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Design of Nonlinear Selectively Invariant Control Systems Based on Quasilinear Models

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Abstract—An original analytical method is developed to design the selectively invariant control systems for nonlinear plants with differentiable nonlinearities. To solve the design problem, the method of designing nonlinear control systems is applied on the base of a quasilinear model of nonlinear plants and the internal models principle of external impacts is used, taking into account the requirements for the relative order of the control device and the fast response of the designed system. The system of linear algebraic equations is solved to determine the parameters of the nonlinear control device. The suggested method can be applied to design the control systems for nonlinear plants of various purposes, operating under conditions of regular external impacts of the known form.

Keywords: nonlinear plant, differentiable nonlinearity, quasilinear model, selectively invariant system, impact, spectrum, spectral model, stability, robustness

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1. INTRODUCTION

In practice, we often meet the plants exposed to the effect of regular external impacts of the known form. Such plants include electromechanical systems, electric and pneumatic drives, mobile robots, unmanned aerial vehicles, grain combines and many other plants [1–6]. Usually the control systems for these plants must provide a complete parry to the effect of these impacts in the steady-state mode. As is known, the most effective way to solve this problem is to ensure the invariance of control systems to the external impacts. However, the conditions for ensuring an absolute invariance are most often unattainable; therefore, the selective invariance is used, to ensure which the models of the external impacts are introduced into the system, which causes a significant increase in the order and complexity of the control device.

Traditionally, the design of the selectively invariant control systems is carried out on the basis of the linear models of the plants [1, 2, 7–12] and the internal models principle of the external impacts. In some works, the external impacts of the type under consideration are called "finite-dimensional" impacts [12, 13]; however, the design problem of the control systems is also solved on the basis of the internal models principle.

The increased requirements for the control systems quality lead to the need to apply the nonlinear models of the plants [13–19]. When using the well-known methods for the nonlinear control systems design, such as the plant model conversion to the Brunovsky canonical form, the state feedback linearization, backstepping, the passification and others, it is usually assumed that the plant's nonlinearities are differentiable, and their state variables are measurable. However, the application of these design methods for the nonlinear control systems is complicated by the necessity to bring the nonlinear plants models to the special forms, which requires searching the suitable nonlinear transformations.

The nonlinear systems exposed to the finite-dimensional external impacts are considered in the works of V.O. Nikiforov, A.A. Bobtsov and others (see [13]). The problem of parrying their effect on the system is also solved on the basis of the internal models principle using the Lyapunov function method, but under the condition that for the unperturbed plant the following are known: a) stabilizing control and b) the Lyapunov function, which allows us to prove the stability of equilibrium of a closed unperturbed system.

In the suggested approach to the design of the nonlinear selectively invariant control systems for the nonlinear plants, both the setting and disturbing external impacts are taken into account. To solve the problem, here we use the method for designing nonlinear control systems on the base of the quasilinear models (QLM) of the nonlinear plants, suggested in [20, 21]. In comparison with the above design methods for nonlinear systems [14–19] the advantage of this method is that to construct a QLM, only the differentiability of the plant's nonlinearities in all their arguments is required, and the parameters of the nonlinear control device are determined by the solution of a resolving system of linear algebraic equations (SLAE). The conditions for solving the considered problem of designing nonlinear selectively invariant control system are determined by the property of fullness (controllability and observability) of the "control-output" channel of the QLM of the nonlinear plant and the condition of disjointness of its transmission zeros and spectrums of external impacts.

Very often, the functional matrices of controllability or observability of the QLM of the control plant turn out to be nonsingular only in the bounded neighborhood of its equilibrium. In this case, the equilibrium of the designed nonlinear system is asymptotically locally stable [22, 23]. If the fullness conditions are satisfied in the entire state space of the plant and the feedbacks are chosen in such a way that the functional matrix of the resolving SLAE is also nonsingular in the entire state space of the plant, then the equilibrium of the closed system can be asymptotically globally stable. The latter can be established on the basis of a theorem proved in [20]. The QLM's matrices and vectors are functions of the state variables [20–23], but, as it turned out, this is not an obstacle to the analytical solution of the design problem the nonlinear selectively invariant control systems.

2. STATEMENT OF THE PROBLEM

A control system is called selectively invariant if it contains a model of the external impact, and a system error caused by this impact is zero in the steady-state mode [9–11]. Such a model is called exogenous [8, p. 168] or internal [10, 22]. The mathematical models of the impacts are homogeneous differential equations. They can be represented either by the operators of these equations, or by the corresponding equations in the Cauchy form (in state variables) [7–13]. Some features and examples of the external impact models are given in the Appendix. If there is an external impact model in a stable system, then as soon as this impact begins to affect the system, the model generates a signal that completely parries its effect in the steady-state mode. In this case, the corresponding initial conditions of this model are set automatically during the transient that occurs when this impact is applied to the system.

Considering the problem of designing the nonlinear selectively invariant control systems, for definiteness we will assume that the plant does not have internal models of the impacts or their separate components. We also assume that the nonlinearities of the plant are differentiable, and the state variables are measurable, which allows us to apply the method of designing the nonlinear control systems based on the QLM [20–22].

Let the QLM of the nonlinear plant in deviations from some steady-state mode has the following form

$$\dot{x} = A(x)x + b(x)u + b_f(x)f, \quad y = c^{\mathrm{T}}(x)x, \tag{1}$$

where $x \subset \mathrm{R}^n$ is state vector of the plant; u, y and f are scalar control action, controlled output variable and external unmeasured disturbance; $A(x)$ and $b(x)$, $b_f(x)$, $c(x)$ are known functional $n \times n$-matrix and n-vectors. In [22, 24], a method is given for constructing the quasilinear models of type (1) for the nonlinear plants given by the equations $\dot{x} = \varphi(x, u)$, $y = \psi(x)$, if $\varphi(\mathbf{0}, 0) = \mathbf{0}$, $\psi(\mathbf{0}) = 0$ and $\partial\varphi(x, u)/\partial u = \varphi'_u(x)$, i.e. under conditions that $\varphi(x, u)$ and $\psi(x)$ are functions differentiable with respect to all arguments; $x = \mathbf{0}$ is an equilibrium of plant (1); the partial derivative with respect to u of the vector-function $\varphi(x, u)$ does not depend on u. Here $\mathbf{0}$ is a zero n-vector.

Further, the full plants are considered, i.e. the plants, QLM (1) of which satisfies the conditions of controllability and observability:

$$\begin{aligned} &\left|\det\begin{bmatrix} b(x) & A(x)b(x) & \ldots & A^{n-1}(x)b(x)\end{bmatrix}\right| \geq \varepsilon_{\mathrm{c}} > 0, \\ &\left|\det\begin{bmatrix} c(x) & A^{\mathrm{T}}(x)c(x) & \ldots & (A^{\mathrm{T}}(x))^{n-1}c(x)\end{bmatrix}\right| \geq \varepsilon_{\mathrm{o}} > 0, \quad \forall\, x \subset \Omega_{\mathrm{CO}} \in \mathrm{R}^n, \end{aligned} \tag{2}$$

where $\varepsilon_{\mathrm{c}}, \varepsilon_{\mathrm{o}}$ are some constants; Ω_{CO} is some neighborhood of the point $x = \mathbf{0}$ [22].

