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NONLINEAR SYSTEMS

Boundary Control of Some Distributed Heterogeneous Vibrating System with Given States at Intermediate Time Instants

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Abstract—This paper considers the boundary control problem for a distributed heterogeneous vibrating system described by a one-dimensional wave equation with piecewise constant characteristics. The travel time of a wave through each homogeneous section is assumed the same. The control is implemented by displacement at the two ends. A constructive control design approach is proposed to transfer the vibrations on a given time interval from the initial state through the multipoint intermediate states to the terminal state. The control design scheme is as follows: the original problem is reduced to a control problem with distributed actions and zero boundary conditions. Then the variable separation method and control methods for finite-dimensional systems with multipoint intermediate conditions are used. The results are illustrated by an example.

Keywords: vibration control, boundary control, heterogeneous vibrating process, wave equation, piecewise constant characteristics, separation of variables

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1. INTRODUCTION

Boundary and optimal control problems for distributed vibrating systems were studied by many researchers; in particular, see [1–15]. For a distributed homogeneous vibrating system described by a homogeneous wave equation with multipoint intermediate conditions, boundary control problems were considered in [2–6]. Their solutions were constructed based on Fourier methods and control methods for finite-dimensional systems with multipoint intermediate conditions.

The solutions of control problems for heterogeneous distributed compound systems were analyzed in [7–15]. The paper [8] was one of the first works in this field; the author solved the control problem for a distributed vibrating system consisting of two piecewise homogeneous media, originally formulated by A.G. Butkovskii. The solution was constructed using the method of propagating waves. In [9, 10] and other publications, the same author and his students studied similar boundary control problems for heterogeneous vibrating processes. Those boundary control problems were examined using the d'Alembert method, and d'Alembert-type formulas were derived. The papers [13–19] were devoted to boundary problems for the equation describing the longitudinal vibrations of a rod with piecewise constant characteristics (consisting of at least two sections) with a free or fixed right end. The studies were carried out in the class of generalized solutions. A mechanical system consisting of two equal-length pieces of a string connected by a spring was considered in [20]. The boundary control problem for the vibrations of a complexly coupled system with singularities was investigated using the d'Alembert formula.

The need to model and control distributed vibrating processes of compound systems with piecewise constant (heterogeneous) characteristics arises in many theoretical and applied fields of science and technology. However, control of heterogeneous elastic vibrations is still insufficiently investigated: this research area is at the stage of formation.

Below, we consider the boundary control problem for some distributed heterogeneous vibrating system with given states at intermediate time instants. This system is described by the homogeneous wave equation and reflects the transverse vibrations of a heterogeneous string and also the longitudinal vibrations of a heterogeneous rod. The vibrating process is characterized by different elastic properties and densities of the section. Their lengths are such that the wave travels through each section in equal time.

The conditions determining contact interactions between the materials of heterogeneous bodies are important. In mathematical modeling, these conditions of coupling (connection or gluing) for two sections with different physical characteristics of materials must be considered properly to match the continuous flow of excited wave processes.

This paper aims at developing an analytical boundary control design approach to one-dimensional vibrating heterogeneous processes that transfer vibrations on a given time interval from an initial state through multipoint intermediate states to a terminal state.

2. PROBLEM STATEMENT

Consider the vibrations of a distributed piecewise homogeneous medium along a segment $-l_1 \le x \le l$ that consists of two sections, $-l_1 \le x \le 0$ and $0 \le x \le l$. Let $a_i = \sqrt{\frac{k_i}{\rho_i}}$ denote the wave velocity along the sections, where $\rho_i = \text{const}$ is the density and $k_i = \text{const}$ is Young's modulus, $i = 1, 2$. Following [9], assume that the lengths l_1 and l of the sections are chosen so that the wave travels through the sections $l_1 \le x \le 0$ and $0 \le x \le l$ in the same time, i.e.,

$$\frac{l_1}{a_1} = \frac{l}{a_2}. \tag{2.1}$$

Note that the vibrating heterogeneous process under consideration can be the longitudinal vibrations of a piecewise homogeneous rod (with the density ρ and the modulus of elasticity k) or the transverse vibrations of a piecewise homogeneous string (with the density ρ and the string tension k).

Let the state (longitudinal vibrations) of the rod (or the transverse vibrations of the string) be represented by a function $Q(x,t)$, $-l_1 \le x \le l$, $0 \le t \le T$. The deviation from the equilibrium is described by the equation

$$\frac{\partial^2 Q(x,t)}{\partial t^2} = \begin{cases} a_1^2 \dfrac{\partial^2 Q(x,t)}{\partial x^2}, & -l_1 \le x \le 0, \quad 0 \le t \le T \\ a_2^2 \dfrac{\partial^2 Q(x,t)}{\partial x^2}, & 0 \le x \le l, \qquad 0 \le t \le T \end{cases} \tag{2.2}$$

with boundary conditions

$$Q(-l_1,t) = \mu(t), \; Q(l,t) = \nu(t), \; 0 \le t \le T, \tag{2.3}$$

and coupling conditions at the junction point $x = 0$ of the form

$$Q(0-0,t) = Q(0+0,t),$$
$$a_1^2 \rho_1 \left. \frac{\partial Q(x,t)}{\partial x} \right|_{x=0-0} = a_2^2 \rho_2 \left. \frac{\partial Q(x,t)}{\partial x} \right|_{x=0+0}. \tag{2.4}$$

