsilon;t))' H_{xx}\left[t\right](\varepsilon\delta x(t) + o(\varepsilon;t))$$

$$+ 2\varepsilon\delta u(t)' H_{ux}\left[t\right](\varepsilon\delta x(t) + o(\varepsilon;t)) + \varepsilon^2\delta u(t)'(t) H_{uu}\left[t\right]\delta u(t)\Big] + o(\varepsilon^2)$$

$$= -\varepsilon\sum_{t=t_0}^{t_1-1} H_u'\left[t\right]\delta u(t) + \frac{\varepsilon^2}{2}\delta x'(t_1)\varphi_{xx}(x(t_1))\delta x(t_1)$$

$$-\frac{\varepsilon^2}{2}\sum_{t=t_0}^{t_1-1}\Big[\delta x'(t) H_{xx}\left[t\right]\delta x(t) + 2\varepsilon\delta u'(t) H_{ux}\left[t\right]\delta x(t) + \delta u'(t) H_{uu}\left[t\right]\delta u(t)\Big] + o(\varepsilon^2). \tag{16}$$

5. NECESSARY OPTIMALITY CONDITIONS

The special second-order expansion (16) of the performance criterion allows establishing first- and second-order necessary optimality conditions.

The following result is well-known from classical variational calculus; for example, see [9, 10]. If the expansion

$$S(u+\varepsilon\delta u) - S(u) = \varepsilon A_1 + \frac{\varepsilon^2}{2}A_2 + o\left(\varepsilon^2\right) \tag{17}$$

holds with some numbers A_1 and A_2 that are independent of ε, then these numbers are called the first and second variations, respectively, of the functional $S(u)$ at the point u and denoted by

$$A_1 = \delta^1 S(u, \delta u),$$
$$A_2 = \delta^2 S(u, \delta u).$$

According to this definition, due to the expansion (17), the first and second variations of the functional $S(u)$ are given by

$$\delta^1 S(u, \delta u) = -\sum_{t=t_0}^{t_1-1} H_u' [t] \, \delta u(t), \tag{18}$$

$$\delta^2 S(u, \delta u) = \delta x'(t_1) \varphi_{xx} (x(t_1)) \, \delta x(t_1)$$
$$- \sum_{t=t_0}^{t_1-1} \Big[\delta x'(t) H_{xx} [t] \, \delta x(t) + 2\varepsilon \delta u'(t) H_{ux} [t] \, \delta x(t) + \delta u'(t) H_{uu} [t] \, \delta u(t) \Big]. \tag{19}$$

In classical variational calculus, if a functional $S(u)$ achieves minimum at a point $u = u(t)$, then for any $\delta u(t)$:

—Its first variation at u vanishes, i.e.,

$$\delta^1 S(u, \delta u) = 0. \tag{20}$$

—Its second variation at u is nonnegative, i.e.,

$$\delta^2 S(u, \delta u) \geq 0. \tag{21}$$

Therefore, due to (20) and (21), we have

$$\sum_{t=t_0}^{t_1-1} H_u' [t] \, \delta u(t) = 0, \tag{22}$$

$$\delta x'(t_1) \varphi_{xx} (x(t_1)) \, \delta x(t_1)$$
$$- \sum_{t=t_0}^{t_1-1} \Big[\delta x'(t) H_{xx} [t] \, \delta x(t) + 2\delta u(t)' H_{ux} [t] \, \delta x'(t) + \delta u'(t) H_{uu} [t] \, \delta u(t) \Big] \geq 0 \tag{23}$$

along the optimal process $(u(t), x(t))$ for any $\delta u(t) \in R^r$, $t \in T$.

Evidently, identity (22) and inequality (23) are implicit necessary optimality conditions of the first and second order, respectively.

Nevertheless, they will serve for deriving constructively verifiable necessary optimality conditions of the first and second order as follows. Since $\delta u(t)$ is arbitrary, let

$$\delta u(t) = \begin{cases} v, & t = \theta \in T \\ 0, & t \neq \theta \in T, \end{cases} \tag{24}$$

where $\theta \in T$ and $v \in R^r$ is an arbitrary vector.

In view of (24) and (22), we obtain

$$H_u'[\theta]\, v = 0$$

for all $v \in R^r$ and $t = \theta \in T$.

Due to the arbitrariness of the vector v, the latter relation yields the identity

$$H_u[\theta] = 0. \tag{25}$$

Thus, we have arrived at the following result.

Theorem 3. *For an admissible control $u(t)$ to be optimal in problem* (3)–(6), *it is necessary to have the relation* (25) *for any $\theta \in T$.*

The relation (25) is an analog of the Euler equation for the optimal control problem under consideration. Clearly, the Euler equation is a constructively verifiable necessary optimality condition. In turn, inequality (23) is an implicit necessary optimality condition of the second order.

An admissible control $u(t)$ satisfying the Euler equation will be called the classical extremal.

As is well known, $\delta x(t)$ (variation of the trajectory) is the solution of problem (14)–(15). Hence, by Theorem 2, $\delta x(t)$ can be written as

$$\delta x(t) = \sum_{j=t_0}^{t-1} R_\alpha(t-1, j)\, f_u[j]\, \delta u(j) \prod_{k=j+1}^{t-1} [1 + R_\alpha(t-1,k)\, f_x[k]]. \tag{26}$$

Using the representation (26), we transform some terms in inequality (23).