The nonlinear control device (NCD) suggested in [20, 21] is used in the designed selectively invariant system. In this case, its equations have the following form:

$$\dot{z} = R(x)z + q(x)g - l(x)y - \sum_{i=1}^{q} l_i(x)\tilde{x}_i, \quad u = k^{\mathrm{T}}(x)z, \tag{3}$$

where $z \subset \mathrm{R}^r$ is state vector of the NCD, g is scalar setting impact of designed system; $R(x)$ and $q(x)$, $l(x)$, $l_i(x)$, $i = 1, \ldots, q$ are functional $r \times r$-matrix and r-vectors; q is number of state variables $\tilde{x}_i \in x$ used in NCD (3) and renumbered in the ascending order $\tilde{x}_1, \ldots \tilde{x}_q$, $q \leq n$. The value of r, the variables $\tilde{x}_i$ and the number q are determined during the formation of matrix G_y (22) of the resolving SLAE (see below). Equations (3) differ from those given in [20] only in the following: the setting impact and the controlled output variable are present here, and feedbacks are not introduced on all state variables.

Bearing in mind the design of nonlinear selectively invariant control systems (1)–(3), we will assume that the spectral models are known in the form of K_p-images of the setting impact $g = g(t)$ and the disturbance $f = f(t)$, i.e. the operators-polynomials $G(p)$ and $F(p)$ of the degrees $\nu_g = \deg G(p)$ and $\nu_f = \deg F(p)$ are known, where p is an operator d/dt, such that $G(p)g(t) \equiv 0$ and $F(p)f(t) \equiv 0$. Let a polynomial be $\Phi(p) = \mathrm{lcm}\{G(p)F(p)\}$, where the lcm is the least common multiple [10, 11]. The operator equation "input-output" of closed system (1), (3) with respect to the deviation $\varepsilon = g - y$ can be written as follows:

$$\begin{aligned} H(p, x)\,\varepsilon &= H_{\varepsilon g}(p, x)g - H_f(p, x)f, \\ H_{\varepsilon g}(p, x) &= H(p, x) - H_g(p, x), \end{aligned} \tag{4}$$

where $H(p, x)$, $H_g(p, x)$, $H_f(p, x)$ are some polynomials of p the coefficients of which are functions of the state variables x_i, $i = \overline{1, n}$ [20, 22]. The derivation of these polynomials from equations (1) and (3) is given in the Appendix. On the base of equation (4), the conditions for selective invariance of system (1), (3) with respect to the impacts $g = g(t)$ and $f = f(t)$ have the following form:

$$\begin{aligned} &H_{\varepsilon g}(p, x) = \tilde{H}_{\varepsilon g}(p, x)G(p), \\ &H_f(p, x) = \tilde{H}_f(p, x)F(p), \quad \forall\, x \subset \Omega_{\mathrm{CO}} \in \mathrm{R}^n, \end{aligned} \tag{5}$$

where $\tilde{H}_{\varepsilon g}(p,x)$, $\tilde{H}_f(p,x)$ are polynomials of the same type as in (4), but of lower degrees. In this case, the design problem has a solution if conditions (2) are met and

$$\begin{aligned} &\mathrm{GCD}\{B(p,x),\Phi(p)\} = \mathrm{const}, \\ &\mathrm{GCD}\{H(p,x),\Phi(p)\} = \mathrm{const}, \quad \forall\, x \subset \Omega_{\mathrm{CO}} \in \mathrm{R}^n. \end{aligned} \tag{6}$$

where GCD is greatest common divisor.

Conditions (2) and the first condition of (6) are necessary to solve the design problem, since they include the characteristics of the given plant and impacts. The physical meaning of the first condition of (6) lies in the spectrum disjointness of the impacts $g(t)$ and $f(t)$ with the zeros of the plant transmission over the channel $u \to y$, which makes possible to reproduce the setting impact $g(t)$ at the system output and parry the disturbance $f(t)$ [10]. The second condition of (6) is a condition of the spectrum disjointness of the impacts $g(t)$ and $f(t)$ with the roots of the closed system's characteristic polynomial. This condition is constructive and can always be satisfied if the above necessary solvability conditions are met.

Thus, to solve the design problem, it is necessary to choose the parameters of the functional matrices and vectors in (3) for satisfying the conditions of selective invariance (5), the conditions of stability, the conditions of the requirement to duration of the transients, and the conditions of physical feasibility, taking into account a relative order μ_{NCD} of the NCD [22, 25].

3. SOLUTION OF THE PROBLEM

If the control action u is excluded from equations (1), (3) and the resulting equations are written in a vector-matrix form, then the closed system QLM will have the form:

$$\dot{w} = H(x)w + h(x)g + h_f(x)f, \quad y = [c^{\mathrm{T}}(x) \quad \bar{\mathbf{0}}^{\mathrm{T}}]w, \tag{7}$$

where $w = [x^{\mathrm{T}} \quad z^{\mathrm{T}}]^{\mathrm{T}} \in \mathrm{R}^{\ell}$, $\ell = n + r$, $\bar{\mathbf{0}}$—zero r-vector,

$$H(x) = \begin{bmatrix} A(x) & b(x)k^{\mathrm{T}}(x) \\ -\Pi(x) & R(x) \end{bmatrix}, \quad h(x) = \begin{bmatrix} \mathbf{0} \\ q(x) \end{bmatrix}, \quad h_f(x) = \begin{bmatrix} b_f(x) \\ \bar{\mathbf{0}} \end{bmatrix}, \tag{8}$$

and $\Pi(x) = l(x)c^{\mathrm{T}}(x) + \sum\limits_{i=1}^{q} l_i(x)e_i$; e_i—ith row of the $n \times n$-identity matrix E.

It is shown in the Appendix that the equation "input-output" of the closed system follows from equations (7), taking into account (8),

$$H(p,x)y = H_g(p,x)g + H_f(p,x)f, \tag{9}$$

where

$$H(p,x) = A(p,x)R(p,x) + B(p,x)L(p,x) + \sum_{i=1}^{q} L_i(p,x)V_i(p,x), \tag{10}$$

$$H_g(p,x) = B(p,x)Q(p,x), \tag{11}$$

$$H_f(p,x) = B_f(p,x)R(p,x) + \sum_{i=1}^{q} L_i(p,x)\tilde{N}_i(p,x), \tag{12}$$

$$\tilde{N}_i(p,x) = \Big(B_f(p,x)V_i(p,x) - B(p,x)W_i(p,x)\Big)A^{-1}(p,x). \tag{13}$$

In expressions (10)–(13):

$$A(p,x) = \det[pE - A(x)],$$
$$B(p,x) = c^{\mathrm{T}}(x)\,\mathrm{adj}\,[pE - A(x)]\,b(x),$$
$$B_f(p,x) = c^{\mathrm{T}}(x)\,\mathrm{adj}\,[pE - A(x)]\,b_f(x); \tag{14}$$

$$R(p,x) = \det[pE - R(x)],$$
$$L(p,x) = k^{\mathrm{T}}(x)\,\mathrm{adj}\,[pE - R(x)]\,l(x),$$
$$Q(p,x) = k^{\mathrm{T}}(x)\,\mathrm{adj}\,[pE - R(x)]\,q(x); \tag{15}$$

$$V_i(p,x) = e_i\mathrm{adj}\,[pE - A(x)]\,b(x),$$
$$L_i(p,x) = k^{\mathrm{T}}(x)\,\mathrm{adj}\,[pE - R(x)]\,l_i(x),$$
$$W_i(p,x) = e_i\mathrm{adj}\,[pE - A(x)]\,b_f(x), \quad i = \overline{1,q}. \tag{16}$$

Note that in (13) there is an exact division by the polynomial $A(p,x)$. Let's turn to solving the above problems on choosing the parameters of equation (3).