The initial ($t = t_0 = 0$) and terminal ($t = T$) conditions are given by

$$Q(x,0) = \varphi_0(x), \ \left.\frac{\partial Q(x,t)}{\partial t}\right|_{t=0} = \psi_0(x), \ -l_1 \le x \le l, \tag{2.5}$$

$$Q(x,T) = \varphi_T(x), \ \left.\frac{\partial Q}{\partial t}\right|_{t=T} = \psi_T(x), \ -l_1 \le x \le l. \tag{2.6}$$

In addition, at some intermediate time instants t_k ($k = 1, \ldots, m$) such that

$$0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T,$$

the state function takes given values

$$Q(x, t_i) = \varphi_i(x), \ -l_1 \le x \le l, \ i = 1, \ldots, m. \tag{2.7}$$

The functions $\mu(t)$ and $\nu(t)$ in formula (2.3) *are control actions (boundary controls).*

By assumption, $Q(x,t) \in C^2(\Omega_T)$, where $\Omega_T = \{(x,t) : x \in [-l_1, l], t \in [0,T]\}$, and $\varphi_i(x) \in C^2[-l_1, l]$, $i = 0, 1, \ldots, m, m+1$, and $\psi_0(x)$, $\psi_T(x) \in C^1[-l_1, l]$.

Also, all these functions satisfy the following matching conditions:

$$\begin{gathered}\mu(0) = \varphi_0(-l_1), \ \dot\mu(0) = \psi_0(-l_1), \ \nu(0) = \varphi_0(l), \ \dot\nu(0) = \psi_0(l), \\ \mu(t_i) = \varphi_i(-l_1), \ \nu(t_i) = \varphi_i(l), \ i = 1, \ldots, m, \\ \mu(T) = \varphi_T(-l_1), \ \dot\mu(T) = \psi_T(-l_1), \ \nu(T) = \varphi_T(l), \ \dot\nu(T) = \psi_T(l).\end{gathered} \tag{2.8}$$

The boundary control problem. It is required to find controls $\mu(t)$ and $\nu(t)$ ($0 \le t \le T$) transferring the vibrations of system (2.2) from the given initial state (2.5) through the intermediate states (2.7) to the terminal state (2.6) ($t = T$).

Note that the coupling conditions (2.4) *at the junction point $x = 0$ also hold for the functions $\varphi_0(x)$, $\varphi_T(x)$, and $\varphi_i(x)$, $i = 1, \ldots, m$.*

3. REDUCTION TO A PROBLEM WITH ZERO BOUNDARY CONDITIONS

To solve the problem, we pass to the new variable [21]

$$\xi = \begin{cases} \dfrac{a_2}{a_1} x, & -l_1 \le x \le 0 \\ x, & 0 \le x \le l, \end{cases} \tag{3.1}$$

which extends or compresses the segment $-l_1 \le x \le 0$ with respect to the point $x = 0$. Due to (2.1), in this case, the segment $-l_1 \le x \le 0$ is transformed to the segment $-l \le \xi \le 0$. On equal-length sections, the function $Q(\xi, t)$ satisfies the same equation

$$\frac{\partial^2 Q(\xi,t)}{\partial t^2} = \begin{cases} a_2^2 \dfrac{\partial^2 Q(\xi,t)}{\partial \xi^2}, & -l \le \xi \le 0, \ \ 0 \le t \le T \\ a_2^2 \dfrac{\partial^2 Q(\xi,t)}{\partial \xi^2}, & 0 \le \xi \le l, \ \ \ 0 \le t \le T \end{cases}$$

or, equivalently,

$$\frac{\partial^2 Q(\xi,t)}{\partial t^2} = a_2^2 \frac{\partial^2 Q(\xi,t)}{\partial \xi^2}, \ -l \le \xi \le l, \ 0 \le t \le T, \tag{3.2}$$

with the corresponding boundary conditions

$$Q(-l,t) = \mu(t), \ Q(l,t) = \nu(t), \ 0 \le t \le T, \tag{3.3}$$

the initial conditions

$$Q(\xi,0) = \varphi_0(\xi), \ \left.\frac{\partial Q(\xi,t)}{\partial t}\right|_{t=0} = \psi_0(\xi), \ -l \le x \le l, \tag{3.4}$$

the intermediate conditions

$$Q(\xi,t_i) = \varphi_i(\xi), \ -l \le \xi \le l, \ i = 1,\ldots,m, \tag{3.5}$$

the terminal conditions

$$Q(\xi,T) = \varphi_T(\xi), \ \left.\frac{\partial Q(\xi,t)}{\partial t}\right|_{t=T} = \psi_T(\xi), \ -l \le \xi \le l, \tag{3.6}$$

and the coupling conditions

$$Q(0-0,t) = Q(0+0,t), \ a_1\rho_1 \frac{\partial Q(\xi,t)}{\partial \xi}\,|_{\xi=0-0} = a_2\rho_2 \frac{\partial Q(\xi,t)}{\partial \xi}\,|_{\xi=0+0} \tag{3.7}$$

at the junction point $\xi = 0$.

