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About the Method for Constructing External Estimates of the Limit Controllability Set for the Linear Discrete-Time System with Bounded Control

A. V. Berendakova[*,a] **and D. N. Ibragimov**[*,b]

[*]*Moscow Aviation Institute (National Research University), Moscow, Russia*
e-mail: [a]*abv1998@yandex.ru,* [b]*rikk.dan@gmail.com*

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Abstract—This article discusses the problem of constructing an external estimate of the limit set of controllability for a linear discrete system with convex control constraints. We have proposed a decomposition method that allows us to reduce the problem for the initial system to subsystems of smaller dimension by switching to the normal Jordan basis of the matrix of the system. The statement about the structure of the reference hyperplane to the limit set of controllability is formulated and proved. A method for constructing an external estimate of the limit set of controllability with an arbitrary order of accuracy in the sense of the Hausdorff distance is proposed based on the principle of contraction mappings. The paper provides examples.

Keywords: discrete control system, limit set of controllability, reference half-space, principle of contraction mappings, convex set, polyhedral approximation

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1. INTRODUCTION

The issues of constructing reachability and controllability sets [1–6] are closely related to the tasks of managing dynamic systems. In most mechanical systems, the control action is limited in its capabilities: jet engines of an aircraft have limited thrust and a finite fuel reserve, servos of various robotic systems are also capable of developing some fixed force. These limitations lead to the fact that the controlled object can be brought to the desired mode of operation, generally speaking, not from all initial states. In this regard, it turns out to be an urgent task to analyze each individual initial state on the issue of controllability and reachability [7].

For discrete-time control systems, there is an approach aimed at constructing limit sets of controllability and reachability. However, often, even in the linear case, it is only possible to formulate sufficient conditions that these sets will be limited. At the same time, only the most general estimates of their structure are given: in [1] it is demonstrated that the limit sets of controllability and reachability of linear systems are a cylinder with a certain convex profile. In [2], also in the case of a certain matrix structure of a linear system based on the maximum principle, a method for estimating the limit set of reachability is proposed.

Methods of constructing and estimating the limit set of 0-controllability are of particular interest in the case of solving the speed-in-action problem [8–12]. This problem has a certain specificity for discrete time, while in the continuous case its solution has been known for a long time and is reduced to the use of relay control [8, 9]. For systems described by finite-difference relations, the use of standard methods (the maximum principle [13, 14] and the dynamic programming method [15]) lead either to a full iteration procedure or to a degenerate situation, since the extremum for almost all initial states is not regular, and the Lagrange multipliers on the optimal solution simultaneously turn into zero [3, 16–19]. Due to this fact, an approach based on the use of 0-controllability sets is applied. The methods for solving this problem are described in [20–23].

Nevertheless, these works postulate the solvability of the initial speed-in-action problem, but do not provide the necessary and sufficient conditions for the feasibility of this fact. In turn, having the opportunity to construct a limit set of 0-controllability or its evaluation, it is possible to determine for a number of initial states whether the speed problem is solvable.

The structure of the article is presented as follows. Section 2 provides a statement of the problem and introduces the basic concepts. In Section 3, the main lemmas are formulated and proved, which allow, based on the transition to the normal Jordan basis of the matrix of the system, to decompose the original system into subsystems of smaller dimensions. Section 4 provides necessary and sufficient conditions for the limitation of the limit sets of 0-controllability of the system, and also suggests their external evaluation based on the use of the apparatus of reference hyperplanes. In Section 5, a method that allows to construct an external estimate of the limit set of 0-controllability of a discrete linear system with an arbitrary degree of accuracy in the sense of the Hausdorff distance is presented, based on the principle of contraction mappings. Various numerical examples demonstrating the effectiveness of proven theorems and lemmas are presented in Section 6.

2. PROBLEM STATEMENT

We consider an n-dimensional linear autonomous discrete control system $(A, \mathcal{U})$ with limited control:

$$\begin{aligned} &x(k+1) = Ax(k) + u(k), \\ &x(0) = x_0,\ u(k) \in \mathcal{U},\ k \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{1}$$

where $x(k), u(k) \in \mathbb{R}^n$ are vectors of state and control, respectively, $\mathcal{U} \subset \mathbb{R}^n$ is a convex compact set of acceptable control values, $A \in \mathbb{R}^{n\times n}$—matrix of the system (1). It is assumed that $0 \in \operatorname{int} \mathcal{U}$.

Let's define a family of 0-controllability sets $\{\mathcal{X}(N)\}_{N=0}^{\infty}$, where each $\mathcal{X}(N)$ represents a set of those initial states from which the system (1) can be translated to the origin in N steps by choosing an acceptable control:

$$\mathcal{X}(N) = \begin{cases} \{x_0 \in \mathbb{R}^n \colon \exists u(0), \ldots, u(N-1) \in \mathcal{U} \colon x(N) = 0\}, & N \in \mathbb{N} \\ \{0\}, & N = 0. \end{cases} \tag{2}$$

It is required to construct a limit set of 0-controllability $\mathcal{X}_\infty$, i.e. the set of those initial states from which the system $(A, \mathcal{U})$ can be translated to the origin in any finite number of steps:

$$\mathcal{X}_\infty = \{x_0 \in \mathbb{R}^n \colon \exists N \in \mathbb{N},\ u(0), \ldots, u(N-1) \in \mathcal{U} \colon x(N) = 0\}.$$

Taking into account (2), the representation is also valid

$$\mathcal{X}_\infty = \bigcup_{N=0}^{\infty} \mathcal{X}(N). \tag{3}$$

3. DECOMPOSITION OF A LINEAR SYSTEM

As will be shown below, the structure of the limit set of 0-controllability of the system (1) is determined by the properties of the system matrix A. In [1] it is proved that $\mathcal{X}_\infty$ is a cylindrical set oriented along the eigenvectors of the matrix A corresponding to eigenvalues that do not exceed 1 in absolute value. For this reason, the process of constructing $\mathcal{X}_\infty$ is connected with the transition to the normal Jordan basis A. Due to this fact, in the Section 3, we consider the properties of the system (1) and sets of the form (2) and (3) associated with various linear transformations of the coordinate system.

Let $(A_1,\mathcal{U}_1)$ and $(A_2,\mathcal{U}_2)$ be n_1-dimensional and n_2-dimensional systems of the form (1). Denote by $(A_1,\mathcal{U}_1)\times(A_2,\mathcal{U}_2)$ the system $(A,\mathcal{U})$ of dimension n_1+n_2, where

$$A=\begin{pmatrix}A_1 & O\\ O & A_2\end{pmatrix}\in\mathbb{R}^{(n_1+n_2)\times(n_1+n_2)},\ \mathcal{U}=\mathcal{U}_1\times\mathcal{U}_2\in\mathbb{R}^{n_1+n_2}.$$

Lemma 1. *Let $\{\mathcal{X}_i(N)\}_{N=0}^{\infty}$ and $\mathcal{X}_{i,\infty}$ denote the class of 0-controllability sets and the limit set of 0-controllability, respectively, of the system $(A_i,\mathcal{U}_i)$, $i\in\{1,2\}$, also $(A,\mathcal{U})=(A_1,\mathcal{U}_1)\times(A_2,\mathcal{U}_2)$.*

Then

1) $\mathcal{X}(N)=\mathcal{X}_1(N)\times\mathcal{X}_2(N)$, $N\in\mathbb{N}\cup\{0\}$;

2) $\mathcal{X}_\infty=\mathcal{X}_{1,\infty}\times\mathcal{X}_{2,\infty}$.

The proofs of the Lemma 1 and all subsequent assertions are given in the Appendix.

Lemma 2. *Let $S\in\mathbb{R}^{n\times n}$, $\det S\neq 0$, $(A,\mathcal{U})$ be an n-dimensional system of the form (1), $\{\mathcal{Y}(N)\}_{N=0}^{\infty}$ and $\mathcal{Y}_\infty$ denote the class of 0-controllability sets and the limit set of 0-controllability, respectively, of the system $(S^{-1}AS,S^{-1}\mathcal{U})$.*

Then

1) $\mathcal{X}(N)=S\mathcal{Y}(N)$, $N\in\mathbb{N}\cup\{0\}$;

2) $\mathcal{X}_\infty=S\mathcal{Y}_\infty$.

Lemma 3. *Let $A_1\in\mathbb{R}^{n_1\times n_1}$, $A_2\in\mathbb{R}^{n_2\times n_2}$,*

$$A=\begin{pmatrix}A_1 & O\\ O & A_2\end{pmatrix}\in\mathbb{R}^{(n_1+n_2)\times(n_1+n_2)},$$

moreover, all eigenvalues of the matrix A_1 do not exceed 1 in absolute value. $\mathcal{U}_2\subset\mathbb{R}^{n_2}$ denotes the projection of a convex compact body $\mathcal{U}\subset\mathbb{R}^{n_1+n_2}$ onto an n_2-dimensional subspace:

$$\mathcal{U}_2=\underbrace{\begin{pmatrix}0 & \cdots & 0 & 1 & \cdots & 0\\ \vdots & & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & 0 & 0 & \cdots & 1\end{pmatrix}}_{n_2\times(n_1+n_2)}\mathcal{U}\in\mathbb{R}^{n_2}.$$

Then the (n_1+n_2)-dimensional system $(A,\mathcal{U})$ satisfies the equality

$$\mathcal{X}_\infty=\mathbb{R}^{n_1}\times\mathcal{X}_{2,\infty},$$

where $\mathcal{X}_{2,\infty}$ is the limit set of 0-controllability of the system $(A_2,\mathcal{U}_2)$.

The lemmas proved in the Section 3 define the structure of the limit set of 0-controllability of an arbitrary system of the form (1). According to the Lemma 3, each set $\mathcal{X}_\infty$ is a cylindrical set oriented along the eigenvectors and associated vectors of the matrix A corresponding to eigenvalues not exceeding 1 in absolute value. To pass to a normal Jordan basis of the matrix A, we can use the Lemma 2. At the same time, the procedure for constructing $\mathcal{X}_\infty$, due to the block-diagonal form of the normal Jordan form of the matrix, taking into account the Lemma 1, can be reduced to constructing similar sets for subsystems of lower dimension with Jordan cells as matrices.

4. CONSTRUCTION OF ESTIMATES FOR LIMIT SETS OF 0-CONTROLLABILITY

In the 4 section, we consider a method for constructing polyhedral estimates for the $\mathcal{X}_\infty$ set based on the apparatus of supporting half-spaces and properties of convex sets. To do this, we formulate and prove a theorem that guarantees that for an arbitrary system of the form (1) the set $\mathcal{X}_\infty$ is convex.