Ensuring the selective invariance. According to the definition the system has this property if it contains the internal models of impacts. Under the statement of the problem, the plant does not contain them, so they must be introduced into the NCD. For this purpose, its characteristic polynomial is taken as $R(p,x) = \tilde{R}(p,x)\Phi(p)$. According to (4), the action $f(t)$ is multiplied by polynomial (12), equal to the sum of two summands; where the spectral model $F(p)$ there is in $R(p,x)$, so we assume that $L_i(p,x) = \tilde{L}_i(p,x)\Phi(p)$. In this case, in equation (4), the disturbance $f(t)$ will be multiplied by $F(p)$; thus, the second condition of (5) will be satisfied and the effect of $f(t)$ on the system error will be parried, since $F(p)f(t) \equiv 0$. Similarly, the setting impact $g(t)$ according to (4) is multiplied by the polynomial $H_{\varepsilon g}(p,x) = H(p,x) - H_g(p,x)$, therefore, for $R(p,x) = \tilde{R}(p,x)\Phi(p)$ and $L_i(p,x) = \tilde{L}_i(p,x)\Phi(p)$ to fulfill the first condition in (5), it is necessary that $L(p,x) - Q(p,x) = \tilde{Q}(p,x)G(p)$. Here $\tilde{R}(p,x)$, $\tilde{Q}(p,x)$ and $\tilde{L}_i(p,x)$ are some polynomials of lower degrees compared to the degrees of the polynomials $R(p,x)$, $Q(p,x)$ and $L_i(p,x)$, $i = \overline{1,\, q}$ respectively.

Ensuring the stability. For this purpose, according to the design method based on the QLM, the functional characteristic polynomial $H(p,x)$ of $\ell = n + r$ degree is replaced in (10) by a Hurwitz polynomial $H^*(p)$ of the same degree, the roots of which are constant, real, and various numbers [20, 22, 23]. As a result, taking into account the polynomials $R(p,x)$ and $\tilde{L}_i(p,x)$ chosen above, equality (10) takes the form:

$$H^*(p) = \bar{A}(p,x)\tilde{R}(p,x) + B(p,x)L(p,x) + \sum_{i=1}^{q}\bar{V}_i(p,x)\tilde{L}_i(p,x), \tag{17}$$

where $\bar{A}(p,x) = A(p,x)\Phi(p)$; $\bar{V}_i(p,x) = V_i(p,x)\Phi(p)$ are polynomials with the known coefficients.

The roots p_i^* of the polynomial $H^*(p)$ can be chosen using, in particular, the conditions:

$$\left|\mathrm{Re}(p_j^*)\right| \geq (5 \div 7)/t_s^*, \quad p_j^* = -\sigma_j^*, \quad \sigma_j^* > \varepsilon_\sigma > 0,$$
$$\left|\sigma_j^* - \sigma_\varsigma^*\right| \geq \Delta_\sigma > 0, \quad j \neq \varsigma, \quad j,\varsigma = \overline{1,\ell}, \tag{18}$$

here t_s^* is required duration of the transient [25]; ε_σ, Δ_σ are some numbers.

Ensuring the solvability of the design problem. Expression (17) is actually a polynomial equation with respect to the unknown polynomials $\tilde{R}(p,x) = \rho_0(x) + \rho_1(x)p + \ldots + \rho_{\tilde{r}}(x)p^{\tilde{r}}$, $L(p,x) = \lambda_0(x) + \lambda_1(x)p + \ldots + \lambda_l(x)p^l$ and $\tilde{L}_i(p,x) = \tilde{\lambda}_{i,0}(x) + \tilde{\lambda}_{i,1}(x)p + \ldots + \tilde{\lambda}_{i,\tilde{l}_i}(x)p^{\tilde{l}_i}$. According to [20, 22, 26], equation (17) is solved by turning to the SLAE equivalent to it:

$$G_y d = h_\gamma, \tag{19}$$

where the vectors d, h_γ are defined by the expressions:

$$d = [\lambda_0 \lambda_1 \ldots \lambda_l \;\; \tilde{\lambda}_{1,0}\; \tilde{\lambda}_{1,1} \ldots \tilde{\lambda}_{1,\tilde{l}_1} \;\; \ldots \;\; \tilde{\lambda}_{q,0}\; \tilde{\lambda}_{q,1} \ldots \tilde{\lambda}_{q,\tilde{l}_q} \;\; \rho_0\; \rho_1 \ldots \rho_{\tilde{r}}]^{\mathrm{T}}, \tag{20}$$

$$h_\gamma = [\delta_0^* \;\; \delta_1^* \;\; \ldots \;\; \delta_\ell^*]^{\mathrm{T}}, \tag{21}$$

δ_j^*—coefficients of the Hurwitz polynomial $H^*(p)$, and the matrix has the form

$$G_y = \left[\begin{array}{cccc|cccc|c}
\beta_0 & & & 0 & v_{10} & & & 0 & \\
\beta_1 & \beta_0 & & & v_{1,1} & v_{10} & & & \\
\vdots & \beta_1 & \ddots & & \vdots & v_{1,1} & \ddots & & \cdots \\
\beta_m & \vdots & \ddots & \beta_0 & v_{1,\varsigma_1} & \vdots & \ddots & v_{10} & \cdots \\
0 & \beta_m & \ddots & \beta_1 & & v_{1,\varsigma_1} & \ddots & v_{1,1} & \cdots \\
\vdots & 0 & \ddots & \vdots & & & \ddots & \vdots & \cdots \\
 & \vdots & \ddots & \beta_m & & & & v_{1,\varsigma_1} & \\
0 & & & & 0 & & & &
\end{array}\right.$$
$$\qquad\qquad l+1\text{—columns} \qquad\qquad \tilde{l}_1+1\text{—columns}$$

$$\left.\begin{array}{|cccc|cccc}
v_{q0} & & & 0 & \alpha_0 & & & 0 \\
v_{q1} & v_{q0} & & & \alpha_1 & \alpha_0 & & \\
\vdots & v_{q1} & \ddots & & \vdots & \alpha_1 & \ddots & \\
v_{q,\varsigma_q} & \vdots & \ddots & v_{q0} & \vdots & \vdots & \ddots & \alpha_0 \\
 & v_{q,\varsigma_q} & \ddots & v_{q1} & \alpha_{\tilde{n}} & \vdots & \ddots & \alpha_1 \\
 & & \ddots & \vdots & & \alpha_{\tilde{n}} & \ddots & \vdots \\
 & & & v_{q,\varsigma_q} & & & \ddots & \vdots \\
0 & & & & 0 & & & \alpha_{\tilde{n}}
\end{array}\right]. \tag{22}$$
$$\qquad\qquad \tilde{l}_q+1\text{—columns} \qquad \tilde{r}+1\text{—columns}$$

In expressions (19), (20) $G_y = G_y(x)$, $d = d(x)$ and in (22) $\beta_j = \beta_j(x)$, $v_{i,j} = v_{i,j}(x)$ and $\alpha_j = \alpha_j(x)$ are functional coefficients of the polynomials $B(p,x)$, $\bar{V}_i(p,x)$ and $\bar{A}(p,x)$ at p^j; for brevity, the argument x in (19), (20) and (22) is omitted.