For the sake of convenience, after the change of variable (3.1), all the functions are written in the original notations.

Note that the boundary conditions (3.3) are heterogeneous. Therefore, we construct the solution of equation (3.2) as the sum

$$Q(\xi,t) = V(\xi,t) + W(\xi,t), \tag{3.8}$$

where a function $V(\xi,t)$ with the boundary conditions

$$V(-l,t) = V(l,t) = 0 \tag{3.9}$$

has to be determined and the function $W(\xi,t)$ is the solution of equation (3.2) with the conditions

$$W(-l,t) = \mu(t), \ W(l,t) = \nu(t); \tag{3.10}$$

it has the form

$$W(\xi,t) = \frac{1}{2l}\left[(l-\xi)\mu(t) + (l+\xi)\nu(t)\right]. \tag{3.11}$$

In view of (3.11), substituting (3.8) into (3.2) gives the following equation for the function $V(\xi,t)$:

$$\frac{\partial^2 V(\xi,t)}{\partial t^2} = a_2^2 \frac{\partial^2 V(\xi,t)}{\partial \xi^2} + F(\xi,t), \ -l \le \xi \le l, \ 0 \le t \le T, \tag{3.12}$$

where

$$F(\xi,t)=\frac{1}{2l}\left[(\xi-l)\ddot{\mu}(t)-(\xi+l)\ddot{\nu}(t)\right]. \tag{3.13}$$

The function $V(\xi,t)$ satisfies the coupling condition at the junction point $\xi=0$ that corresponds to (3.7). According to (3.1), we have

$$\begin{gathered}\varphi_0(-l_1)=\varphi_0(-l),\ \varphi_i(-l_1)=\varphi_i(-l),\ \varphi_T(-l_1)=\varphi_T(-l),\\ \psi_0(-l_1)=\psi_0(-l),\ \psi_T(-l_1)=\psi_T(-l).\end{gathered} \tag{3.14}$$

Due to the initial, intermediate, and terminal conditions, (3.4)–(3.6), the function $V(\xi,t)$ satisfies:

the initial conditions

$$\begin{gathered}V(\xi,0)=\varphi_0(\xi)-\frac{1}{2l}\left[(l-\xi)\mu(0)+(l+\xi)\nu(0)\right],\\ \left.\frac{\partial V(\xi,t)}{\partial t}\right|_{t=0}=\psi_0(\xi)-\frac{1}{2l}\left[(l-\xi)\dot{\mu}(0)+(l+\xi)\dot{\nu}(0)\right],\end{gathered} \tag{3.15}$$

the intermediate conditions

$$V(\xi,t_i)=\varphi_i(\xi)-\frac{1}{2l}\left[(l-\xi)\mu(t_i)+(l+\xi)\nu(t_i)\right],\ i=1,\ldots,m, \tag{3.16}$$

and the terminal conditions

$$\begin{gathered}V(\xi,T)=\varphi_T(\xi)-\frac{1}{2l}\left[(l-\xi)\mu(T)+(l+\xi)\nu(T)\right],\\ \left.\frac{\partial V(\xi,t)}{\partial t}\right|_{t=T}=\psi_T(\xi)-\frac{1}{2l}\left[(l-\xi)\dot{\mu}(T)+(l+\xi)\dot{\nu}(T)\right].\end{gathered} \tag{3.17}$$

Considering (2.8) and (3.14), conditions (3.15)–(3.17) can be written as follows:

$$\begin{gathered}V(\xi,0)=\varphi_0(\xi)-\frac{1}{2l}\left[(l-\xi)\varphi_0(-l)+(l+\xi)\varphi_0(l)\right],\\ \left.\frac{\partial V(\xi,t)}{\partial t}\right|_{t=0}=\psi_0(\xi)-\frac{1}{2l}\left[(l-\xi)\psi_0(-l)+(l+\xi)\psi_0(l)\right],\end{gathered} \tag{3.18}$$

$$V(\xi,t_i)=\varphi_i(\xi)-\frac{1}{2l}\left[(l-\xi)\varphi_i(-l)+(l+\xi)\varphi_i(l)\right],\ i=1,\ldots,m, \tag{3.19}$$

$$\begin{gathered}V(\xi,T)=\varphi_T(\xi)-\frac{1}{2l}\left[(l-\xi)\varphi_T(-l)+(l+\xi)\varphi_T(l)\right],\\ \left.\frac{\partial V(\xi,t)}{\partial t}\right|_{t=T}=\psi_T(\xi)-\frac{1}{2l}\left[(l-\xi)\psi_T(-l)+(l+\xi)\psi_T(l)\right].\end{gathered} \tag{3.20}$$

Thus, the original problem has been reduced to the following vibration control problem: find boundary controls $\mu(t)$ and $\nu(t)$, $0\le t\le T$, under which the vibration described by (3.12) with the homogeneous boundary conditions (3.9) will pass from the given initial state (3.18) through the intermediate states (3.19) to the terminal state (3.20).