It is obvious that

$$\delta x'(t_1)\, \varphi_{xx}(x(t_1))\, \delta x(t_1) = \sum_{\tau=t_0}^{t_1-1} \sum_{s=t_0}^{t_1-1} R_\alpha(t-1,\tau)\, f_u[\tau]\, \delta u(\tau)$$
$$\times \prod_{k=\tau+1}^{t_1-1} [1 + R_\alpha(t-1,k)\, f_x[k]]\, \varphi_{xx}(x(t_1))$$
$$\times R_\alpha(t-1,s)\, f_u[s]\, \delta u(s) \prod_{k=s+1}^{t_1-1} [1 + R_\alpha(t-1,k)\, f_x[k]], \tag{27}$$

$$\sum_{t=t_0}^{t_1-1} \delta x'(t) H_{xx}[t]\, \delta x(t)$$
$$= \sum_{t=t_0}^{t_1-1} \Bigg[\sum_{\tau=t_0}^{t_1-1} \sum_{s=t_0}^{t_1-1} R_\alpha(t-1,\tau) f_u[\tau] \delta u(\tau) \prod_{k=\max(\tau+1,s+1)}^{t_1-1} [1 + R_\alpha(t-1,\tau) f_x[\tau]]$$
$$\times H_{xx}[t]\, [1 + R_\alpha(t-1,s)\, f_x[s]]\, R_\alpha(t-1,s)\, f_u[s]\, \delta u(s) \Bigg], \tag{28}$$

$$\sum_{t=t_0}^{t_1-1} \delta u(t)' H_{ux}[t]\, \delta x'(t)$$
$$= \sum_{t=t_0}^{t_1-1} \delta u'(t) H_{ux}[t] \left[\sum_{\tau=t_0}^{t-1} R_\alpha(t-1,\tau) \prod_{k=\tau+1}^{t-1} [1 + R_\alpha(t-1,k) f_x[k]] f_u[\tau] \delta u(\tau) \right]. \tag{29}$$

Utilizing identities (27)–(29) in equality (23), we obtain

$$\sum_{\tau=t_0}^{t_1-1}\sum_{s=t_0}^{t_1-1} R_\alpha(t-1,\tau) f_u[\tau]\,\delta u(\tau) \prod_{k=\tau+1}^{t_1-1}\left[1+R_\alpha(t-1,k) f_x[k]\right]\varphi_{xx}(x(t_1))$$
$$\times R_\alpha(t-1,s) f_u[s]\,\delta u(s)\prod_{k=s+1}^{t_1-1}\left[1+R_\alpha(t-1,k) f_x[k]\right]$$
$$-\sum_{t=t_0}^{t_1-1}\Bigg[\sum_{\tau=t_0}^{t_1-1}\sum_{s=t_0}^{t_1-1} R_\alpha(t-1,\tau) f_u[\tau]\,\delta u(\tau)$$
$$\times \prod_{k=\max(\tau+1,s+1)}^{t_1-1}\left[1+R_\alpha(t-1,\tau) f_x[\tau]\right] H_{xx}[t]$$
$$\times\left[1+R_\alpha(t-1,s) f_x[s]\right] R_\alpha(t-1,s) f_u[s]\,\delta u(s)\Bigg]$$
$$+2\sum_{t=t_0}^{t_1-1}\delta u'(t) H_{ux}[t]\left[\sum_{\tau=t_0}^{t-1} R_\alpha(t-1,\tau)\prod_{k=\tau+1}^{t-1}\left[1+R_\alpha(t-1,k) f_x[k]\right] f_u[\tau]\delta u(\tau)\right]$$
$$+\sum_{t=t_0}^{t_1-1}\delta u'(t) H_{uu}[t]\,\delta u(t)\geq 0. \tag{30}$$

Let $M(\tau,s)$ be a matrix function of dimensions $(n\times n)$ given by

$$M(\tau,s) = -R_\alpha(t-1,\tau)\prod_{k=\tau+1}^{t_1-1}\left[1+R_\alpha(t-1,k) f_x[k]\right]\varphi_{xx}(x(t_1))$$
$$\times R_\alpha(t-1,\tau)\prod_{k=s+1}^{t_1-1}\left[1+R_\alpha(t-1,k) f_x[k]\right]$$
$$-R_\alpha(t-1,\tau)\prod_{k=\max(\tau+1,s+1)}^{t_1-1}\left[1+R_\alpha(t-1,\tau) f_x[\tau]\right] H_{xx}[t]\left[1+R_\alpha(t-1,s) f_x[s]\right]. \tag{31}$$

In view of formula (31), inequality (30) reduces to

$$2\sum_{t=t_0}^{t_1-1}\delta u'(t) H_{ux}[t]\sum_{\tau=t_0}^{t-1} R_\alpha(t-1,\tau)\prod_{k=\tau+1}^{t-1}\left[1+R_\alpha(t-1,k) f_x[k]\right] f_u[\tau]\delta u(\tau)$$
$$+\sum_{\tau=t_0}^{t_1-1}\sum_{s=t_0}^{t_1-1}\delta u(\tau) f_u'[\tau] M(\tau,s) f_u[s]\delta u(s) + \sum_{t=t_0}^{t_1-1}\delta u(t)' H_{uu}[t]\delta u(t)\leq 0. \tag{32}$$

Theorem 4 (second-order necessary optimality condition). *For the classical extremal to be optimal in problem* (3)–(6), *it is necessary to have inequality* (32) *for all* $\delta u(t)\in U$, $t\in T$, *where* $M(\tau,s)$ *is given by* (31).

This necessary optimality condition is quite general. Using the arbitrariness of the variations $\delta u(t)$ of control functions $u(t)$, we can obtain several even simpler optimality conditions from it.

For example, the result below is a direct consequence of Theorem 4.

Corollary. *For the classical extremal to be optimal in problem* (3)–(6), *it is necessary to have the inequality*

$$v' \left[f'_u [\theta] \, M(\theta, \theta) \, f_u [\theta] + H_{uu} [\theta] \right] v \leq 0$$

for all $v \in R^r$ *and* $\theta \in T$.

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