To ensure the solvability of system (22), in equation (17) only those polynomials $\bar{V}_i(p,x)$, are taken into account for which the degree ℓ of the polynomial $H^*(p)$ will be minimal, the matrix G_y is square, and $\det G_y \neq 0$ [22, 26]. In [26], a method is given for determining the necessary degrees of the polynomials $\tilde{R}(p)$, $L(p)$ and $\tilde{L}_i(p)$ in the linear case, taking into account μ_{NCD}. However, this method can also be applied in the case of quasilinear models. Therefore, it is not considered here, but it will be illustrated below when solving a numerical example.

The solution of system (22) determines the polynomials $\tilde{R}(p)$, $L(p)$ and q of the polynomials $\tilde{L}_i(p,x)$, by which the characteristic polynomial of matrix $H(x)$ (8) is equal to the polynomial $H^*(p)$.

Implementation of the NCD. For this purpose, by turning to (3), the corresponding equation "input-output" of the NCD is written to the operator form:

$$R(p,x)u = Q(p,x)g - L(p,x)y - \sum_{i=1}^{q} L_i(p,x)\tilde{x}_i. \tag{23}$$

The polynomials $R(p,x)$ and $L_i(p,x)$ are found by the formulas $R(p,x) = \tilde{R}(p,x)\Phi(p)$, $L_i(p,x) = \tilde{L}_i(p,x)\Phi(p)$, and the polynomial $L(p,x)$ is determined by the solution of system (22). The polynomial $Q(p,x)$ of degree $\kappa = \nu_g - 1$, where $\nu_g = \deg G(p)$, is found from the above expression $L(p,x) - Q(p,x) = \tilde{Q}(p,x)G(p)$ in the following way. If the polynomial $G(p) \neq p^{\nu_g}$, then the following polynomial equation is written:

$$\tilde{Q}(p,x)G(p) + Q(p,x) = L(p,x), \tag{24}$$

where the polynomials $\tilde{Q}(p,x)$ and $Q(p,x)$ are its minimal solution, which is found by turning to the equivalent SLAE [11, 22]. If $G(p) \equiv p^{\nu_g}$, then the following polynomial is taken:

$$Q(p,x) = \lambda_0(x) + \lambda_1(x)p + \ldots + \lambda_{\nu_g-1}(x)p^{\nu_g-1}. \tag{25}$$

Thus, all polynomials of the equation "input-output" NCD are defined. To be sure that the NCD with the accepted μ_{NCD} can be physically implemented, it is sufficient to turn from equation (25) to the equations equivalent to it in state variables, for example, using the formulas given in [22, p. 346]. In this case, to ensure the parametric robustness of the selective invariance property, it is required to ensure the formation of the spectral models in an explicit form. This point is shown in details in the example below.

Matrix $H(x)$ (8) of system (7) is generally functional, the roots of its characteristic polynomial are real, negative and different in the area $x \subset \Omega_{\mathrm{CO}} \in \mathrm{R}^n$, $\|x\| < \infty$. If the area $\Omega_{\mathrm{CO}} = \mathrm{R}^n$, $\|x\| < \infty$, then to make the equilibrium of system (7) globally stable, the existence of an ℓ-vector of $b_1(w)$ with differentiable components or constants is sufficient, under which the following conditions are satisfied:

$$|\det U_s(w)| \geq \varepsilon_s > 0, \quad U_s(w) = \left[b_1(w) \quad H(x)b_1(w) \quad \ldots \quad H^{\ell-1}(x)b_1(w)\right], \tag{26}$$

$$\operatorname*{Sup}_{w} \frac{\mathrm{Sp}P_1(w)}{(\det P_1(w))^{1/\ell}} \leq K < \infty, \quad \forall\, w \subset \mathrm{R}^l,\ \|w\| < \infty, \tag{27}$$

where $\mathrm{Sp}(\cdot)$ is trace of the matrix $(\cdot)$; $P_1(w) = (U_s(w)M_s)(U_s(w)M_s)^{\mathrm{T}}$; ε_s, K—positive numbers;

$$M_s = \begin{bmatrix} \delta_1 & \delta_2 & \cdots & \delta_{\ell-1} & 1 \\ \delta_2 & \cdot & \cdot^{\cdot^{\cdot}} & 1 & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots & \vdots \\ \delta_{\ell-1} & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix}, \tag{28}$$

δ_i are polynomial coefficients $H(p,x) = \det(pE - H(x)) = p^\ell + \delta_{\ell-1}p^{\ell-1} + \ldots + \delta_1 p + \delta_0$ [20].

The conditions for the roots of the characteristic polynomial $H(p,x)$ of the functional matrix $H(x)$ are constructive and are satisfied by choosing the polynomial $H^*(p)$. If conditions (2) and the first condition of (6) are met in the area $\Omega_{\mathrm{CO}} = \mathrm{R}^n$, $\|x\| < \infty$, then the system equilibrium will be asymptotically globally stable when conditions (26) and (27) are satisfied [20]. If, in this case, matrix $H(x)$ (8) turns out to be constant, then the equilibrium $w = \mathbf{0}$ of system (7) will be asymptotically globally stable [22], regardless of conditions (26), (27). If the area $\Omega_{\rm CO} \in {\rm R}^n$, $\|x\| < \infty$ is bounded, then the equilibrium $w = \mathbf{0}$ of system (7) will be asymptotically locally stable [22], also independently of conditions (26), (27).

Let us show the effectiveness of the developed method for the design of nonlinear selectively invariant control systems using a numerical example.

4. EXAMPLE

Suppose a nonlinear plant is described by the equations:

$$\dot{x}_1 = 2x_1 + 3\sin x_2 + 1.5u + f, \quad \dot{x}_2 = 4\sin x_2 + 2u + 3f, \quad y = 3x_1 - 2.25x_2, \tag{29}$$

where x_1, x_2 and y are measured state variables and an controlled output variable; the disturbance $f(t) = f_0 + f_m \sin(0.5t + \varphi_0)$, $t \geq 0$ is not measured; the setting impact $g(t) = g_0 1(t)$ is measured; f_0, f_m, φ_0, g_0 are unknown bounded constants. Design a nonlinear selectively invariant system to $g(t)$ and $f(t)$ in such a way that the settling time $t_s \leq t_s^* = 1.5$ s; the relative order of the desired NCD $\mu_{\rm NCD} = 0$ [22, 25].