4. SOLUTION

We find the solution of equation (3.12) under the boundary conditions (3.9) and the matching conditions in the form

$$V(\xi,t)=\sum_{k=1}^{\infty}V_k(t)\sin\frac{\pi k\xi}{l},\ \ V_k(t)=\frac{1}{l}\int_{-l}^{l}V(\xi,t)\sin\frac{\pi k\xi}{l}d\xi. \tag{4.1}$$

Let the functions $F(\xi,t)$, $\varphi_i(\xi)$ $(i=0,1,\ldots,m+1)$, $\psi_0(\xi)$, and $\psi_T(\xi)$ be expanded in the Fourier series in the basis $\left\{\sin\frac{\pi k\xi}{l}\right\}$ $(k=1,2,\ldots)$. Substituting their values and $V(\xi,t)$ into equations (3.12), (3.13) and conditions (3.18)–(3.20) yields

$$\ddot{V}_k(t)+\lambda_k^2V_k(t)=F_k(t),\ \ \lambda_k^2=\left(\frac{a_2\pi k}{l}\right)^2,\ \ k=1,2,\ldots, \tag{4.2}$$

$$F_k(t)=\frac{a_2}{\lambda_k l}\left[\ddot{\nu}(t)\left(2(-1)^k-1\right)-\ddot{\mu}(t)\right], \tag{4.3}$$

$$V_k(0)=\varphi_k^{(0)}-\frac{a_2}{\lambda_k l}\left[\varphi_0(-l)-\varphi_0(l)\left(2(-1)^k-1\right)\right],$$

$$\dot{V}_k(0)=\psi_k^{(0)}-\frac{a_2}{\lambda_k l}\left[\psi_0(-l)-\psi_0(l)\left(2(-1)^k-1\right)\right], \tag{4.4}$$

$$V_k(t_i)=\varphi_k^{(i)}-\frac{a_2}{\lambda_k l}\left[\varphi_i(-l)-\varphi_i(l)\left(2(-1)^k-1\right)\right], \tag{4.5}$$

$$V_k(T)=\varphi_k^{(T)}-\frac{a_2}{\lambda_k l}\left[\varphi_T(-l)-\varphi_T(l)\left(2(-1)^k-1\right)\right],$$

$$\dot{V}_k(T)=\psi_k^{(T)}-\frac{a_2}{\lambda_k l}\left[\psi_T(-l)-\psi_T(l)\left(2(-1)^k-1\right)\right]. \tag{4.6}$$

Here, $F_k(t)$, $\varphi_k^{(i)}$ $(i=0,1,\ldots,m,m+1)$, $\psi_k^{(0)}$, and $\psi_k^{(T)}$ denote the Fourier coefficients of the functions $F(\xi,t)$, $\varphi_i(\xi)$ $(i=0,1,\ldots,m,m+1)$, $\psi_0(\xi)$, and $\psi_T(\xi)$, respectively.

The general solution of equation (4.2) with conditions (4.4) and its derivative have the form

$$V_k(t)=V_k(0)\cos\lambda_k t+\frac{1}{\lambda_k}\dot{V}_k(0)\sin\lambda_k t+\frac{1}{\lambda_k}\int_0^t F_k(\tau)\sin\lambda_k(t-\tau)d\tau,$$

$$\dot{V}_k(t)=-\lambda_k V_k(0)\sin\lambda_k t+\dot{V}_k(0)\cos\lambda_k t+\int_0^t F_k(\tau)\cos\lambda_k(t-\tau)d\tau. \tag{4.7}$$

According to [2–6, 22], due to conditions (4.5) and (4.6) and formula (4.7), the control functions $\mu(t)$ and $\nu(t)$ satisfy the following integral relations for each k :

$$\int_0^T\mu(\tau)\sin\lambda_k\left(T-\tau\right)d\tau+E_k\int_0^T\nu(\tau)\sin\lambda_k\left(T-\tau\right)d\tau=C_{1k}(T),$$

$$\int_0^T\mu(\tau)\cos\lambda_k\left(T-\tau\right)d\tau+E_k\int_0^T\nu(\tau)\cos\lambda_k\left(T-\tau\right)d\tau=C_{2k}(T), \tag{4.8}$$

$$\int_0^T\mu(\tau)h_k^{(i)}\left(\tau\right)d\tau+E_k\int_0^T\nu(\tau)h_k^{(i)}\left(\tau\right)d\tau=C_{1k}(t_i),\ \ i=1,\ldots,m,$$

where

$$C_{1k}(T)=\frac{1}{\lambda_k^2}\left[\frac{\lambda_k l}{a_2}\tilde{C}_{1k}+X_{1k}+E_kY_{1k}\right],$$