Theorem 1. *For any n-dimensional system $(A,\mathcal{U})$ of the form* (1) *it is true that $\mathcal{X}_\infty$ is an open and convex set.*

By virtue of the Lemmas 1, 2, and 3, the problem of constructing the limit set of 0-controllability can be considered only for systems whose matrix eigenvalues are strictly greater than 1. Since Theorem 1 $\mathcal{X}_\infty$ is convex by the theorem, its closure can be represented as the intersection of all supporting half-spaces [24, Theorem 18.8]. Let us formulate the structure of the supporting half-space to $\mathcal{X}_\infty$ as a Lemma 4.

Lemma 4. *Let all eigenvalues of the matrix $A \in \mathbb{R}^{n\times n}$ be strictly greater than 1 in absolute value, $\mathcal{X}_\infty$ is defined by the relations* (3).

Then for all $p \in \mathbb{R}^n \setminus \{0\}$ the following relations hold:

$$1)\quad \mathcal{X}_\infty \subset \mathcal{H}_p = \left\{x \in \mathbb{R}^n\colon (p,x) \leqslant \sum_{k=1}^{\infty} \max_{u_k\in\mathcal{U}} \left(-(A^{-k})^{\mathrm{T}} p, u_k\right)\right\};$$

$$2)\quad x^* = -\sum_{k=1}^{\infty} A^{-k}u_k^* \in \overline{\mathcal{X}_\infty} \cap \partial\mathcal{H}_p, \quad \textit{where}$$

$$u_k^* = \arg \max_{u_k\in\mathcal{U}} \left(-(A^{-k})^{\mathrm{T}} p, u_k\right).$$

Since, according to the Lemma 2, it is permissible to assume that the matrix A is reduced to the normal Jordan form, it suffices to consider only the case when A has the form of a Jordan cell to construct the basic outer estimates for $\mathcal{X}_\infty$.

Lemma 5. *Let an n-dimensional system $(A,\mathcal{U})$ satisfy the condition*

$$A = \begin{pmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \lambda & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & \lambda \end{pmatrix} \in \mathbb{R}^{n\times n},$$

where $|\lambda| > 1$, $u_{i,\max} = \max\limits_{u\in\mathcal{U}} u_i$, $u_{i,\min} = \min\limits_{u\in\mathcal{U}} u_i$, $i = \overline{1,n}$.

Then

$$\mathcal{X}_\infty \subset \bigcap_{i=1}^{n} \{x \in \mathbb{R}^n\colon x_i \in (x_{i,\min}; x_{i,\max})\}.$$

And

1) *if $\lambda > 1$ then*

$$x_{i,\min} = \sum_{j=0}^{n-i} \frac{\min\{(-1)^{j+1}u_{i+j,\min}; (-1)^{j+1}u_{i+j,\max}\}}{(\lambda-1)^{j+1}},$$

$$x_{i,\max} = \sum_{j=0}^{n-i} \frac{\max\{(-1)^{j+1}u_{i+j,\min}; (-1)^{j+1}u_{i+j,\max}\}}{(\lambda-1)^{j+1}};$$

2) *if* $\lambda < -1$ *then*

$$x_{i,\min} = \sum_{j=0}^{n-i} \left(\frac{u_{i+j,\min} - u_{i+j,\max}}{2(|\lambda| - 1)^{j+1}} + \frac{u_{i+j,\min} + u_{i+j,\max}}{2(|\lambda| + 1)^{j+1}} \right),$$

$$x_{i,\max} = \sum_{j=0}^{n-i} \left(\frac{u_{i+j,\max} - u_{i+j,\min}}{2(|\lambda| - 1)^{j+1}} + \frac{u_{i+j,\min} + u_{i+j,\max}}{2(|\lambda| + 1)^{j+1}} \right).$$

Corollary 1. *Let* $n = 1$ *under the conditions of the Lemma 5.*
Then

$$\mathcal{X}_\infty = \left(\frac{-u_{1,\max} - \max\{\lambda u_{1,\max}; \lambda u_{1,\min}\}}{|\lambda|^2 - 1}; \frac{-u_{1,\min} - \min\{\lambda u_{1,\max}; \lambda u_{1,\min}\}}{|\lambda|^2 - 1} \right).$$

Lemma 6. *Let* $2n$*-dimensional system* $(A,\mathcal{U})$ *satisfy the condition*

$$A = \begin{pmatrix} rA_\varphi & I & \cdots & O \\ O & rA_\varphi & \ddots & O \\ \vdots & \vdots & \ddots & I \\ O & O & \cdots & rA_\varphi \end{pmatrix} \in \mathbb{R}^{2n\times 2n}, \; A_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}, \; I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $r > 1$, $\varphi \in [0; 2\pi)$, $r_{i,\max} = \max\limits_{u\in\mathcal{U}} \|(u_{2i-1} \;\; u_{2i})^{\mathrm{T}}\|_{\mathbb{R}^2}$, $i = \overline{1,n}$.
Then

$$R_{i,\max} = \sum_{j=0}^{n-i} \frac{r_{i+j,\max}}{(r-1)^{j+1}},$$

$$\mathcal{X}_\infty \subset \bigcap_{i=1}^{n} \{x \in \mathbb{R}^{2n} \colon \|(x_{2i-1} \;\; x_{2i})^{\mathrm{T}}\|_{\mathbb{R}^2} < R_{i,\max}\}.$$

The Lemmas 5 and 6 allow us to construct outer estimates for the limit set of 0-controllability of the system (1) in the direction of each of the eigenvectors and associated vectors. To construct the corresponding supporting hyperplanes bounding $\mathcal{X}_\infty$, it suffices to compute the eigenvalues of the matrix A. If the obtained restrictions on $\mathcal{X}_\infty$ are not enough, you can apply the Lemma 4 to construct an arbitrary reference hyperplane.

5. EXTERNAL ESTIMATE OF THE 0-CONTROLLABILITY LIMIT SET BASED ON THE CONTRACTION MAPPING PRINCIPLE

In the Section 5, we consider the case when the limit set of $\mathcal{X}_\infty$ 0-controllability of the $(A,\mathcal{U})$ system is bounded, which, by virtue of the Lemma 4, is equivalent to the fact that all the eigenvalues of the matrix A are strictly greater than 1 in absolute value. Whence it follows that the matrix A is invertible [25] and the following lemma holds, which defines the structure of the 0-controllability sets of the system $(A,\mathcal{U})$.

Lemma 7 [26, Lemma 1]. *Let* $A \in \mathbb{R}^{n\times n}$, $\det A \neq 0$. *Then for all* $N \in \mathbb{N}$ *the 0-controllability set* (2) *of the system* $(A,\mathcal{U})$ *satisfies the relation*

$$\mathcal{X}(N) = -\sum_{k=1}^{N} A^{-k}\mathcal{U}.$$

Lemma 7 can also be reduced to equivalent recurrence relations of the following form:

$$\mathcal{X}(N) = A^{-1}\mathcal{X}(N-1) + (-A^{-1}\mathcal{U}).$$

Denote by $\mathbb{K}_n$ the set of all compact sets in $\mathbb{R}^n$, and by ρ_H the Hausdorff distance [27]:

$$\mathbb{K}_n = \{\mathcal{X} \subset \mathbb{R}^n \colon \mathcal{X} - \text{compact}\},$$

$$\rho_H(\mathcal{X}, \mathcal{Y}) = \max\left\{\sup_{x\in\mathcal{X}} \inf_{y\in\mathcal{Y}} \|x-y\|; \sup_{y\in\mathcal{Y}} \inf_{x\in\mathcal{X}} \|x-y\|\right\}.$$

If we take into account that $\mathcal{U}$ is a convex compact set in $\mathbb{R}^n$, then every set of the form (2) is also a convex compact set, because representable as an algebraic sum of linear transformations of compact sets [24]. Then in the metric space $(\mathbb{K}_n, \rho_H)$ one can define a mapping $T\colon \mathbb{K}_n \to \mathbb{K}_n$ of the following form:

$$T(\mathcal{X}) = A^{-1}\mathcal{X} + (-A^{-1}\mathcal{U}). \tag{4}$$

Taking into account the Lemma 7 and the relation (4), if the mapping T or $\underbrace{T\circ\ldots\circ T}_{M}$ for some $M \in \mathbb{N}$ are contractive, the limit of a sequence of 0-controllability sets (2) in the space $(\mathbb{K}_n, \rho_H)$ can be determined by the contraction mapping principle [28]. Also, the principle of contraction mappings makes it possible to estimate the error of the limit point approximation using the fixed point iteration method. On the other hand, the limit point, up to closure due to (3), must be c $\mathcal{X}_\infty$. We formulate this fact in the form of a theorem.

Theorem 2. *Let all eigenvalues of the matrix $A \in \mathbb{R}^{n\times n}$ be strictly greater than 1 in absolute value, the family $\{\mathcal{X}(N)\}_{N=0}^{\infty}$ is defined by the relations* (2)*, the set $\mathcal{X}_\infty$ is defined by* (3)*, the mapping T has the form* (4)*.*

Then

1) *there exists $M \in \mathbb{N}$ such that the mapping $T_M = \underbrace{T\circ\ldots\circ T}_{M}$ is contractive with some compression ratio $\alpha \in [0;1)$;*
2) *$\overline{\mathcal{X}_\infty}$ is the only fixed point of the mapping T in the space $(\mathbb{K}_n, \rho_H)$;*
3) *valid estimate*

$$\rho_H(\overline{\mathcal{X}_\infty}, \mathcal{X}(NM)) \leqslant \frac{\alpha^N}{1-\alpha}\rho_H(\mathcal{X}(M), \{0\}).$$

The value of the contraction factor α from the Theorem 2 generally depends on the choice of the norm in the space $\mathbb{R}^n$ and, as a consequence, on the associated operator norm of the matrix A^{-1}. For example, the following estimates for the value of α are known under the choice of different norms in $\mathbb{R}^n$ [28]:

$$\alpha_1 = \max_{1\leqslant j\leqslant n} \sum_{i=1}^{n} |a_{ij}|, \ \alpha_2 = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2}; \ \alpha_\infty = \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} |a_{ij}|. \tag{5}$$

Methods that make it possible in the general case to determine at what value of $M \in \mathbb{N}\cup\{0\}$ the mapping T_M is contractive are currently unknown. However, taking into account the estimates (5), the value of M can be determined numerically by sequentially calculating α for different values of $M \in \mathbb{N}\cup\{0\}$.

Also, the choice of the norm in the space $\mathbb{R}^n$ affects the value of the Hausdorff distance in $\mathbb{K}_n$, which ultimately determines the structure of the external estimates of the set $\mathcal{X}_\infty$. This fact is formulated as the following theorem.