Decision. First of all, let's construct the QLM of the plant. For this purpose, following [22], we find the derivative $d\sin x_2/dx_2 = \cos x_2$ and integrate it with respect to the auxiliary variable:

$$a_s(x_2) = \int_0^1 \cos(x_2\theta)d\theta = x_2^{-1}\sin(x_2\theta)\Big|_0^1 = x_2^{-1}\sin x_2 = \omega(x_2).$$

Replacing the function $\sin x_2$ in (29) by its QLM $a_s(x_2)x_2$, we obtain the QLM of the plant:

$$\dot{x} = \begin{bmatrix} 2 & 3\omega(x_2) \\ 0 & 4\omega(x_2) \end{bmatrix} x + \begin{bmatrix} 1.5 \\ 2 \end{bmatrix} u + \begin{bmatrix} 1 \\ 3 \end{bmatrix} f, \quad y = [3 \quad -2.25]x, \tag{30}$$

where $x = [\ x_1 \quad x_2\ ]^{\rm T}$. Comparing systems (30) and (1), we conclude that in this case

$$A(x) = \begin{bmatrix} 2 & 3\omega(x_2) \\ 0 & 4\omega(x_2) \end{bmatrix}, \quad b(x) = \begin{bmatrix} 1.5 \\ 2 \end{bmatrix}, \quad b_f(x) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad c(x) = \begin{bmatrix} 3 \\ -2.25 \end{bmatrix}. \tag{31}$$

The determinants of the matrices from condition (2) are found according to (31):

$$\det \begin{bmatrix} 1.5 & 3 + 6\omega(x_2) \\ 2 & 8\omega(x_2) \end{bmatrix} = -6, \quad \det \begin{bmatrix} 3 & -2.25 \\ 6 & 0 \end{bmatrix} = 13.5;$$

thus, conditions (2) are satisfied and QLM (30) is full in the area $\Omega_{\rm CO} = {\rm R}^2$, $\|x\| < \infty$.

In this case, K_p-images of the external impacts have the following form: $G(p) = p$, $F(p) = p(p^2 + 0.25)$, i.e. $\Phi(p) = p(p^2 + 0.25)$. According to formulas (14)–(16), the polynomials are found: $B(p, x) = 9$, $A(p, x) = (p - 2)(p - 4\omega(x_2))$, $B_f(p, x) = -3.75p + 15\,\omega(x_2) + 13.5$, $V_1(p, x) = 1.5p$, $V_2(p, x) = 2(p - 2)$, $W_1(p, x) = p + 5\,\omega(x_2)$, $W_2(p, x) = 3(p - 2)$. First condition (6) is obviously satisfied.

Following [26], we establish that, in order to obtain a square matrix G_y with a minimal ℓ, it is sufficient to provide the feedback only on one state variable, i.e., $q = 1$ and $\tilde{x}_1 = x_1$. In this case, polynomial equation (17) takes the form:

$$H^*(p) = \bar{A}(p, x)\tilde{R}(p, x) + B(p, x)L(p, x) + \bar{V}_1(p, x)\tilde{L}_1(p, x), \tag{32}$$

where $\bar{A}(p, x) = [p^2 - (4\omega(x_2) + 2)\,p + 8\omega(x_2)]\,(p^3 + 0.25p)$, $\bar{V}_1(p, x) = 1.5p\,(p^3 + 0.25p)$.

From the above form of the polynomials $R(p,x)$, $L(p,x)$, $Q(p,x)$, $L_i(p,x)$ and $q=1$, $\mu_{\rm NCD}=0$, the equalities follow: $\tilde r=\deg\tilde R(p,x)=r-3$, $l=\deg L(p,x)=r$, $\tilde l_1=\deg\tilde L_1(p,x)=r-3$, $\ell=\deg H^*(p)=r+2$. Moreover, in algebraic system (19), which is equivalent to polynomial equation (32), the number of equations is $N_e=\ell+1=2+r+1$, and the number of unknown coefficients is $N_c=3r-3$. Then the condition $N_c=N_e$ implies $r=3$, and, therefore, $\ell=5$, $\tilde r=0$, $\tilde l_1=0$, $l=3$. At the same time, $\tilde R(p,x)=\rho_0(x)$, $\tilde L_1(p,x)=\tilde\lambda_{10}(x)$, $\det G_y(x)\neq 0$ and $L(p,x)=\lambda_0(x)+\lambda_1(x)p+\lambda_2(x)p^2+\lambda_3(x)p^3$.

In this case, $t_s^*=1.5$ s, $\ell=5$, so the first inequality in (18) takes the form $\left|\mathrm{Re}(p_j^*)\right|\geq 3.33\div 4.67$, $j=\overline{1,5}$. Taking into account this inequality and other conditions of (18), we take: $p_1^*=-4$, $p_2^*=-6$, $p_3^*=-9$, $p_4^*=-12$, $p_5^*=-15$, which leads to the polynomial $H^*(p)=p^5+46p^4+807p^3+6714p^2+26\,352p+38\,880$. As a result of substituting the obtained values into expressions (19)–(22), together with the polynomials $B(p,x)=9$, $\bar{\mathrm{A}}(p,x)$ and $\bar V_1(p,x)$ we obtain a system of linear algebraic equations:

$$\begin{bmatrix}9&0&0&0&0&0\\0&9&0&0&0&2\omega\\0&0&9&0&0.375&-\omega-0.5\\0&0&0&9&0&8\omega+0.25\\0&0&0&0&1.5&-4\omega-2\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}\lambda_0\\\lambda_1\\\lambda_2\\\lambda_3\\\tilde\lambda_{10}\\\rho_0\end{bmatrix}=\begin{bmatrix}38\,880\\26\,352\\6714\\807\\46\\1\end{bmatrix}. \tag{33}$$

The solution of system (33) allows us to write polynomials:

$$L(p,x)=[(806.75-8\omega(x_2))p^3+6702.5p^2+(26\,352-2\omega(x_2))p+38\,880]/9,$$
$$R(p,x)=p(p^2+0.25),\quad L_1(p,x)=[48+4\omega(x_2)](p^3+0.25p)/1.5.$$

In this case, $G(p)\equiv p$, i.e. $\nu_g=1$, therefore, from expression (25) we find $\kappa=0$ and $Q(p,x)=4320$. The obtained data lead to equation "input-output" (23) of the required NCD:

$$p\left(p^2+0.25\right)u=4320g-\left(\lambda_0+\lambda_1p+\lambda_2p^2+\lambda_3p^3\right)y-2\left[48+4\omega(x_2)\right]p\left(p^2+0.25\right)x_1/3. \tag{34}$$

In order to form the internal spectral models of the external impacts in an explicit form in the NCD, which is necessary to ensure the parametric robustness of the selective invariance property [22], equation (34) is reduced to the form:

$$u=\left(\frac{17\,280}{p}-\frac{17\,280p}{p^2+0.25}\right)g-\left(\frac{806.75-8\omega(x_2)}{9}+\frac{17\,280}{p}-\frac{148\,817.5p-26\,150.3125}{9\,(p^2+0.25)}\right)y$$
$$-\frac{2}{3}\left[48+4\omega\left(x_2\right)\right]x_1.$$

Applying the relations (A.2.6) and (A.2.7)) from [22, p. 347] to this expression we will arrive to a quasilinear model of the desired NCD:

$$\dot z=\begin{bmatrix}0&0&0\\0&0&-0.25\\0&1&0\end{bmatrix}z+\begin{bmatrix}17\,280\\0\\0\end{bmatrix}\varepsilon-\begin{bmatrix}0\\0\\17\,280\end{bmatrix}g-\frac{1}{9}\begin{bmatrix}0\\26\,150.3125\\-148\,817.5\end{bmatrix}y, \tag{35}$$

$$u=z_1+z_3-\left\{\left[806.75-8\omega(x_2)\right]y+6\left[48+4\omega(x_2)\right]x_1\right\}/9. \tag{36}$$

As can be seen, the obtained NCD contains internal spectral models of both the constant components of the external impacts and the harmonic component with a frequency of $\omega=0.5$ rad/s.