$$\tilde{C}_{1k}=\lambda_kV_k(T)-\lambda_kV_k(0)\cos\lambda_kT-\dot{V}_k(0)\sin\lambda_kT,$$

$$C_{2k}(T)=\frac{1}{\lambda_k^2}\left[\frac{\lambda_k l}{a_2}\tilde{C}_{2k}+X_{2k}+E_kY_{2k}\right],$$

$$\tilde{C}_{2k}=\dot{V}_k(T)+\lambda_kV_k(0)\sin\lambda_kT-\dot{V}_k(0)\cos\lambda_kT,$$

$$C_{1k}(t_i)=\frac{1}{\lambda_k^2}\left[\frac{\lambda_k l}{a_2}\tilde{C}_{1k}(t_i)+X_{1k}^{(i)}+E_kY_{1k}^{(i)}\right],$$

$$\tilde{C}_{1k}(t_i)=\lambda_kV_k(t_i)-\lambda_kV_k(0)\cos\lambda_kt_i-\dot{V}_k(0)\sin\lambda_kt_i,$$

$$X_{1k}=\lambda_k\varphi_T(-l)-\psi_0(-l)\sin\lambda_kT-\lambda_k\varphi_0(-l)\cos\lambda_kT,\ E_k=1-2(-1)^k, \tag{4.9}$$

$$X_{2k}=\psi_T(-l)-\psi_0(-l)\cos\lambda_kT+\lambda_k\varphi_0(-l)\sin\lambda_kT,$$

$$Y_{1k}=\lambda_k\varphi_T(l)-\psi_0(l)\sin\lambda_kT-\lambda_k\varphi_0(l)\cos\lambda_kT,$$

$$Y_{2k}=\psi_T(l)-\psi_0(l)\cos\lambda_kT+\lambda_k\varphi_0(l)\sin\lambda_kT,$$

$$X_{1k}^{(i)}=\lambda_k\varphi_i(-l)-\psi_0(-l)\sin\lambda_kt_i-\lambda_k\varphi_0(-l)\cos\lambda_kt_i,$$

$$Y_{1k}^{(i)}=\lambda_k\varphi_i(l)-\psi_0(l)\sin\lambda_kt_i-\lambda_k\varphi_0(l)\cos\lambda_kt_i,$$

$$h_k^{(i)}(\tau)=\begin{cases}\sin\lambda_k(t_i-\tau) & \text{for } 0\le\tau\le t_i\\ 0 & \text{for } t_i<\tau\le T.\end{cases}$$

We introduce the notations

$$\bar{H}_k(\tau)=\begin{pmatrix}\sin\lambda_k\,(T-\tau) & E_k\sin\lambda_k\,(T-\tau)\\ \cos\lambda_k\,(T-\tau) & E_k\cos\lambda_k\,(T-\tau)\\ h_k^{(1)}\,(\tau) & E_kh_k^{(1)}\,(\tau)\\ \dots & \dots\\ h_k^{(m)}\,(\tau) & E_kh_k^{(m)}\,(\tau)\end{pmatrix},$$

$$C_k(t_1,\dots,t_m,T)=\begin{pmatrix}C_{1k}(T)\\ C_{2k}(T)\\ C_{1k}(t_1)\\ \vdots\\ C_{1k}(t_{m-1})\end{pmatrix},$$

$$U\,(\tau)=\begin{pmatrix}\mu(\tau)\\ \nu(\tau)\end{pmatrix}. \tag{4.10}$$

Then equality (4.8) takes the form

$$\int\limits_0^T\bar{H}_k(\tau)U(\tau)d\tau=C_k(t_1,\dots,t_m,T),\ k=1,2,\dots. \tag{4.11}$$

Hence, the function $U\,(\tau)$, $\tau\in[0,T]$, is found from the infinite integral relations (4.11).

In practice, the control design problem for distributed systems is solved using control methods for finite-dimensional systems [1, 22, 23]. For the first n harmonics, from (4.11) we obtain

$$\int_0^T H_n(\tau)\, U_n(\tau) d\tau = \eta_n, \tag{4.12}$$

where the block matrices

$$H_n(\tau) = \begin{pmatrix} \bar{H}_1(\tau) \\ \bar{H}_2(\tau) \\ \vdots \\ \bar{H}_n(\tau) \end{pmatrix}, \ \eta_n = \begin{pmatrix} C_1(t_1, \ldots, t_m, T) \\ C_2(t_1, \ldots, t_m, T) \\ \vdots \\ C_n(t_1, \ldots, t_m, T) \end{pmatrix} \tag{4.13}$$

have the dimensions $(n\,(m+2) \times 2)$ and $(n\,(m+2) \times 1)$, respectively. *From this point onwards, the subscript "n" means "for the first n harmonics."*

Thus, see (4.12), the first n harmonics of system (4.2) with conditions (4.3)–(4.6) are completely controllable iff for any vector η_n (4.13), there is a control action $U_n(t)$, $t \in [0, T]$, satisfying condition (4.12).