Theorem 3. *Let all eigenvalues of the matrix $A \in \mathbb{R}^{n\times n}$ be strictly greater than 1 in absolute value, the family $\{\mathcal{X}(N)\}_{N=0}^{\infty}$ is defined by the relations (2), the set $\mathcal{X}_\infty$ is defined by the relation (3), the value $M \in \mathbb{N}$ is chosen so that T_M is a contraction mapping with compression ratios $\alpha_1, \alpha_2, \alpha_\infty \in [0;1)$, which are associated with the norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$ in the space $\mathbb{R}^n$ respectively. Then*

$$\mathcal{X}_\infty \subset \mathcal{X}(NM) + \operatorname{conv}\left\{(\underbrace{0,\ldots,0}_{i},r,0,\ldots,0)^{\mathrm{T}} : r \in \{-R_1, R_1\},\ i = \overline{0,n-1}\right\},$$

$$\mathcal{X}_\infty \subset \mathcal{X}(NM) + \left\{x \in \mathbb{R}^n :\ \sqrt{\sum_{i=1}^{n}|x_i|^2} \leqslant R_2\right\},$$

$$\mathcal{X}_\infty \subset \mathcal{X}(NM) + \left\{x \in \mathbb{R}^n :\ \max_{i=\overline{1,n}}|x_i| \leqslant R_\infty\right\},$$

$$R_p = \frac{\alpha_p^N}{1-\alpha_p}\max_{x\in\mathcal{X}(M)}\|x\|_p,\ \ p \in \{1,2,\infty\},\ \ N \in \mathbb{N}.$$

Theorem 3 allows one to construct external estimates for the set $\mathcal{X}_\infty$ with any predetermined precision. Unlike the results of Section 4, these estimates are not tangent to the limit set of 0-controllability and have a very complex structure, since they are a Minkowski sum of various convex sets. To obtain more accurate estimates, you can use both approaches at the same time:

$$\mathcal{X}_\infty \subset \hat{\mathcal{X}}_\infty^1 \cap \hat{\mathcal{X}}_\infty^2,$$

where $\hat{\mathcal{X}}_\infty^1$ is the external estimate of $\mathcal{X}_\infty$ based on the Lemmas 4, 5, 6, $\hat{\mathcal{X}}_\infty^2$ is the outer estimate of $\mathcal{X}_\infty$ constructed according to the Theorem 3.

6. EXAMPLES

Let us demonstrate the theoretical results obtained in Sections 4 and 5 using the example of constructing a limit set of 0-controllability for various linear discrete systems of the form (1).

Example 1. Let the system matrix $A \in \mathbb{R}^{5\times 5}$ be of the form

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0\\ 0 & 2 & 1 & 0 & 0\\ 0 & 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 3\sqrt{2} & 3\sqrt{2}\\ 0 & 0 & 0 & -3\sqrt{2} & 3\sqrt{2}\end{pmatrix}.$$

Consider the cube $\mathcal{U} = [-1;1]^5$. Let us construct an external estimate for the limit set of 0-controllability of the system $(A,\mathcal{U})$. The matrix A can be represented as

$$A = \begin{pmatrix} A_1 & O & O\\ O & A_2 & O\\ O & O & A_3\end{pmatrix},\quad A_1 = \begin{pmatrix}\frac{1}{2}\end{pmatrix},\quad A_2 = \begin{pmatrix} 2 & 1\\ 0 & 2\end{pmatrix},\quad A_3 = \begin{pmatrix} 3\sqrt{2} & 3\sqrt{2}\\ -3\sqrt{2} & 3\sqrt{2}\end{pmatrix}.$$

By the Lemma 3, the limit set of 0-controllability of the system $(A,\mathcal{U})$ satisfies the equality

$$\mathcal{X}_\infty = \mathbb{R} \times \mathcal{X}_{23,\infty},$$

where $\mathcal{X}_{23,\infty}$ is the limit set of 0-controllability of the system $(A_{23}, \mathcal{U}_{23})$,

$$A_{23} = \begin{pmatrix} A_2 & O \\ O & A_3 \end{pmatrix}, \ \mathcal{U}_{23} = [-1; 1]^4.$$

The limit set of 0-controllability $\mathcal{X}_{23,\infty}$ can be represented by the Lemma 1 in the form

$$\mathcal{X}_{23,\infty} = \mathcal{X}_{2,\infty} \times \mathcal{X}_{3,\infty},$$

where $\mathcal{X}_{2,\infty}$ is the limit set of 0-controllability of subsystem $(A_2, \mathcal{U}_2)$, $\mathcal{U}_2 = [-1; 1]^2$, $\mathcal{X}_{3,\infty}$—limit set of 0-controllability of subsystem $(A_3, \mathcal{U}_3)$, $\mathcal{U}_3 = [-1; 1]^2$.

Consider the subsystem $(A_2, \mathcal{U}_2)$. The matrix A_2 has a unique eigenvalue $\lambda_2 = 2$ of multiplicity 2. Then $(A_2, \mathcal{U}_2)$ satisfies the conditions of the Lemma 5. Whence it follows that

$$\mathcal{X}_{2,\infty} \subset \bigcap_{i=1}^{2} \{x \in \mathbb{R}^2 \colon x_i \in (x_{i,\min}; x_{i,\max})\}.$$

$$\begin{aligned} x_{1,\min} &= \sum_{j=0}^{2-1} \frac{\min\{(-1)^{j+1} u_{1+j,\min}; (-1)^{j+1} u_{1+j,\max}\}}{(\lambda_2 - 1)^{j+1}} \\ &= \min\{(-1)u_{1,\min}; (-1)u_{1,\max}\} + \min\{(-1)^2 u_{2,\min}; (-1)^2 u_{2,\max}\} = -2, \end{aligned}$$

$$\begin{aligned} x_{1,\max} &= \sum_{j=0}^{2-1} \frac{\max\{(-1)^{j+1} u_{1+j,\min}; (-1)^{j+1} u_{1+j,\max}\}}{(\lambda_2 - 1)^{j+1}} \\ &= \max\{(-1)u_{1,\min}; (-1)u_{1,\max}\} + \max\{(-1)^2 u_{2,\min}; (-1)^2 u_{2,\max}\} = 2, \end{aligned}$$

$$x_{2,\min} = \sum_{j=0}^{2-2} \frac{\min\{(-1)^{j+1} u_{2+j,\min}; (-1)^{j+1} u_{2+j,\max}\}}{(\lambda_2 - 1)^{j+1}} = \min\{(-1)u_{2,\min}; (-1)u_{2,\max}\} = -1,$$

$$x_{2,\max} = \sum_{j=0}^{2-2} \frac{\max\{(-1)^{j+1} u_{2+j,\min}; (-1)^{j+1} u_{2+j,\max}\}}{(\lambda_2 - 1)^{j+1}} = \max\{(-1)u_{2,\min}; (-1)u_{2,\max}\} = 1.$$

Then

$$\mathcal{X}_{2,\infty} \subset \{x \in \mathbb{R}^2 \colon x_1 \in (-2; 2)\} \cap \{x \in \mathbb{R}^2 \colon x_2 \in (-1; 1)\}(-2; 2) \times (-1; 1).$$

Consider the subsystem $(A_3, \mathcal{U}_3)$. The matrix A_3 has two complex conjugate eigenvalues $\lambda_3 = (3 - 3i)\sqrt{2}$, $\lambda_4 = (3 + 3i)\sqrt{2}$. The matrix A_3 can be represented as

$$A_3 = rA_\varphi = r \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix},$$

where $r = 6$, $\varphi = \frac{\pi}{4}$. Then by the Lemma 6

$$\mathcal{X}_{3,\infty} \subset \left\{ x \in \mathbb{R}^2 \colon \|(x_1 \ x_2)^{\mathrm{T}}\|_2 \leqslant R_{1,\max} \right\},$$

$$r_{1,\max} = \max_{u \in \mathcal{U}_3} \|(u_1 \ u_2)^{\mathrm{T}}\|_2 = \max_{u \in \mathcal{U}_3} \sqrt{u_1^2 + u_2^2} = \sqrt{2},$$

$$R_{1,\max} = \sum_{j=0}^{1-1} \frac{r_{1+j,\max}}{(r-1)^{j+1}} = \frac{r_{1,\max}}{(6-1)} = \frac{\sqrt{2}}{5}.$$

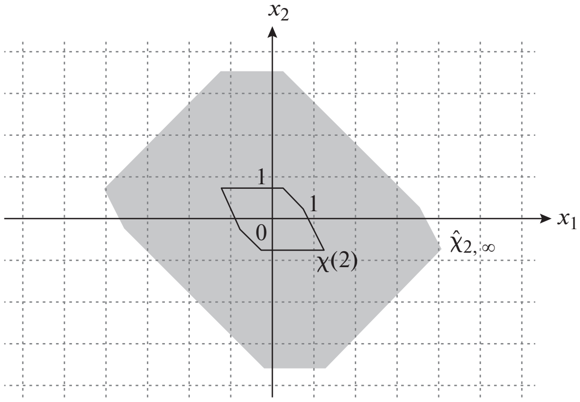

Fig. 1. The estimate $\hat{\mathcal{X}}_{2,\infty}$ for $N = 2$ (grey).

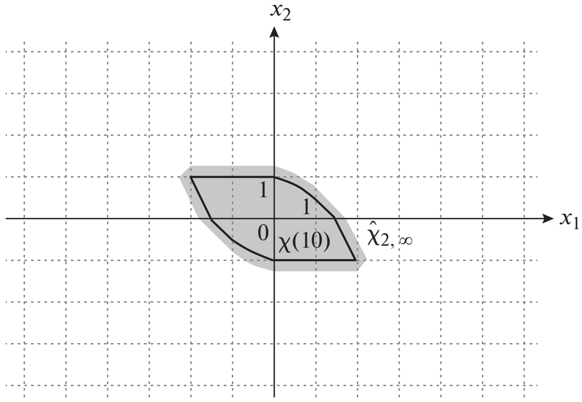

Fig. 2. The estimate $\hat{\mathcal{X}}_{2,\infty}$ for $N = 10$ (grey).

It follows that $\mathcal{X}_{3,\infty} \subset \{x \in \mathbb{R}^2 \colon \|(x_1\ x_2)^{\mathrm{T}}\|_2 \leqslant \frac{\sqrt{2}}{5}\}$.