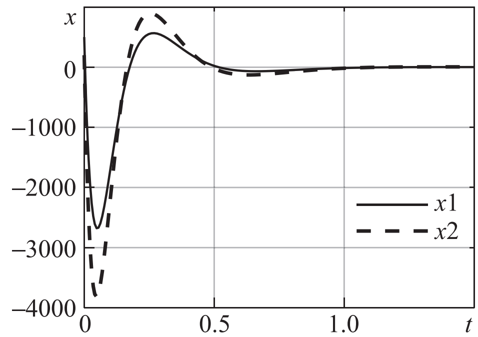

Fig. 1.

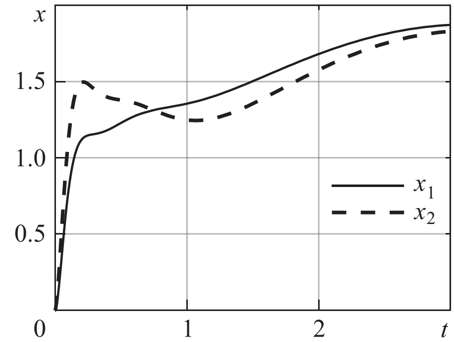

Fig. 2.

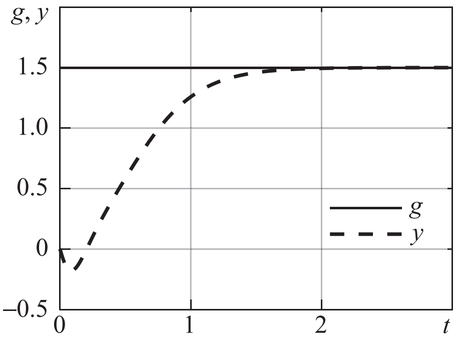

Fig. 3.

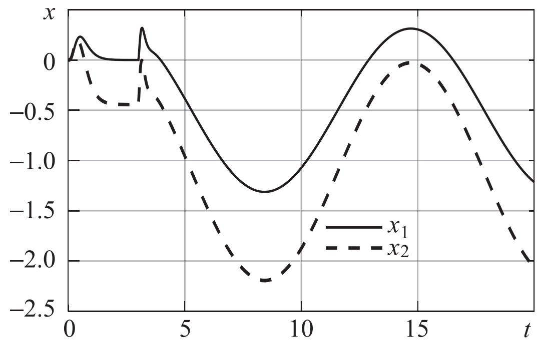

Fig. 4.

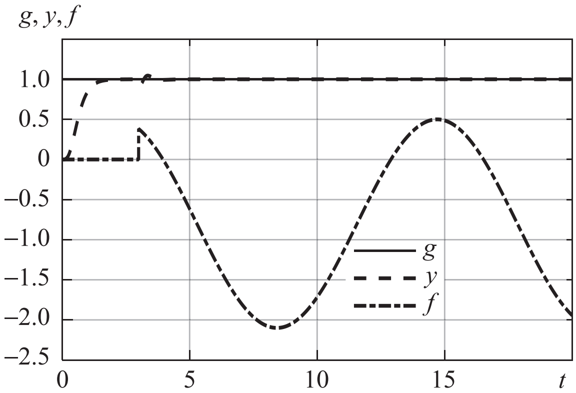

Fig. 5.

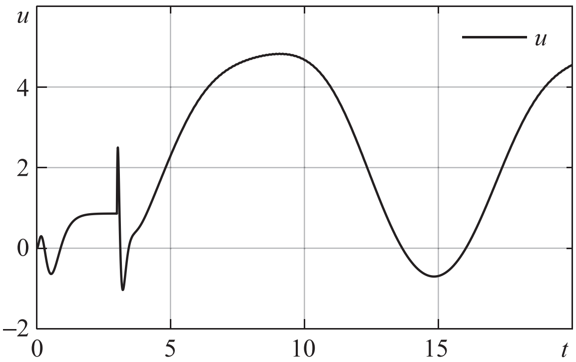

Fig. 6.

Combining equations (30), (35) and (36) into one system, we find that the parameters of this system are constant numbers, and the roots of its characteristic polynomial are strictly less than zero, therefore, the resulting nonlinear system is asymptotically globally stable.

The simulation results of system (29), (35), (36) in MATLAB are shown in Figs. 1–6. Figure 1 shows the graphs of changes in the state variables $x_1(t)$ and $x_2(t)$ of plant (29) in the absence of the external impacts and under "large" initial conditions, i.e. $x_0 = [500 \;\; 200]^{\mathrm{T}}$, $z_0 = [0 \;\; 0 \;\; 0]^{\mathrm{T}}$ and $g(t) = f(t) = 0$. These graphs testify to the asymptotic stability of the designed nonlinear system.

Figures 2 and 3 show the system transients with the simultaneous occurrence of the setting impact $g(t) = 1.5 \times 1(t)$ and a shifted harmonic disturbance $f(t) = 1(t) + 2\sin(0.5\,t)$ at $t \geq 0$ and zero initial conditions. Despite the presence of the disturbance, the deviation of the system $\varepsilon = g - y$ in the steady state is zero (Fig. 3).

For better understanding the nature of the processes in the designed selectively invariant system, Figs. 4–6 show the graphs of changes in the state variables, the setting impact, the output value of the plant and the disturbance, as well as the control action in the time interval from zero to 20 s.

In this case, the disturbance $f(t) = -0.8 + 1.3\sin(0.5(t-3)+2)$, $3 \le t$ (Fig. 5) occurs 3 seconds later than the setting impact $g(t) = 1(t)$, therefore, after the end of the transient ($0 \le t < 3$), caused by the setting impact, the state variables of the plant (Fig. 4), the output variable $y(t)$ (Fig. 5) and the control action $u(t)$ (Fig. 6) take constant values, which corresponds to the constant setting impact.

When a disturbance occurs $f(t)$ ($t = 3$ s) the transient process begins in the system, which is especially noticeable in the graphs in Figs. 4 and 6. After its completion, the output signal of the control device becomes similar in shape to an external disturbance, and its constant and harmonic components are in antiphase with similar components of the external disturbance.

5. CONCLUSION

The suggested method for designing nonlinear selectively invariant control systems is analytical and makes possible to design the control systems with zero errors both in terms of setting and disturbing external impacts of the known form. The solution of the design problem is obtained on the basis of the internal models principle using the original design method for nonlinear control systems. This method uses quasilinear models, which are exact representations of nonlinear differential equations in Cauchy form with differentiable right-hand sides. The developed method is applicable for the design of nonlinear selectively invariant control systems for the plants with differentiable nonlinearities. The selective invariance property is robust to all system parameters, except for the spectrum-setting parameters of internal models.