Following [22, 23], we write the control action $U_n(t)$ satisfying the integral relation (4.12) as

$$U_n(t) = H_n^T(t) S_n^{-1} \eta_n + f_n(t), \tag{4.14}$$

where $H_n^T(t)$ is the transposed matrix and $f_n(t)$ is a vector function such that

$$\int_0^T H_n(t) f_n(t) dt = 0, \ S_n = \int_0^T H_n\,(t)\, H_n^T\,(t)\, dt. \tag{4.15}$$

In this formula, S_n is a known matrix of dimensions $(n\,(m+2) \times n\,(m+2))$ with the property $\det S_n \neq 0$.

Due to (4.14), we have a set of control functions solving the boundary control problem.

Considering the notations of the function $h_k^{(i)}(\tau)$ for the time intervals $[t_{i-1}, t_i]$, $i = 1, \ldots, m+1$, the control functions $\mu_n(t)$ and $\nu_n(t)$ under $f_n(t) = 0$ can be represented as

$$\mu_n(t) = \begin{cases} \mu_n^{(1)}(t), & 0 \le t \le t_1 \\ \mu_n^{(2)}(t), & t_1 < t \le t_2 \\ \ldots \\ \mu_n^{(m)}(t), & t_{m-1} < t \le t_m \\ \mu_n^{(m+1)}(t), & t_m < t \le T, \end{cases} \qquad \nu_n(t) = \begin{cases} \nu_n^{(1)}(t), & 0 \le t \le t_1 \\ \nu_n^{(2)}(t), & t_1 < t \le t_2 \\ \ldots \\ \nu_n^{(m)}(t), & t_{m-1} < t \le t_m \\ \nu_n^{(m+1)}(t), & t_m < t \le T. \end{cases} \tag{4.16}$$

Substituting these expressions for the functions $\mu_n(t)$ and $\nu_n(t)$ into (4.3) and the expression for $F_k(t)$ into (4.7), we obtain the function $V_k\,(t)$, $t \in [0, T]$. Next, formula (4.1) implies

$$V_n(\xi, t) = \sum_{k=1}^{n} V_k(t) \sin \frac{\pi k}{l} \xi. \tag{4.17}$$

In view of (3.7) and (3.10), the vibration function $Q_n(\xi, t)$, $-l \le \xi \le l$, for the first n harmonics is given by

$$Q_n(\xi, t) = V_n(\xi, t) + W_n(\xi, t), \tag{4.18}$$

where

$$W_n(\xi, t) = \frac{1}{2l}\left[(l-\xi)\mu_n(t) + (l+\xi)\nu_n(t)\right]. \tag{4.19}$$

With the notations (3.1), the function $Q_n(x,t)$, $-l_1 \le x \le l$, can be written as

$$Q_n(x,t) = \begin{cases} \sum\limits_{k=1}^{n} V_k(t)\sin\frac{\pi k}{l_1}x + \frac{1}{2}\left[\left(1-\frac{x}{l_1}\right)\mu_n(t) + \left(1+\frac{x}{l_1}\right)\nu_n(t)\right], \\ \qquad\qquad -l_1 \le x \le 0,\ 0 \le t \le T \\ \sum\limits_{k=1}^{n} V_k(t)\sin\frac{\pi k}{l}x + \frac{1}{2}\left[\left(1-\frac{x}{l}\right)\mu_n(t) + \left(1+\frac{x}{l}\right)\nu_n(t)\right], \\ \qquad\qquad 0 \le x \le l,\ 0 \le t \le T. \end{cases} \tag{4.20}$$

Here, the control functions $\mu_n(t)$ and $\nu_n(t)$ have the form (4.17).

5. AN ILLUSTRATIVE EXAMPLE

To illustrate the proposed control design approach, we fix the right end in the boundary conditions (2.3): $Q(l,t) = 0$, $0 \le t \le T$. (In other words, $\nu(t) = 0$.) Consider the case $m = 1$, i.e., the vibration state

$$Q(x, t_1) = \varphi_1(x),\ -l_1 \le x \le l$$

is given at one intermediate time instant t_1 $(0 < t_1 < T.)$

In this case, formula (4.3) implies $F_k(t) = -\frac{a_2}{\lambda_k l}\ddot{\mu}(t)$; according to (4.8), we obtain the integral relations

$$\int_0^T \mu(\tau)\sin\lambda_k(T-\tau)\,d\tau = C_{1k}(T),\ \int_0^T \mu(\tau)\cos\lambda_k(T-\tau)\,d\tau = C_{2k}(T),$$

$$\int_0^T \mu(\tau)h_k^{(1)}(\tau)\,d\tau = C_{1k}(t_1),\ k = 1, 2, \ldots,$$

where

$$C_{1k}(T) = \frac{1}{\lambda_k^2}\left[\frac{\lambda_k l}{a_2}\tilde{C}_{1k} + X_{1k}\right],\ C_{2k}(T) = \frac{1}{\lambda_k^2}\left[\frac{\lambda_k l}{a_2}\tilde{C}_{2k} + X_{2k}\right],$$

$$C_{1k}(t_1) = \frac{1}{\lambda_k^2}\left[\frac{\lambda_k l}{a_2}\tilde{C}_{1k}(t_1) + X_{1k}^{(1)}\right].$$