Then, according to the Lemmas 1 and 3, the limit set of 0-controllability of the system $(A, \mathcal{U})$ can be estimated as follows:

$$\mathcal{X}_\infty \subset \mathbb{R} \times (-2; 2) \times (-1; 1) \times \left\{ x \in \mathbb{R}^2 \colon \sqrt{x_1^2 + x_2^2} \leqslant \frac{\sqrt{2}}{5} \right\}.$$

Example 2. For the subsystem $(A_2, \mathcal{U}_2)$ from the Example 1, we construct an estimate for the limit set of 0-controllability $\mathcal{X}_{2,\infty}$ according to the Theorem 3. As the value of the parameter that defines the norm in $\mathbb{R}^2$, we choose $p = 1$, i.e.

$$\|x\|_1 = |x_1| + |x_2|.$$

Then

$$A_2^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \|A_2^{-1}\| = \alpha_1 = \frac{3}{4},$$

i.e. $M = 1$. According to the Lemma 7

$$\mathcal{X}(M) = -A_2^{-1}\mathcal{U}_2 = \operatorname{conv} \left\{ \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{3}{4} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{3}{4} \\ -\frac{1}{2} \end{pmatrix} \right\},$$

$$\max_{x \in \mathcal{X}(M)} \|x\|_1 = \frac{5}{4},$$

$$R_1(N) = \frac{\left(\frac{3}{4}\right)^N}{1 - \frac{3}{4}} \cdot \frac{5}{4} = 5 \left(\frac{3}{4}\right)^N.$$

Let us construct external estimates for various N.

$$\hat{\mathcal{X}}_{2,\infty} = \sum_{k=1}^{N} A_2^{-k}\mathcal{U}_2 + \operatorname{conv} \left\{ \begin{pmatrix} R_1(N) \\ 0 \end{pmatrix}, \begin{pmatrix} -R_1(N) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ R_1(N) \end{pmatrix}, \begin{pmatrix} 0 \\ -R_1(N) \end{pmatrix} \right\}.$$

The estimates for $\hat{\mathcal{X}}_{2,\infty}$ for the cases $N = 2$ and $N = 10$ are shown in Figs. 1 and 2.

Example 3. Consider a three-dimensional system $(A, \mathcal{U})$, where

$$A = \begin{pmatrix} -3 & 0 & 1 \\ 0.5 & -3.5 & 0.5 \\ -0.5 & 0.5 & -2.5 \end{pmatrix}, \quad \mathcal{U} = [-1; 1]^3.$$

Matrix A has a single eigenvalue $\lambda_1 = -3$ of multiplicity 3, which corresponds to a single linearly independent eigenvector h_1 and associated vectors h_2, h_3:

$$h_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \; h_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \; h_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

The normal Jordan form of the matrix A has the form

$$J = S^{-1}AS = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix},$$

According to the Lemma 2

$$\mathcal{X}_\infty = S\mathcal{Y}_\infty, \tag{6}$$

where $\mathcal{X}_\infty$ is the 0-controllability limit set of $(A, \mathcal{U})$, $\mathcal{Y}_\infty$ is the 0-controllability limit set of $(J, S^{-1}\mathcal{U})$.

$$S^{-1}\mathcal{U} = \operatorname{conv}\left\{ \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}, \begin{pmatrix} -0.5 \\ -0.5 \\ 1.5 \end{pmatrix}, \begin{pmatrix} -0.5 \\ 1.5 \\ -0.5 \end{pmatrix}, \begin{pmatrix} 1.5 \\ -0.5 \\ -0.5 \end{pmatrix}, \begin{pmatrix} -1.5 \\ 0.5 \\ 0,5 \end{pmatrix}, \begin{pmatrix} 0.5 \\ 0.5 \\ -1.5 \end{pmatrix}, \begin{pmatrix} 0.5 \\ -1.5 \\ 0.5 \end{pmatrix}, \begin{pmatrix} -0.5 \\ -0.5 \\ -0.5 \end{pmatrix} \right\}.$$

The system $(J, S^{-1}\mathcal{U})$ satisfies the conditions of the Lemma 5. Whence it follows that

$$\mathcal{Y}_\infty \subset \bigcap_{i=1}^{3} \{y \in \mathbb{R}^3 \colon y_i \in (y_{i,\min}; y_{i,\max})\},$$

$$y_{i,\min} = \sum_{j=0}^{3-i} \left(\frac{u_{i+j,\min} - u_{i+j,\max}}{2 \cdot 2^{j+1}} + \frac{u_{i+j,\min} + u_{i+j,\max}}{2 \cdot 4^{j+1}} \right),$$

$$y_{i,\max} = \sum_{j=0}^{3-i} \left(\frac{u_{i+j,\max} - u_{i+j,\min}}{2 \cdot 2^{j+1}} + \frac{u_{i+j,\min} + u_{i+j,\max}}{2 \cdot 4^{j+1}} \right),$$

where $u_{i,\max} = \max\limits_{u \in S^{-1}\mathcal{U}} u_i = 1.5$, $u_{i,\min} = \min_{u \in S^{-1}\mathcal{U}} u_i = -1.5$, $i = \overline{1, 3}$.

$$y_{1,\min} = -\frac{21}{16}, \quad y_{1,\max} = \frac{21}{16}, \quad y_{2,\min} = -\frac{9}{8}, \quad y_{2,\max} = \frac{9}{8}, \quad y_{3,\min} = -\frac{3}{4}, \quad y_{3,\max} = \frac{3}{4}.$$

Then

$$\mathcal{Y}_\infty \subset \left(-\frac{21}{16}; \frac{21}{16} \right) \times \left(-\frac{9}{8}; \frac{9}{8} \right) \times \left(-\frac{3}{4}; \frac{3}{4} \right).$$

By virtue of (6), the inclusion

$$\mathcal{X}_\infty \subset \operatorname{intconv}\left\{ \begin{pmatrix} -\frac{39}{16} \\ -\frac{33}{16} \\ -\frac{15}{8} \end{pmatrix}, \begin{pmatrix} -\frac{39}{16} \\ -\frac{9}{16} \\ -\frac{3}{8} \end{pmatrix}, \begin{pmatrix} -\frac{3}{16} \\ -\frac{33}{16} \\ \frac{3}{8} \end{pmatrix}, \begin{pmatrix} -\frac{3}{16} \\ -\frac{9}{16} \\ \frac{15}{8} \end{pmatrix}, \begin{pmatrix} \frac{3}{16} \\ \frac{9}{16} \\ -\frac{15}{8} \end{pmatrix}, \begin{pmatrix} \frac{3}{16} \\ \frac{33}{16} \\ -\frac{3}{8} \end{pmatrix}, \begin{pmatrix} \frac{39}{16} \\ \frac{9}{16} \\ \frac{3}{8} \end{pmatrix}, \begin{pmatrix} \frac{39}{16} \\ \frac{33}{16} \\ \frac{15}{8} \end{pmatrix} \right\}.$$

7. CONCLUSION

The paper considers the problem of constructing an external estimate for the limit set of 0-controllability of a linear discrete system with bounded control. The set of admissible control values is assumed to be a convex compact containing the origin. It is proved that the structure of the 0-controllability limit set depends on the normal Jordan form and the eigenvalues of the system matrix.

Statements that make it possible to reduce the problem of constructing the limit set of 0-controllability of a system with a block-diagonal matrix to the problem of constructing analogous sets for subsystems of lower dimension are formulated and proved. For subsystems whose matrix has all eigenvalues that do not exceed one in modulus, it is proved that the limit set of 0-controllability coincides with the entire phase space. For subsystems whose matrix has all eigenvalues strictly greater than one in absolute value, it is proved that the limit set of 0-controllability is a convex, bounded, and open set. In this case, there has been developed a method for constructing polyhedral estimates for the limit set of 0-controllability based on the apparatus of supporting half-spaces and properties of convex sets. The reference half-spaces oriented along the direction of the eigenvectors and associated vectors of the matrix of the system were constructed explicitly. Also, for the case of a bounded limit set of 0-controllability, there has been developed a method for constructing its outer estimate based on the principle of contraction mappings with any predetermined accuracy.

The obtained theoretical results were tested on examples.

APPENDIX

Proof of Lemma 1. Denote the initial states of the system $(A_1,\mathcal{U}_1)$ and $(A_2,\mathcal{U}_2)$ by $x_{0,1} \in \mathbb{R}^{n_1}$ and $x_{0,2} \in \mathbb{R}^{n_2}$ respectively. Then $x_0 = \begin{pmatrix} x_{0,1} \\ x_{0,2} \end{pmatrix}$ is the initial state of the system $(A,\mathcal{U})$.

By (1) it is true that for all $N \in \mathbb{N}$

$$\begin{aligned} x(N) &= A^N x_0 + A^{N-1}u(0) + A^{N-2}u(1) + \ldots + u(N-1) \\ &= \begin{pmatrix} A_1^N & O \\ O & A_2^N \end{pmatrix}\begin{pmatrix} x_{0,1} \\ x_{0,2} \end{pmatrix} + \begin{pmatrix} A_1^{N-1} & O \\ O & A_2^{N-1} \end{pmatrix}\begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} + \ldots + \begin{pmatrix} u_1(N-1) \\ u_2(N-1) \end{pmatrix} \\ &= \begin{pmatrix} A_1^N x_{0,1} + A_1^{N-1}u_1(0) + \ldots + u_1(N-1) \\ A_2^N x_{0,2} + A_2^{N-1}u_2(0) + \ldots + u_2(N-1) \end{pmatrix} = \begin{pmatrix} x_1(N) \\ x_2(N) \end{pmatrix}. \end{aligned}$$

Then $x(N) = 0$ if and only if there exist $u_1(0), \ldots, u_1(N-1) \in \mathcal{U}_1$ and $u_2(0), \ldots, u_2(N-1) \in \mathcal{U}_2$ such that $x_1(N) = 0$, $x_2(N) = 0$. The equality data is, by virtue of (2), equivalent to including $x_{0,1} \in \mathcal{X}_1(N)$, $x_{0,2} \in \mathcal{X}_2(N)$. Hence,

$$\mathcal{X}(N) = \mathcal{X}_1(N) \times \mathcal{X}_2(N).$$

Let $x_0 \in \mathcal{X}_\infty$. Then according to (3) there exists $\widetilde{N} \in \mathbb{N}$ such that

$$x_0 \in \mathcal{X}(\widetilde{N}) = \mathcal{X}_1(\widetilde{N}) \times \mathcal{X}_2(\widetilde{N}) \subset \bigcup_{N=0}^{\infty} \mathcal{X}_1(N) \times \bigcup_{N=0}^{\infty} \mathcal{X}_2(N) = \mathcal{X}_{1,\infty} \times \mathcal{X}_{2,\infty}.$$

Then $\mathcal{X}_\infty \subset \mathcal{X}_{1,\infty} \times \mathcal{X}_{2,\infty}$.

Let $x_0 \in \mathcal{X}_{1,\infty} \times \mathcal{X}_{2,\infty}$. Therefore, there are $\widetilde{N}_1, \widetilde{N}_2 \in \mathbb{N}$ such that $x_0 \in \mathcal{X}_1(\widetilde{N}_1) \times \mathcal{X}_2(\widetilde{N}_2) \subset \mathcal{X}_1(\widetilde{N}) \times \mathcal{X}_2(\widetilde{N})$, where $\widetilde{N} = \max\{\widetilde{N}_1, \widetilde{N}_2\}$. Then, by point 1 of the Lemma 1

$$x_0 \in \mathcal{X}(\widetilde{N}) \subset \bigcup_{N=0}^{\infty} \mathcal{X}(N) = \mathcal{X}_\infty.$$

It follows that $\mathcal{X}_{1,\infty} \times \mathcal{X}_{2,\infty} \subset \mathcal{X}_\infty$.