APPENDIX

Mathematical models of impacts are homogeneous differential equations of a certain order, may be in combination with the algebraic ones [7–13, 27]. For example, the model of impact $f(t) = f_0 1(t)$ will be the equations $\dot{x}_f(t) = 0$, $x_f(0) = f_0$, $f(t) = x_f(t)$, where $x_f(0)$ is an initial condition. The model of the harmonic impact $f(t) = f_m \sin(\omega_f t + \phi_f)$ with frequency ω_f, arbitrary amplitude f_m and phase ϕ_f will be the equations $\dot{x}_{f1} = -\omega_f^2 x_{f2}$, $\dot{x}_{f2} = x_{f1}$, $f = r_1 x_{f1} + r_2 x_{f2}$ with the initial conditions x_{f10} and x_{f20}. Here r_1, r_2 are some constants.

To parry the effect of the external impact on the system error, it is sufficient to have only its spectral model in system, which unambiguously describes its shape, using only its spectrum. In the general case, the spectral model of the impact $g(t)$ can be represented either by the equation in state variables $\dot{x}_g = G\,x_g$, where G and x_g are numerical matrix and state vector, or by K_p-image, i.e. by the polynomial $G(p) = \det(pE - G)$, where $p = d/dt$. We emphasize that the polynomial $G(p)$ at $p = D$ is a Kulebakin K(D)-image of this impact [7], i.e., the representations of the spectral model by the K_p-image or by the equations in the Cauchy form are equivalent [27].

An important property of the impact's K_p-image is that the product of the K_p-image on this impact is equal to zero [7] for all $t \ge 0$. For example, if the impact $\varphi_1(t) = \varphi_0 \exp(\lambda_\varphi t)$, then its K_p-image $\Phi_1(p) = p - \lambda_\varphi$, and product $\Phi_1(p)\,\varphi(t) = (p - \lambda_\varphi)\varphi_0 \exp(\lambda_\varphi t) = \varphi_0[(d\exp(\lambda_\varphi t)/dt) - \lambda_\varphi \exp(\lambda_\varphi t)] \equiv 0$ for the bounded φ_0, since $d\exp(\lambda_\varphi t)/dt = \lambda_\varphi \exp(\lambda_\varphi t)$.

The equation $\dot{x}_{\tilde{f}} = \tilde{F} x_{\tilde{f}}$, where the matrix $\tilde{F} = \mathrm{diag}\{0 \;\; \lambda_{\tilde{f}}\}$ is a spectral model of the impact $\tilde{f}(t) = \tilde{f}_0 1(t) + \tilde{f}_e \exp(\lambda_{\tilde{f}} t)$, $0 \le t < \infty$, where $\tilde{f}_0$ and $\tilde{f}_e$ are bounded constants. The K_p-image of this impact is a polynomial $\tilde{F}(p) = p^2 - \lambda_{\tilde{f}} p$. It is easy to verify that $(p^2 - \lambda_{\tilde{f}} p)\tilde{f}(t) \equiv 0$. It follows from the above examples that the K_p-image of the sum of impacts is equal to the product of the

K_p-images of each of them. Note also that the K_p-image of the impact $f(t)$ can be easily found from the table of the Laplace images [25, p. 29]: it is equal to the denominator of its image $f(s)$ at $s = p$. The coefficients of the K_p-images or the coefficients of equations in the Cauchy form of impacts are the spectrum-setting parameters of their models.

Derivation of the "input-output" equation of the closed system. Differential equation (7) in the operator form can be written as $[pE - H(x)]w = h(x)g + h_f(x)f$. From here $w = [pE - H(x)]^{-1} \times \{h(x)g + h_f(x)f\}$. Taking into account the equality $[pE-H(x)]^{-1} = \operatorname{adj}[pE-H(x)]/\det[pE-H(x)]$ and substituting this expression into second equation (7), we obtain equation (9), where

$$H(p,x) = \det[pE - H(x)], \tag{A.1}$$

$$H_g(p,x) = [c^{\mathrm{T}}(x) \quad \bar{\mathbf{0}}^{\mathrm{T}}]\operatorname{adj}[pE - H(x)]h(x), \tag{A.2}$$

$$H_f(p,x) = [c^{\mathrm{T}}(x) \quad \bar{\mathbf{0}}^{\mathrm{T}}]\operatorname{adj}[pE - H(x)]\,h_f(x). \tag{A.3}$$

Here the matrix $pE - H(x)$ is defined by the expression

$$pE - H(x) = \begin{bmatrix} pE - A(x) & -b(x)k^{\mathrm{T}}(x) \\ \Pi(x) & pE - R(x) \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}. \tag{A.4}$$

Let us show that the operators of equation (9) are directly related by expressions (10)–(13) with operators (14)–(16) of the equations "input-output" of quasilinear models (1) and (3). Expressions (14)–(16) are derived from indicated equations (1) and (3) in exactly the same way as the above derivation of equation (9). In the general case, NCD output equation (3) can have the form $u = k^{\mathrm{T}}(x)z + \lambda_r(x)y + \sum_{i=1}^{q}\tilde{\lambda}_{i\,r}(x)\tilde{x}_i$. In this case, the calculations below will become much more complicated, but their meaning will not change [22, p. 349–353]. Therefore, for greater clarity, it is further assumed that $\lambda_r(x) \equiv 0$ and $\tilde{\lambda}_{i\,r}(x) \equiv 0$, $i = \overline{1,\ q}$.

Derivation of operator $H(p,x)$ (10). Accordance to the formula (A.8), given in [28, p. 223], the expression: $H(p,x) = \det[pE - H(x)] = \det\tilde{A}\det(\tilde{D} - \tilde{C}\tilde{A}^{-1}\tilde{B})$ follows from (A.4). From here, taking into account the notation (A.4), we derive the equality:

$$H(p,x) = \det[pE - A(x)]\det\left\{pE - R(x) + \Pi(x)\,[pE - A(x)]^{-1}\,b(x)k^{\mathrm{T}}(x)\right\}.$$

Since $[pE - A(x)]^{-1} = \operatorname{adj}[pE - A(x)]/\det[pE - A(x)]$, then, taking into account (14), (16) and the notation $\Pi(x)$, we have

$$H(p,x) = A(p,x)\det\left[pE - R(x) + \psi_l(p,x)l(x)k^{\mathrm{T}}(x) + \sum_{i=1}^{q}\psi_i(p,x)l_i(x)\,k^{\mathrm{T}}(x)\right]. \tag{A.5}$$

Here, it is indicated that

$$\psi_l(p,x) = B(p,x)/A(p,x), \quad \psi_i(p,x) = V_i(p,x)/A(p,x). \tag{A.6}$$

Applying identity (A.25) from [28, p. 233] to the second factor in (A.5) and taking into account (15), we get:

$$H(p,x) = A(p,x)\begin{bmatrix} R(p,x) + \psi_l(p,x)k^{\mathrm{T}}(x)\operatorname{adj}[pE - R(x)]l(x) + \\ \displaystyle\sum_{i=1}^{q}\psi_i(p,x)k^{\mathrm{T}}(x)\operatorname{adj}[pE - R(x)]l_i(x) \end{bmatrix}.$$

From here, with taking into account the notation (A.6), (15), operator (10) follows.