The constants $\tilde{C}_{1k}, \tilde{C}_{2k}, \tilde{C}_{1k}(t_1), X_{1k}, X_{2k}$, and $X_{1k}^{(1)}$ are calculated by (4.9). Hence,

$$\bar{H}_k(\tau) = \begin{pmatrix} \sin\lambda_k(T-\tau) \\ \cos\lambda_k(T-\tau) \\ h_k^{(1)}(\tau) \end{pmatrix},\ C_k(t_1, T) = \begin{pmatrix} C_{1k}(T) \\ C_{2k}(T) \\ C_{1k}(t_1) \end{pmatrix},\ k = 1, 2, \ldots.$$

For the sake of simplicity, we construct the boundary control function $\mu_n(t)$ for $n = 1$ (hence, $k = 1$) using formulas (4.12) (or (4.14)) and (4.13). According to (4.10) and (4.15), we obtain

$H_1(\tau) = \bar{H}_1(\tau)$, $\eta_1 = C_1$, and the matrix

$$S_1 = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}.$$

In the notations (4.15), the matrix S_1 has the elements

$$s_{11} = \frac{T}{2} - \frac{1}{4\lambda_1}\sin 2\lambda_1 T,\ s_{12} = s_{21} = \frac{1}{2\lambda_1}\sin^2\lambda_1 T,\ s_{22} = \frac{T}{2} + \frac{1}{4\lambda_1}\sin 2\lambda_1 T,$$

$$s_{13} = s_{31} = \frac{t_1}{2}\cos\lambda_1(T - t_1) - \frac{1}{2\lambda_1}\sin\lambda_1 t_1 \cos\lambda_1 T,$$

$$s_{23} = s_{32} = \frac{1}{2\lambda_1}\sin\lambda_1 t_1 \sin\lambda_1 T - \frac{t_1}{2}\sin\lambda_1(T - t_1),\ s_{33} = \frac{t_1}{2} - \frac{1}{4\lambda_1}\sin 2\lambda_1 t_1.$$

In addition, $\Delta = \det S_1 \neq 0$.

We introduce the notations $S_1^{-1} = \begin{pmatrix} \widehat{s}_{11} & \widehat{s}_{12} & \widehat{s}_{13} \\ \widehat{s}_{21} & \widehat{s}_{22} & \widehat{s}_{23} \\ \widehat{s}_{31} & \widehat{s}_{32} & \widehat{s}_{33} \end{pmatrix}$,

where

$$\widehat{s}_{11} = \frac{1}{\Delta}\left[\left(\frac{T}{2} + \frac{1}{4\lambda_1}\sin 2\lambda_1 T\right)\left(\frac{t_1}{2} - \frac{1}{4\lambda_1}\sin 2\lambda_1 t_1\right) - \left(\frac{1}{2\lambda_1}\sin\lambda_1 t_1 \sin\lambda_1 T - \frac{t_1}{2}\sin\lambda_1(T - t_1)\right)^2\right],$$

$$\widehat{s}_{12} = \widehat{s}_{21} = \frac{1}{\Delta}\left[\left(\frac{t_1}{2}\cos\lambda_1(T - t_1) - \frac{1}{2\lambda_1}\sin\lambda_1 t_1 \cos\lambda_1 T\right) \times \left(\frac{1}{2\lambda_1}\sin\lambda_1 t_1 \sin\lambda_1 T - \frac{t_1}{2}\sin\lambda_1(T - t_1)\right) - \left(\frac{1}{2\lambda_1} - \frac{1}{2\lambda_1}\cos^2\lambda_1 T\right)\left(\frac{t_1}{2} - \frac{1}{4\lambda_1}\sin 2\lambda_1 t_1\right)\right],$$

$$\widehat{s}_{13} = \widehat{s}_{31} = \frac{1}{\Delta}\left[\left(\frac{1}{2\lambda_1} - \frac{1}{2\lambda_1}\cos^2\lambda_1 T\right)\left(\frac{1}{2\lambda_1}\sin\lambda_1 t_1 \sin\lambda_1 T - \frac{t_1}{2}\sin\lambda_1(T - t_1)\right) - \left(\frac{t_1}{2}\cos\lambda_1(T - t_1) - \frac{1}{2\lambda_1}\sin\lambda_1 t_1 \cos\lambda_1 T\right)\left(\frac{T}{2} + \frac{1}{4\lambda_1}\sin 2\lambda_1 T\right)\right],$$

$$\widehat{s}_{22} = \frac{1}{\Delta}\left[\left(\frac{T}{2} - \frac{1}{4\lambda_1}\sin 2\lambda_1 T\right)\left(\frac{t_1}{2} - \frac{1}{4\lambda_1}\sin 2\lambda_1 t_1\right) - \left(\frac{t_1}{2}\cos\lambda_1(T - t_1) - \frac{1}{2\lambda_1}\sin\lambda_1 t_1 \cos\lambda_1 T\right)^2\right],$$