Finally, we get that $\mathcal{X}_\infty = \mathcal{X}_{1,\infty} \times \mathcal{X}_{2,\infty}$. Lemma 1 is proved.

Proof of Lemma 2. Let $\{y(k)\}_{k=0}^{N}$ be the trajectory of the system $(S^{-1}AS, S^{-1}\mathcal{U})$, i.e. $y(N)$ according to (1) for the initial state $y_0 \in \mathbb{R}^n$ admits the following representation:

$$y(N) = S^{-1}ASy(N-1) + v(N-1) = \ldots = S^{-1}A^N Sy_0 + S^{-1}A^{N-1}Sv(0) + \ldots + v(N-1),$$

where $v(0), \ldots, v(N-1) \in S^{-1}\mathcal{U}$.

By virtue of (2) $y_0 \in \mathcal{Y}(N)$ if and only if $y(N) = 0$, i.e.

$$\begin{gathered} S^{-1}A^N Sy_0 + S^{-1}A^{N-1}Sv(0) + \ldots + v(N-1) = 0, \\ A^N Sy_0 + A^{N-1}Sv(0) + \ldots + Sv(N-1) = 0, \end{gathered}$$

which, by virtue of (2), is equivalent to including $Sy_0 \in \mathcal{X}(N)$, since by construction $Sv(0), \ldots, Sv(N-1) \in \mathcal{U}$. Whence follows the equality $\mathcal{X}(N) = S\mathcal{Y}(N)$.

Let $x_0 \in \mathcal{X}_\infty$. By (3), there exists $\widetilde{N} \in \mathbb{N} \cup \{0\}$ such that $x_0 \in \mathcal{X}(\widetilde{N})$, which is equivalent to including $x_0 \in S\mathcal{Y}(\widetilde{N})$ according to point 1 of the Lemma 2. Hence $S^{-1}x_0 \in \mathcal{Y}(\widetilde{N})$. Then $S^{-1}x_0 \in \bigcup\limits_{N=0}^{\infty} \mathcal{Y}(N) = \mathcal{Y}_\infty$, i.e. $x_0 \in S\mathcal{Y}_\infty$. Then $\mathcal{X}_\infty \subset S\mathcal{Y}_\infty$.

Let $x_0 \in S\mathcal{Y}_\infty$, then $S^{-1}x_0 \in \mathcal{Y}_\infty$. By (3) there exists $\widetilde{N} \in \mathbb{N} \cup \{0\}$ such that $S^{-1}x_0 \in \mathcal{Y}(\widetilde{N})$. Then $x_0 \in S\mathcal{Y}(\widetilde{N})$, which is equivalent to including $x_0 \subset \mathcal{X}(\widetilde{N})$ by point 1 of the Lemma 2. According to the (3) relations, the inclusion $x_0 \in \mathcal{X}_\infty$ is also true. Then $S\mathcal{Y}_\infty \subset \mathcal{X}_\infty$.

Finally, we get that $S\mathcal{Y}_\infty = \mathcal{X}_\infty$. Lemma 2 is proved.

Proof of Lemma 3. Let $x_0 \in \mathbb{R}^{n_1} \times \mathcal{X}_{2,\infty}$. Then $x_0 = \begin{pmatrix} x_{0,1} \\ x_{0,2} \end{pmatrix}$, where $x_{0,1} \in \mathbb{R}^{n_1}$, $x_{0,2} \in \mathcal{X}_{2,\infty}$, whence according to (3) there exists $\widetilde{N} \in \mathbb{N} \cup \{0\}$ such that $x_{0,2} \in \mathcal{X}_2(\widetilde{N})$, which according to (2) is equivalent to the existence of $u_2^*(0), \ldots, u_2^*(\widetilde{N}-1) \in \mathcal{U}_2$ such that $x_2(\widetilde{N}) = 0$. Then for the system $(A, \mathcal{U})$ there are $u(0), \ldots, u(\widetilde{N}-1) \in \mathcal{U}$ such that $u(k) = \begin{pmatrix} u_1(k) \\ u_2^*(k) \end{pmatrix}$, $k = \overline{0, \widetilde{N}-1}$. By (1), $x(\widetilde{N})$ has the representation

$$x(\widetilde{N}) = \begin{pmatrix} A_1^{\widetilde{N}} x_{0,1} + A_1^{\widetilde{N}-1} u_1(0) + \ldots + u_1(\widetilde{N}-1) \\ 0 \end{pmatrix} = \begin{pmatrix} \widetilde{x}_1 \\ 0 \end{pmatrix}.$$

According to the Lemma 1, it suffices to show that there exists $\mathcal{U}_1 \subset \mathbb{R}^{n_1}$ such that $\mathcal{U}_1 \times \{0\} \subset \mathcal{U}$ and $\mathcal{X}_{1,\infty} = \mathbb{R}^{n_1}$, where $\mathcal{X}_{1,\infty}$ is the limit set of 0-controllability of the system $(A_1, \mathcal{U}_1)$.

Denote by $S \in \mathbb{R}^{n_1 \times n_1}$ the transition matrix to the normal Jordan basis of the matrix A_1. Since $0 \in \operatorname{int} \mathcal{U}$, there exists $u_{\max} > 0$ such that $S[-u_{\max}; u_{\max}]^{n_1} \times \{0\} \subset \mathcal{U}$. Moreover, due to the non-degeneracy of the matrix S and the Lemma 2, the equality $\mathcal{X}_{1,\infty} = \mathbb{R}^{n_1}$ is true for the case $\mathcal{U}_1 = S[-u_{\max}; u_{\max}]^{n_1}$ if and only if $S^{-1}\mathcal{X}_{1,\infty} = \mathbb{R}^{n_1}$, where $S^{-1}\mathcal{X}_{1,\infty}$ is the limit set of 0-controllability of the system $(S^{-1}A_1 S, [-u_{\max}; u_{\max}]^{n_1})$. Moreover, according to the normal Jordan form theorem [25], the equality

$$S^{-1}A_1S = \begin{pmatrix} J_1 & 0 & 0 & \dots \\ 0 & J_2 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{\widetilde{n}_1} \end{pmatrix},$$

where the Jordan cells J_i corresponding to the real eigenvalues $\lambda_i \in \mathbb{R}$ of the matrix A_1 have the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_i \end{pmatrix} \in \mathbb{R}^{m_i \times m_i}, \tag{A.1}$$

and the Jordan cells J_i corresponding to the complex eigenvalues $\lambda_i \in \mathbb{C}$ of the matrix A_1 have the form

$$J_i = \begin{pmatrix} r_iA_{\varphi_i} & I & 0 & \dots & 0 \\ 0 & r_iA_{\varphi_i} & I & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \dots & r_iA_{\varphi_i} & I \\ 0 & 0 & 0 & \dots & r_iA_{\varphi_i} \end{pmatrix} \in \mathbb{R}^{2m_i \times 2m_i}, \quad A_{\varphi_i} = \begin{pmatrix} \cos\varphi_i & \sin\varphi_i \\ -\sin\varphi_i & \cos\varphi_i \end{pmatrix}, \tag{A.2}$$

where $r_i = |\lambda_i|$, $\varphi_i = \arg(\lambda_i)$.

Hence, by virtue of Lemma 1, it suffices to show that for $|\lambda_i| \leqslant 1$, the limit sets of null-controllability of the system are $(J_i, [-u_{\max}; u_{\max}]^{m_i})$ for the case of (A.1) and the systems $(J_i, [-u_{\max}; u_{\max}]^{2m_i})$ for the case of (A.2) coincide with $\mathbb{R}^{m_i}$ and $\mathbb{R}^{2m_i}$ correspondingly, for all $i = \overline{1, \tilde{n}_1}$.

Let $J \in \mathbb{R}^{m\times m}$ satisfy (A.1). Then for all $N \geqslant m$ the following relations hold

$$J^N = \begin{pmatrix} \lambda^N & N\lambda^{N-1} & C_N^2\lambda^{N-2} & \dots & C_N^{m-1}\lambda^{N-m+1} \\ 0 & \lambda^N & N\lambda^{N-1} & \dots & C_N^{m-2}\lambda^{N-m+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda^N \end{pmatrix}, \tag{A.3}$$

where we denote the number of combinations of N choose m by C_N^m:

$$C_N^m = \frac{N!}{(N-m)!m!}.$$

Denote by $\{y(k), v(k-1), y_0\}_{k=1}^N$ the process of controlling the system $(J, [-u_{\max}; u_{\max}]^m)$. Hence

$$y(N) = J^Ny_0 + \sum_{k=0}^{N-1} J^kv(N-k-1).$$

If we denote $z_0 = J^Ny_0$, then by (A.3) it is right for each ith coordinate of z_0, that

$$z_{0,i} = \sum_{j=0}^{m-i} \lambda^{N-j}C_N^j y_{0,j+i}, \ i = \overline{1, m}.$$

Let us assume, that $|\lambda| < 1$. Then for all $N \geqslant 2m$ the following relations hold

$$|z_{0,i}| \leqslant \sum_{j=0}^{m-i} |\lambda^{N-j} y_{0,j+i} C_N^j| \leqslant \sum_{j=0}^{m-1} |\lambda^{N-j}| |\max_{i=\overline{1,m}} y_{0,i}| |C_N^j|$$
$$\leqslant m|\lambda^{N-m+1}| \max_{i=\overline{1,m}} |y_{0,i}| C_N^{m-1} \leqslant m \max_{i=\overline{1,m}} |y_{0,i}| |\lambda|^{N-m+1} \frac{N \cdot (N-1) \cdot \ldots \cdot (N-m+2)}{(m-1)!}$$
$$\leqslant m \max_{i=\overline{1,m}} |y_{0,i}| |\lambda|^{N-m+1} \frac{N^{m-1}}{(m-1)!} \xrightarrow{N\to\infty} 0.$$

Then there exists $\widetilde{N} \in \mathbb{N}$ such, that for all $i = \overline{1,m}$

$$-u_{\max} < z_{0,i} < u_{\max}.$$

Let us take $v(0) = \ldots = v(\widetilde{N}-2) = 0$ and $v(\widetilde{N}-1) = -z_0 \in [-u_{\max}; u_{\max}]^m$. Then we obtain $y(\widetilde{N}) = 0$, i.e. $y_0 \in \mathcal{Y}(\widetilde{N})$. Therefore, by choosing arbitrary $y_0 \in \mathbb{R}^m$ and equation (3) we obtain the result $\mathcal{Y}_\infty = \mathbb{R}^m$.