Derivation of operator $H_g(p,x)$ (11). For this purpose, we use formula (A.12) from [28, p. 223], which for the block matrix (A.4) allows us to write the equality:

$$\operatorname{adj}[pE - H(x)] = \begin{bmatrix} \det M \operatorname{adj} \tilde{A} + \alpha^{-1}(\operatorname{adj} A)\tilde{B}(\operatorname{adj} M)\tilde{C}(\operatorname{adj} \tilde{A}) & -(\operatorname{adj} \tilde{A})\tilde{B}(\operatorname{adj} M) \\ -(\operatorname{adj} M)\tilde{C}(\operatorname{adj} \tilde{A}) & \alpha \times \operatorname{adj} M \end{bmatrix}, \quad \text{(A.7)}$$

where $\alpha = \det \tilde{A} \neq 0$, $M = \tilde{D} - \tilde{C}\tilde{A}^{-1}\tilde{B}$. Substituting expressions (A.7) and vector $h(x)$ from (8) into (A.2), and taking into account notation (14), we obtain the following equality:

$$H_g(p,x) = c^{\mathrm{T}}(x)\operatorname{adj}[pE - A(x)]b(x) \times k^{\mathrm{T}}(x) \operatorname{adj} Mq(x) = B(p,x)\, k^{\mathrm{T}}(x) \operatorname{adj} Mq(x). \quad \text{(A.8)}$$

Since the matrix $M = \tilde{D} - \tilde{C}\tilde{A}^{-1}\tilde{B}$, then, taking into account the notation (A.4), we derive

$$M = pE - R(x) + A^{-1}(p,x)\left\{ l(x)c^{\mathrm{T}}(x) + \sum_{i=1}^{q} l_i(x) e_i \operatorname{adj}[pE - R(x)] \right\} b(x)k^{\mathrm{T}}(x).$$

Opening the curly brackets here and taking into account the notation (A.6), we obtain

$$M = pE - R(x) + \psi_l(p,x) l(x) k^{\mathrm{T}}(x) + \sum_{i=1}^{q} \psi_i(p,x)\, l_i(x)\, k^{\mathrm{T}}(x). \quad \text{(A.9)}$$

Consequently, the product $k^{\mathrm{T}}(x) \operatorname{adj} M\, q(x)$ in equality (A.8) has the form

$$k^{\mathrm{T}}(x) \operatorname{adj} Mq(x) = k^{\mathrm{T}}(x) \operatorname{adj}\left[pE - R(x) + \psi_l(p,x) l(x) k^{\mathrm{T}}(x) + \sum_{i=1}^{q} \psi_i(p,x)\, l_i(x) k^{\mathrm{T}}(x) \right] q(x).$$

Hence, by formula (A.27) from [28, p. 233] and third notation (15), we have

$$k^{\mathrm{T}}(x) \operatorname{adj} Mq(x) = k^{\mathrm{T}}(x) \operatorname{adj}[pE - R(x)]q(x) = Q(p,x). \quad \text{(A.10)}$$

Substituting this equality into expression (A.8), we obtain operator (11).

Derivation of operator $H_f(p,x)$ (12). From expressions (A.3) and (A.7), we deduce

$$H_f(p,x) = c^{\mathrm{T}}(x) \left\{ (\det M) \operatorname{adj} \tilde{A} + \alpha^{-1}(\operatorname{adj} \tilde{A})\tilde{B}(\operatorname{adj} M)\tilde{C} \operatorname{adj} \tilde{A} \right\} b_f(x). \quad \text{(A.11)}$$

Opening the brackets here and substituting the value $\tilde{B}$ from (A.4), we obtain

$$H_f(p,x) = c^{\mathrm{T}}(x) \operatorname{adj} \tilde{A}\, b_f(x) \det M - \alpha^{-1} c^{\mathrm{T}}(x) \operatorname{adj} \tilde{A}\, b(x)\, \Lambda, \quad \text{(A.12)}$$

where it is indicated that

$$\Lambda = k^{\mathrm{T}}(x)(\operatorname{adj} M)\tilde{C}(\operatorname{adj} \tilde{A}) b_f(x). \quad \text{(A.13)}$$

Taking into account equalities $\tilde{A} = pE - A(x)$ and (14), we find

$$c^{\mathrm{T}}(x)\operatorname{adj} \tilde{A} b_f(x) = B_f(p,x), \quad c^{\mathrm{T}}(x)\operatorname{adj} \tilde{A}\, b(x) = B(p,x). \quad \text{(A.14)}$$

Applying formula (A.25) from [28, p. 233)] to (A.9) and taking into account (15), (16) and (A.6), we have

$$\det M = \det\left\{pE - R(x) + \psi_l(p,x)l(x)k^{\mathrm{T}}(x) + \sum_{i=1}^{q}\psi_i(p,x)\,l_i(x)k^{\mathrm{T}}(x)\right\}$$
$$= \det\,[pE - R(x)] + \psi_l(p,x)\,k^{\mathrm{T}}(x)\,\mathrm{adj}\,[pE - R(x)]\,l(x)$$
$$+ \sum_{i=1}^{q}\psi_i(p,x)\left[k^{\mathrm{T}}(x)\mathrm{adj}\,[pE - R(x)]l_i(x)\right]$$
$$= R(p,x) + \psi_l(p,x)L(p,x) + \sum_{i=1}^{q}\psi_i(p,x)\,L_i(p,x). \quad \text{(A.15)}$$

Substituting $\tilde{C}$, $\tilde{A}$ from (A.4) into (A.13), and opening the brackets, taking into account (A.9), we obtain:

$$\Lambda = \left[k^{\mathrm{T}}(x)\,\mathrm{adj}\,Ml(x)\right]B_f(p,x) + \sum_{i=1}^{q}\left[k^{\mathrm{T}}(x)\,\mathrm{adj}\,Ml_i(x)\right]e_i\,\mathrm{adj}\,[pE - A(x)]\,b_f(x). \quad \text{(A.16)}$$

In accordance with the third expression of (16): $e_i\mathrm{adj}\,[pE - A(x)]b_f(x) = W_i(p,x)$; by analogy with (A.10) and taking into account (15) we find $k^{\mathrm{T}}(x)\,\mathrm{adj}M\,l(x) = L(p,x)$, $k^{\mathrm{T}}(x)\,\mathrm{adj}M\,l_i(x) = L_i(p,x)$. Then from (A.16) the equality follows:

$$\Lambda = L(p,x)B_f(p,x) + \sum_{i=1}^{q}L_i(p,x)W_i(p,x). \quad \text{(A.17)}$$

Substituting expressions (A.14), (A.15) and (A.17) into (A.12), we have

$$H_f(p,x) = B_f(p,x)R(p,x) + \psi_l(p,x)L(p,x)B_f(p,x) + \sum_{i=1}^{q}\psi_i(p,x)L_i(p,x)B_f(p,x)$$
$$- \psi_l(p,x)L(p,x)B_f(p,x) - \psi_l(p,x)\sum_{i=1}^{q}L_i(p,x)W_i(p,x).$$

Taking into account (A.6) here, grouping the sums and taking the factor $A^{-1}(p,x)$ out of the bracket, we obtain

$$H_f(p,x) = B_f(p,x)R(p,x) + \sum_{i=1}^{q}L_i(p,x)\left\{V_i(p,x)B_f(p,x) - B(p,x)W_i(p,x)\right\}A^{-1}(p,x).$$

Finally, taking into account notation (13) here, we obtain operator (12).

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