$$\widehat{s}_{23} = \widehat{s}_{32} = \frac{1}{\Delta}\left[\left(\frac{1}{2\lambda_1} - \frac{1}{2\lambda_1}\cos^2\lambda_1 T\right)\left(\frac{t_1}{2}\cos\lambda_1(T - t_1) - \frac{1}{2\lambda_1}\sin\lambda_1 t_1 \cos\lambda_1 T\right) - \left(\frac{T}{2} - \frac{1}{4\lambda_1}\sin 2\lambda_1 T\right)\left(\frac{1}{2\lambda_1}\sin\lambda_1 t_1 \sin\lambda_1 T - \frac{t_1}{2}\sin\lambda_1(T - t_1)\right)\right],$$

$$\widehat{s}_{33} = \frac{1}{\Delta}\left[\left(\frac{T^2}{4} - \frac{1}{4\lambda_1^2}\sin^2\lambda_1 T \cos^2\lambda_1 T\right) - \left(\frac{1}{2\lambda_1} - \frac{1}{2\lambda_1}\cos^2\lambda_1 T\right)^2\right].$$

From (4.14) it follows that $\mu_1(\tau) = H_1^T(\tau)S_1^{-1}\eta_1 + f_1(\tau)$. Letting $f_1(\tau) = 0$ and considering (4.16), we arrive at the following results:
for $\tau \in [0, t_1]$,

$$\mu_1^{(1)}(\tau) = \sin\lambda_1 (T-\tau)\left[\widehat{s}_{11}C_{11}(T) + \widehat{s}_{12}C_{21}(T) + \widehat{s}_{13}C_{11}(t_1)\right]$$
$$+ \cos\lambda_1 (T-\tau)\left[\widehat{s}_{21}C_{11}(T) + \widehat{s}_{22}C_{21}(T) + \widehat{s}_{23}C_{11}(t_1)\right]$$
$$+ \sin\lambda_1 (t_1-\tau)\left[\widehat{s}_{31}C_{11}(T) + \widehat{s}_{32}C_{21}(T) + \widehat{s}_{33}C_{11}(t_1)\right];$$

for $\tau \in (t_1, T]$,

$$\mu_1^{(2)}(\tau) = \sin\lambda_1 (T-\tau)\left[\widehat{s}_{11}C_{11}(T) + \widehat{s}_{12}C_{21}(T) + \widehat{s}_{13}C_{11}(t_1)\right]$$
$$+ \cos\lambda_1 (T-\tau)\left[\widehat{s}_{21}C_{11}(T) + \widehat{s}_{22}C_{21}(T) + \widehat{s}_{23}C_{11}(t_1)\right].$$

Due to the expressions (4.17)–(4.19), the function $Q_1(\xi, t)$ for $-l \le \xi \le l$ takes the form

$$Q_1(\xi, t) = \begin{cases} V_1(t)\sin\dfrac{\pi}{l}\xi + \dfrac{1}{2l}(l-\xi)\mu_1^{(1)}(t), & 0 \le \tau \le t_1 \\ V_1(t)\sin\dfrac{\pi}{l}\xi + \dfrac{1}{2l}(l-\xi)\mu_1^{(2)}(t), & t_1 < \tau \le T. \end{cases}$$

In the notations (3.1), the state function $Q_n(x, t)$ on $-l_1 \le x \le l$ can be represented as follows:
for $\tau \in [0, t_1]$,

$$Q_1(x, t) = \begin{cases} V_1(t)\sin\dfrac{\pi}{l_1}x + \dfrac{1}{2}\left(1 - \dfrac{x}{l_1}\right)\mu_1^{(1)}(t), & -l_1 \le x \le 0 \\ V_1(t)\sin\dfrac{\pi}{l}x + \dfrac{1}{2}\left(1 - \dfrac{x}{l}\right)\mu_1^{(1)}(t), & 0 \le x \le l; \end{cases}$$

for $\tau \in (t_1, T]$,

$$Q_1(x, t) = \begin{cases} V_1(t)\sin\dfrac{\pi}{l_1}x + \dfrac{1}{2}\left(1 - \dfrac{x}{l_1}\right)\mu_1^{(2)}(t), & -l_1 \le x \le 0 \\ V_1(t)\sin\dfrac{\pi}{l}x + \dfrac{1}{2}\left(1 - \dfrac{x}{l}\right)\mu_1^{(2)}(t), & 0 \le x \le l. \end{cases}$$

6. CONCLUSIONS

This paper has considered the boundary control problem for a one-dimensional wave equation describing the transverse vibrations of a piecewise homogeneous string or the longitudinal vibrations of a piecewise homogeneous rod. A constructive boundary control design approach has been proposed for one-dimensional heterogeneous vibrating processes. The boundary control function has been explicitly expressed through the given initial, intermediate, and terminal state functions of the distributed system.

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