Let us assume, that $|\lambda| = 1$. Then by (A.3) for some $N_m \in \mathbb{N}$ and mth coordinate of $y(N_m)$ it is right that

$$y_m(N_m) = \lambda^{N_m} y_m(0) + \sum_{k=0}^{N_m-1} \lambda^k v_m(N_m - k - 1).$$

Here we choose $N_m \in \mathbb{N}$, requiring $|y_m(0)| \leqslant N_m u_{\max}$. Then we take $v(0), \ldots, v(N_m-1) \in [-u_{\max}; u_{\max}]^m$ in accordance with the following condition:

$$v_m(N_m - k - 1) = -\frac{\lambda^{N_m} y_m(0)}{\lambda^k N_m} \in [-u_{\max}; u_{\max}], \quad k = \overline{0, N_m - 1}.$$

We obtain

$$y_m(N_m) = \lambda^{N_m} y_m(0) + \sum_{k=0}^{N_m-1} \lambda^k \frac{-\lambda^{N_m} y_m(0)}{\lambda^k N_m} = 0.$$

Let us assume that for some $N \in \mathbb{N}$ and $i = \overline{1, m-1}$ it is right that $y_m(N) = \ldots = y_{m-i+1}(N) = 0$. Then if $v_m(N + N_{m-i} - k - 1) = \ldots = v_{m-i+1}(N + N_{m-i} - k - 1) = 0$, $k = \overline{0, N_{m-i} - 1}$ is right, then by (A.3) the following result is straightforward

$$y_{m-i}(N + N_{m-i}) = \lambda^{N_{m-i}} y_{m-i}(N) + \sum_{k=0}^{N_{m-i}-1} \lambda^k v_{m-i}(N + N_{m-i} - k - 1),$$

where $N_{m-i} \in \mathbb{N}$ is chosen from the condition $|y_{m-i}(N)| \leqslant N_{m-i} u_{\max}$. Then we determine $v(N), \ldots, v(N + N_{m-i} - 1) \in [-u_{\max}; u_{\max}]^m$ in order that

$$v_{m-i}(N + N_{m-i} - k - 1) = \frac{-\lambda^{N_{m-i}} y_{m-i}(N)}{N_{m-i} \lambda^k}, \quad k = \overline{0, N_{m-i} - 1}.$$

We obtain

$$y_{m-i}(N + N_{m-i}) = \lambda^{N_{m-i}} y_{m-i}(N) + \sum_{k=0}^{N_{m-i}-1} \lambda^k \frac{-\lambda^{N_{m-i}} y_{m-i}(N)}{N_{m-i} \lambda^k} = 0,$$

$$y_m(N + N_{m-i}) = \ldots = y_{m-i+1}(N + N_{m-i}) = 0.$$

Then there exists $N \in \mathbb{N}$ such, that $y(N) = 0$, i.e. $y_0 \in \mathcal{Y}(N)$ by the method of mathematical induction. Therefore, we obtain the result $\mathcal{Y}_\infty = \mathbb{R}^m$ by choosing arbitrary $y_0 \in \mathbb{R}^m$ and equation (3).

Let $J \in \mathbb{R}^{2m\times 2m}$ satisfy the case (A.2). Then for all $N \geqslant m$ the following relations hold

$$J^N = \begin{pmatrix} r^N A_{N\varphi} & Nr^{N-1}A_{(N-1)\varphi} & \dots & C_N^{m-1} r^{N-m+1} A_{(N-m+1)\varphi} \\ 0 & r^N A_{N\varphi} & \dots & C_N^{m-2} r^{N-m+2} A_{(N-m+2)\varphi} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r^N A_{N\varphi} \end{pmatrix}. \tag{A.4}$$

Denote the process of controlling the system $(J, [-u_{\max}; u_{\max}]^{2m})$ by $\{y(k), v(k-1), y_0\}_{k=1}^N$. Hence

$$y(N) = J^N y_0 + \sum_{k=0}^{N-1} J^k v(N-k-1).$$

If we denote $z_0 = J^N y_0$, then by (A.4) for each ith two-dimensional subvector z_0 it is right, that

$$z_{0,i} = \sum_{j=0}^{m-i} C_N^j r^{N-j} A_{(N-j)\varphi} y_{0,j+i} \in \mathbb{R}^2, \quad i = \overline{1,m},$$

where $z_0 = (z_{0,1}^{\mathrm{T}}, \dots, z_{0,m}^{\mathrm{T}})^{\mathrm{T}}$, $y_0 = (y_{0,1}^{\mathrm{T}}, \dots, y_{0,m}^{\mathrm{T}})^{\mathrm{T}}$.

Let us assume, that $r < 1$. Then for all $N \geqslant 2m$ the following relations hold

$$\|z_{0,i}\| \leqslant \sum_{j=0}^{m-i} \|r^{N-j} A_{(N-j)\varphi} y_{0,j+i} C_N^j\| \leqslant \sum_{j=0}^{m-1} r^{N-j} \|A_{(N-j)\varphi} y_{0,i}\| C_N^j$$
$$\leqslant \sum_{j=0}^{m-1} r^{N-j} \max_{i=\overline{1,m}} \|y_{0,i}\| C_N^j \leqslant m r^{N-m+1} \max_{i=\overline{1,m}} \|y_{0,i}\| \frac{N(N-1)\cdot\ldots\cdot(N-m+2)}{(m-1)!}$$
$$\leqslant m r^{N-m+1} \max_{i=\overline{1,m}} \|y_{0,i}\| \frac{N^{m-1}}{(m-1)!} \xrightarrow{N\to\infty} 0.$$

Then there exists $\widetilde{N} \in \mathbb{N}$ such, that for all $i = \overline{1,m}$ it is right that

$$\|z_{0,i}\| < u_{\max}.$$

Let us determine $v(0) = \ldots = v(\widetilde{N}-2) = 0$ and $v(\widetilde{N}-1) = -z_0 \in [-u_{\max}; u_{\max}]^{2m}$. We obtain the result $y(\widetilde{N}) = 0$, i.e. $y_0 \in \mathcal{Y}(\widetilde{N})$. Therefore, we obtain the result $\mathcal{Y}_\infty = \mathbb{R}^{2m}$ by choosing arbitrary $y_0 \in \mathbb{R}^{2m}$ and equation (3).

Let us assume, that $r = 1$. Then by (A.4) for some $N_m \in \mathbb{N}$ and mth two-dimensional subvector $y(N_m)$ it is true, that

$$y_m(N_m) = A_{N_m\varphi} y_m(0) + \sum_{k=0}^{N_m-1} A_{k\varphi} v_m(N_m-k-1).$$

Let us take $N_m \in \mathbb{N}$ which hold the inequality $\|y_m(0)\| \leqslant N_m u_{\max}$. Then we choose $v(0), \ldots, v(N_m-1) \in [-u_{\max}; u_{\max}]^{2m}$ in accordance with the equality

$$v_m(N_m-k-1) = -\frac{A_{(N_m-k)\varphi} y_m(0)}{N_m} \in [-u_{\max}; u_{\max}]^{2m}, \quad k = \overline{0, N_m-1}.$$

We obtain

$$y_m(N_m) = A_{N_m\varphi}y_m(0) + \sum_{k=0}^{N_m-1} \frac{-A_{N_m\varphi}y_m(0)}{N_m} = 0.$$

Let us assume that for some $N \in \mathbb{N}$ and $i = \overline{1, m-1}$ the relation $y_m(N) = \ldots = y_{m-i+1}(N) = 0$ is correct. Then if $v_m(N + N_{m-i} - k - 1) = \ldots = v_{m-i+1}(N + N_{m-i} - k - 1) = 0$, $k = \overline{0, N_{m-i} - 1}$, then according to the (A.4)

$$y_{m-i}(N + N_{m-i}) = A_{N_{m-i}\varphi}y_{m-i}(N) + \sum_{k=0}^{N_{m-i}-1} A_{k\varphi}v_{m-i}(N + N_{m-i} - k - 1),$$

where $N_{m-i} \in \mathbb{N}$ is selected from the condition $\|y_{m-i}(N)\| \leqslant N_{m-i}u_{\max}$. Then we define $v(N), \ldots,$ $v(N + N_{m-i} - 1) \in [-u_{\max}; u_{\max}]^{2m}$ in order that

$$v_{m-i}(N + N_{m-i} - k - 1) = \frac{-A_{(N_{m-i}-k)\varphi}y_{m-i}(N)}{N_{m-i}}, \quad k = \overline{0, N_{m-i} - 1}.$$

We obtain

$$y_{m-i}(N + N_{m-i}) = A_{N_{m-i}\varphi}y_{m-i}(N) + \sum_{k=0}^{N_{m-i}-1} \frac{-A_{N_{m-i}\varphi}y_{m-i}(N)}{N_{m-i}} = 0,$$
$$y_m(N + N_{m-i}) = \ldots = y_{m-i+1}(N + N_{m-i}) = 0.$$

Then there exists $N \in \mathbb{N}$ such, that $y(N) = 0$, i.e. $y_0 \in \mathcal{Y}(N)$ by the method of mathematical induction. Therefore, we obtain the result $\mathcal{Y}_\infty = \mathbb{R}^{2m}$ by choosing arbitrary $y_0 \in \mathbb{R}^{2m}$ and equation (3).

Hence the Lemma 3 is proved.

Proof of Theorem 1. Let $x_0 \in \mathcal{X}_\infty$, which by (3) is equivalent to the existence of $N \in \mathbb{N} \cup \{0\}$ such, that $x_0 \in \mathcal{X}(N)$. By (2) there exist $u(0), \ldots, u(N-1) \in \mathcal{U}$ such, that $x(N) = 0$. Based on (1), for all $h \in \partial B_1(0)$, $\varepsilon > 0$ the following relations hold

$$\widetilde{x}(N) = A^N(x_0 + \varepsilon h) + A^{N-1}u(0) + \ldots + u(N-1)$$
$$= A^N x_0 + A^{N-1}u(0) + \ldots + u(N-1) + A^N h\varepsilon = x(N) + A^N h\varepsilon = A^N h\varepsilon,$$
$$\widetilde{x}(N+1) = A\widetilde{x}(N) + u(N) = A^{N+1}h\varepsilon + u(N),$$

where $u(N) \in \mathcal{U}$. Considering $0 \in \operatorname{int} \mathcal{U}$, there exists $\delta > 0$ such, that $O_\delta(0) \subset \mathcal{U}$, and $\varepsilon > 0$ such, that $\varepsilon A^{N+1}B_1(0) \subset O_\delta(0)$. Let us take

$$u(N) = -A^{N+1}h\varepsilon \in \varepsilon A^{N+1}B_1(0) \subset O_\delta(0) \subset \mathcal{U}.$$

Then $\widetilde{x}(N+1) = 0$, i.e. for all $h \in B_1(0)$ it is true that $x_0 + \varepsilon h \in \mathcal{X}(N+1)$. As a result, $B_\varepsilon(x_0) \subset \mathcal{X}(N+1) \subset \mathcal{X}_\infty$, i.e. $x_0 \in \operatorname{int} \mathcal{X}_\infty$. Hence, $\mathcal{X}_\infty$ is open.

Let $x_{0,1}, x_{0,2} \in \mathcal{X}_\infty, \alpha \in [0; 1]$. Then there exists $N \in \mathbb{N} \cup \{0\}$ such, that $x_{0,1}, x_{0,2} \in \mathcal{X}(N)$, i.e. there exist $u^1(0), u^1(1), \ldots, u^1(N-1), u^2(0), u^2(1), \ldots, u^2(N-1) \in \mathcal{U}$ such, that $x^1(N) = 0$,

$x^2(N) = 0$. According to the (1) it is true that

$$0 = x^1(N) = A^N x_{0,1} + A^{N-1}u^1(0) + A^{N-2}u^1(1) + \ldots + u^1(N-1),$$
$$0 = x^2(N) = A^N x_{0,2} + A^{N-1}u^2(0) + A^{N-2}u^2(1) + \ldots + u^2(N-1),$$
$$0 = \alpha x^1(N) = \alpha A^N x_{0,1} + \sum_{k=0}^{N-1} \alpha A^k u^1(N-k-1),$$
$$0 = (1-\alpha)x^2(N) = (1-\alpha)A^N x_{0,2} + \sum_{k=0}^{N-1} (1-\alpha)A^k u^2(N-k-1),$$
$$0 = A^N(\alpha x_{0,1} + (1-\alpha)x_{0,2}) + \sum_{k=0}^{N-1} A^k(\alpha u^1(N-k-1) + (1-\alpha)u^2(N-k-1)).$$

According to the convexity of $\mathcal{U}$ the relatios $v(N-k-1) = \alpha u^1(N-k-1) + (1-\alpha)u^2(N-k-1) \in \mathcal{U}$, $k = \overline{0, N-1}$ are correct. Then $\alpha x_{0,1} + (1-\alpha)x_{0,2} \in \mathcal{X}(N) \subset \mathcal{X}_\infty$, from which it follows that $\mathcal{X}_\infty$ is convex.

The Theorem 1 is proved.

Proof of Lemma 4. Let $x_0 \in \mathcal{X}_\infty$. Then by (3) there exists $N \in \mathbb{N} \cup \{0\}$ such, that $x_0 \in \mathcal{X}(N)$. According to the (2) there exist $u(0), u(1), \ldots, u(N-1) \in \mathcal{U}$ such, that $x(N) = 0$. Based on (1), it is right that

$$0 = x(N) = A^N x_0 + \sum_{k=0}^{N-1} A^k u(N-k-1),$$
$$0 = x_0 + A^{-N} \sum_{k=0}^{N-1} A^k u(N-k-1),$$
$$x_0 = -\sum_{k=0}^{N-1} A^{-N+k} u(N-k-1) = -\sum_{k=1}^{N} A^{-k} u(k-1).$$

Let p belong to $\mathbb{R}^n \setminus \{0\}$. Then

$$(p, x_0) = \left(p, -\sum_{k=1}^{N} A^{-k} u(k-1)\right) = \sum_{k=1}^{N} \Big(p, -A^{-k} u(k-1)\Big)$$
$$= \sum_{k=1}^{N} \Big((-A^{-k})^{\mathrm{T}} p, u(k-1)\Big) \leqslant \sum_{k=1}^{N} \max_{u_k \in \mathcal{U}} \Big(-(A^{-k})^{\mathrm{T}} p, u_k\Big).$$

Since $0 \in \mathcal{U}$, then for all $k \in \mathbb{N}$

$$\max_{u_k \in \mathcal{U}} \Big(-(A^{-k})^{\mathrm{T}} p, u_k\Big) \geqslant 0.$$

Then

$$(p, x_0) \leqslant \sum_{k=1}^{\infty} \max_{u_k \in \mathcal{U}} (-(A^{-k})^{\mathrm{T}})p, u_k),$$

i.e. $x_0 \in \mathcal{H}_p$. It follows that $\mathcal{X}_\infty \subset \mathcal{H}_p$.

Let us consider the following quantity for some $p \in \mathbb{R}^n \setminus \{0\}$:

$$(p, x^*) = \sum_{k=1}^{\infty} (p, -A^{-k} u_k^*) = \sum_{k=1}^{\infty} (-(A^{-k})^{\mathrm{T}} p, u_k^*) = \sum_{k=1}^{\infty} \max_{u_k \in \mathcal{U}} \Big(-(A^{-k})^{\mathrm{T}} p, u_k\Big).$$

Then $x^* \in \partial \mathcal{H}_p$.

Since all eigenvalues of the matrix A are strictly greater than 1 in absolute value, then the series $\sum\limits_{k=1}^{\infty} A^{-k}u_k^*$ converges. Then $x_N = -\sum\limits_{k=1}^{N} A^{-k}u_k^* \overset{N\to\infty}{\longrightarrow} x^*$. Demonstrate that $x_N \in \mathcal{X}(N) \subset \mathcal{X}_\infty$. Let us take $u(k) = u_{k+1}^* \in \mathcal{U}$. Then

$$x(N) = A^N x_N + \sum_{k=0}^{N-1} A^k u(N-k-1) = -\sum_{k=1}^{N} A^{N-k}u_k^* + \sum_{k=0}^{N-1} A^k u_{N-k-1}^* = 0.$$

Then $x_N \in \mathcal{X}(N) \subset \mathcal{X}_\infty$ for all $N \in \mathbb{N}$. It follows that $x^* = \lim\limits_{N\to\infty} x_N \in \overline{\mathcal{X}_\infty}$.

The Lemma 4 is proved.

Corollary 2. *Let all eigenvalues of the matrix $A \in \mathbb{R}^{n\times n}$ be strictly greater than 1 in absolute value, $\mathcal{X}_\infty$ is defined by (3).*

Then for all $p \in \mathbb{R}^n \setminus \{0\}$ it is right that:

1) $\mathcal{X}_\infty \subset \mathcal{H}_{-p} = \left\{ x \in \mathbb{R}^n \colon (p, x) \geqslant \sum\limits_{k=1}^{\infty} \min\limits_{u_k\in\mathcal{U}} (-(A^{-k})^{\mathrm{T}}p, u_k) \right\}$;

2) $x^* = -\sum\limits_{k=1}^{\infty} A^{-k}u_k^* \in \overline{\mathcal{X}_\infty} \cap \partial\mathcal{H}_{-p}$, *where*

$$u_k^* = \arg \min_{u_k\in\mathcal{U}} \left(-(A^{-k})^{\mathrm{T}}p, u_k \right).$$

Proof of Corollary 2. For proving that it is sufficient to consider the conditions of the Lemma 4 for the vector $-p$.

Proof of Lemma 5. Let $p = \begin{pmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^n$, where 1 corresponds to the ith coordinate of the vector p. Then for arbitrary $k \in \mathbb{N}$

$$A^{-k} = \begin{pmatrix} \lambda^{-k} & (-1)k\lambda^{-k-1} & (-1)^2k(k+1)\lambda^{-k-2}\frac{1}{2} & \ldots & (-1)^{n-1}\dfrac{(k+n-2)!}{(n-1)!(k-1)!}\lambda^{-k-n+1} \\ 0 & \lambda^{-k} & (-1)k\lambda^{-k-1} & \ldots & (-1)^{n-2}\dfrac{(k+n-3)!}{(n-2)!(k-1)!}\lambda^{-k-n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda^{-k} \end{pmatrix},$$

$$-(A^{-k})^{\mathrm{T}}p = -\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda^{-k} \\ (-1)k\lambda^{-k-1} \\ \vdots \\ (-1)^{n-i}\dfrac{(k+n-i-1)!}{(k-1)!(n-i)!}\lambda^{-k-n+i} \end{pmatrix},$$

$$(-(A^{-k})^{\mathrm{T}}p, u) = -(\lambda^{-k}u_i - k\lambda^{-k-1}u_{i+1} + \ldots + (-1)^{n-i}\lambda^{-k-n+i}C_{k+n-i-1}^{n-i}u_n)$$

$$= -\sum_{j=0}^{n-i} \lambda^{-k-j}(-1)^j u_{j+i}C_{k+j-1}^{j} = \sum_{j=0}^{n-i} \lambda^{-k-j}(-1)^{j+1}u_{j+i}C_{k+j-1}^{j}.$$

Let us consider the case $\lambda > 1$. Since $u_i \in [u_{i,\min}; u_{i,\max}]$, then for all $k \in \mathbb{N}$ the following inequalities hold

$$(-(A^{-k})^{\mathrm{T}}p, u) \leqslant \sum_{j=0}^{n-i} \lambda^{-k-j} C_{k+j-1}^{j} \max\{(-1)^{j+1}u_{j+i,\min}; (-1)^{j+1}u_{j+i,\max}\},$$

$$-(-(A^{-k})^{\mathrm{T}}p, u) \leqslant -\sum_{j=0}^{n-i} \lambda^{-k-j} C_{k+j-1}^{j} \min\{(-1)^{j+1}u_{j+i,\min}; (-1)^{j+1}u_{j+i,\max}\}.$$

Additionally, the following equalities hold

$$\sum_{k=1}^{\infty}\sum_{j=0}^{n-i} \lambda^{-k-j} C_{k+j-1}^{j} = \sum_{j=0}^{n-i}\sum_{k=1}^{\infty} \lambda^{-k-j} C_{k+j-1}^{j} = \sum_{j=0}^{n-i} \frac{1}{(\lambda-1)^{j+1}}.$$

By Lemma 4 and Corollary 2, which implies, that for all $x \in \mathcal{X}_\infty$ inequalities hold

$$\sum_{j=0}^{n-i} \frac{\min\{(-1)^{j+1}u_{j+i,\min}; (-1)^{j+1}u_{j+i,\max}\}}{(\lambda-1)^{j+1}} \leqslant (p, x)$$
$$\leqslant \sum_{j=0}^{n-i} \frac{\max\{(-1)^{j+1}u_{j+i,\min}; (-1)^{j+1}u_{j+i,\max}\}}{(\lambda-1)^{j+1}}.$$

Since $\mathcal{X}_\infty$ is open according to the Theorem 1, these inequalities strictly hold, i.e.

$$\mathcal{X}_\infty \subset \bigcap_{i=1}^{n} \{x \in \mathbb{R}^n \colon x_i \in (x_{i,\min}; x_{i,\max})\}.$$

Let us consider the case $\lambda < -1$. For all $k \in \mathbb{N}$ it is true that

$$(-(A^{-k})^{\mathrm{T}}p, u) = \sum_{j=0}^{n-i} |\lambda|^{-k-j}(-1)^{-k-j+j+1}u_{j+i}C_{k+j-1}^{j} = \sum_{j=0}^{n-i} |\lambda|^{-k-j}(-1)^{-k+1}u_{j+i}C_{k+j-1}^{j}.$$

Then

$$\max_{u\in\mathcal{U}}(-(A^{-(2k-1)})^{\mathrm{T}}p, u) = \max_{u\in\mathcal{U}}\left(\sum_{j=0}^{n-i} |\lambda|^{-(2k-1)-j}(-1)^{-(2k-1)+1}u_{j+i}C_{(2k-1)+j-1}^{j}\right)$$
$$= \sum_{j=0}^{n-i} |\lambda|^{-(2k-1)-j}u_{i+j,\max}C_{(2k-1)+j-1}^{j},$$

$$\max_{u\in\mathcal{U}}(-(A^{-2k})^{\mathrm{T}}p, u) = \max_{u\in\mathcal{U}}\left(\sum_{j=0}^{n-i} |\lambda|^{-2k-j}(-1)^{-2k+1}u_{j+i}C_{2k+j-1}^{j}\right)$$
$$= \sum_{j=0}^{n-i} |\lambda|^{-2k-j}(-u_{i+j,\min})C_{2k+j-1}^{j}.$$

Then by Lemma 4 for all $x \in \mathcal{X}_\infty$ it is right, that

$$(p, x) \leqslant \sum_{k=1}^{\infty}\sum_{j=0}^{n-i} |\lambda|^{-(2k-1)-j}u_{i+j,\max}C_{(2k-1)+j-1}^{j} - \sum_{k=1}^{\infty}\sum_{j=0}^{n-i} |\lambda|^{-2k-j}u_{i+j,\min}C_{2k+j-1}^{j}$$
$$= \sum_{j=0}^{n-i} u_{i+j,\max}\left(\frac{1}{2(|\lambda|+1)^{j+1}} + \frac{1}{2(|\lambda|-1)^{j+1}}\right)$$
$$- \sum_{j=0}^{n-i} u_{i+j,\min}\left(\frac{1}{2(|\lambda|-1)^{j+1}} + \frac{1}{2(|\lambda|+1)^{j+1}}\right) = x_{i,\max}.$$

Similarly

$$\min_{u\in\mathcal{U}}\Big(-(A^{-(2k-1)})^{\mathrm{T}}p,u\Big)=\min_{u\in\mathcal{U}}\left(\sum_{j=0}^{n-i}|\lambda|^{-(2k-1)-j}(-1)^{-(2k-1)+1}u_{j+i}C^{j}_{(2k-1)+j-1}\right)$$
$$=\sum_{j=0}^{n-i}|\lambda|^{-(2k-1)-j}u_{j+i,\min}C^{j}_{(2k-1)+j-1},$$
$$\min_{u\in\mathcal{U}}\Big(-(A^{-2k})^{\mathrm{T}}p,u\Big)=\min_{u\in\mathcal{U}}\left(\sum_{j=0}^{n-i}|\lambda|^{-2k-j}(-1)^{-2k+1}u_{j+i}C^{j}_{2k+j-1}\right)$$
$$=\sum_{j=0}^{n-i}|\lambda|^{-2k-j}(-u_{j+i,\max})C^{j}_{2k+j-1}.$$

Then by Corollary 2 for all $x\in\mathcal{X}_\infty$ it is right that

$$(p,x)\geqslant\sum_{k=1}^{\infty}\sum_{j=0}^{n-i}|\lambda|^{-(2k-1)-j}u_{i+j,\min}C^{j}_{(2k-1)+j-1}-\sum_{k=1}^{\infty}\sum_{j=0}^{n-i}|\lambda|^{-2k-j}u_{i+j,\max}C^{j}_{2k+j-1}$$
$$=\sum_{j=0}^{n-i}u_{i+j,\min}\left(\frac{1}{2(|\lambda|+1)^{j+1}}+\frac{1}{2(|\lambda|-1)^{j+1}}\right)$$
$$-\sum_{j=0}^{n-i}u_{i+j,\max}\left(\frac{1}{2(|\lambda|-1)^{j+1}}+\frac{1}{2(|\lambda|+1)^{j+1}}\right)=x_{i,\min}.$$

Since $\mathcal{X}_\infty$ is open according to the 1, then

$$x_{i,\min}<(p,x)<x_{i,\max},$$
$$\mathcal{X}_\infty\subset\bigcap_{i=1}^{n}\{x\in\mathbb{R}^n\colon x_i\in(x_{i,\min};x_{i,\max})\}.$$

The Lemma 5 is proved.

Proof of Lemma 6. Let $p=(0\ 0\ \ldots\ \widetilde{p}^{\mathrm{T}}\ \ldots\ 0)^{\mathrm{T}}\in\mathbb{R}^{2n}$, $\widetilde{p}=(p_1\ p_2)^{\mathrm{T}}\in\mathbb{R}^2$, $p_1^2+p_2^2=1$, where $\widetilde{p}$ corresponds to the $(2i-1)$th and $2i$th coordinates of the vector p. Then for arbitrary $k\in\mathbb{N}$ it is true that

$$A^{-k}=\begin{pmatrix} r^{-k}A_{-k\varphi} & -kr^{-k-1}A_{(-k-1)\varphi} & \ldots & (-1)^{n-1}C^{n-1}_{n+k-2}r^{-k-n+1}A_{(-k-n+1)\varphi}\\ 0 & r^{-k}A_{-k\varphi} & \ldots & (-1)^{n-2}C^{n-2}_{n+k-3}r^{-k-n+2}A_{(-k-n+2)\varphi}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & r^{-k}A_{-k\varphi}\end{pmatrix},$$

$$-(A^{-k})^{\mathrm{T}}p=\begin{pmatrix}0\\ \vdots\\ 0\\ r^{-k}A_{-k\varphi}\widetilde{p}\\ -kr^{-k-1}A_{(-k-1)\varphi}\widetilde{p}\\ \vdots\\ (-1)^{n-i}C^{n-i}_{k+n-i-1}r^{-k-n+i}A_{(-k-n+i)\varphi}\widetilde{p}\end{pmatrix}.$$

Let $u^i \in \mathbb{R}^2$, $i = \overline{1,n}$, $u = ({u^1}^{\mathrm{T}}, \ldots, {u^n}^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{2n}$. Then

$$\Big(-(A^{-k})^{\mathrm{T}}p, u\Big) = -\Big(r^{-k}(A_{-k\varphi}\widetilde{p}, u^i) + (-1)kr^{-k-1}(A_{(-k-1)\varphi}\widetilde{p}, u^{i+1}) + \ldots$$
$$+ (-1)^{n-i}C_{k+n-i-1}^{n-i}r^{-k-n+i}(A_{(-k-n+i)\varphi}\widetilde{p}, u^n)\Big)$$
$$= -\sum_{j=0}^{n-i} r^{-k-j}(A_{(-k-j)\varphi}\widetilde{p}, u^{i+j})C_{k+j-1}^{j} \leqslant \sum_{j=0}^{n-i} r^{-k-j}\|(A_{(-k-j)\varphi}\widetilde{p}\|\|u^{i+j}\|C_{k+j-1}^{j}$$
$$= \sum_{j=0}^{n-i} r^{-k-j}\|u^{i+j}\|C_{k+j-1}^{j} \leqslant \sum_{j=0}^{n-i} r^{-k-j}r_{i+j,\max}C_{k+j-1}^{j}.$$

Then by Lemma 4 for arbitrary $x \in \mathcal{X}_\infty$ it is right that

$$(p, x) \leqslant \sum_{k=1}^{\infty}\sum_{j=0}^{n-i} r^{-k-j}r_{i+j,\max}C_{k+j-1}^{j} = \sum_{j=0}^{n-i}\sum_{k=1}^{\infty} r^{-k-j}r_{i+j,\max}C_{k+j-1}^{j} = \sum_{j=0}^{n-i}\frac{r_{i+j,\max}}{(r-1)^{j+1}}.$$

Since $\mathcal{X}_\infty$ according to the Theorem 1 is open, then

$$\mathcal{X}_\infty \subset \bigcap_{i=1}^{n}\{x \in \mathbb{R}^{2n} \colon \|(x_{2i-1}x_{2i})^{\mathrm{T}}\|_{\mathbb{R}^2} < R_{i,\max}\}.$$

The Lemma 6 is proved.

Proof of Theorem 2. Let us consider for some $B \in \mathbb{R}^{n\times n}$ and $\mathcal{C} \in \mathbb{K}_n$ the mapping

$$\widetilde{T}(\mathcal{X}) = B\mathcal{X} + \mathcal{C}.$$

Let us demonstrate, that if $B\colon \mathbb{R}^n \to \mathbb{R}^n$ is a contraction mapping with the compression ratio $\beta \in [0; 1)$, then $\widetilde{T}\colon \mathbb{K}_n \to \mathbb{K}_n$ is also a contraction mapping.

$$\rho_H(\widetilde{T}(\mathcal{X}), \widetilde{T}(\mathcal{Y})) = \max\left\{\sup_{x\in\widetilde{T}(\mathcal{X})}\inf_{y\in\widetilde{T}(\mathcal{Y})}\rho(x,y);\ \sup_{y\in\widetilde{T}(\mathcal{Y})}\inf_{x\in\widetilde{T}(\mathcal{X})}\rho(x,y)\right\}$$
$$= \max\left\{\sup_{\substack{x\in\mathcal{X}\\ c_1\in\mathcal{C}}}\inf_{\substack{y\in\mathcal{Y}\\ c_2\in\mathcal{C}}}\|Bx + c_1 - By - c_2\|; \sup_{\substack{y\in\mathcal{Y}\\ c_2\in\mathcal{C}}}\inf_{\substack{x\in\mathcal{X}\\ c_1\in\mathcal{C}}}\|Bx + c_1 - By - c_2\|\right\}$$
$$\leqslant \max\left\{\sup_{\substack{x\in\mathcal{X}\\ c_1\in\mathcal{C}}}\inf_{\substack{y\in\mathcal{Y}\\ c_2\in\mathcal{C}}}(\|B(x-y)\| + \|c_1 - c_2\|); \sup_{\substack{y\in\mathcal{Y}\\ c_2\in\mathcal{C}}}\inf_{\substack{x\in\mathcal{X}\\ c_1\in\mathcal{C}}}(\|B(x-y)\| + \|c_1 - c_2\|)\right\}$$
$$= \max\left\{\sup_{x\in\mathcal{X}}\inf_{y\in\mathcal{Y}}\|B(x-y)\|; \sup_{y\in\mathcal{Y}}\inf_{x\in\mathcal{X}}\|B(x-y)\|\right\}$$
$$\leqslant \max\left\{\sup_{x\in\mathcal{X}}\inf_{y\in\mathcal{Y}}\beta\|x-y\|; \sup_{y\in\mathcal{Y}}\inf_{x\in\mathcal{X}}\beta\|x-y\|\right\} = \beta\rho_H(\mathcal{X}, \mathcal{Y}).$$

Then $\widetilde{T}$ is a contraction mapping with the compression ratio β.

According to the (4) the following equality holds

$$T^M(\mathcal{X}) = A^{-M}\mathcal{X} + \sum_{k=1}^{M}(-A^{-k}\mathcal{U}) = \widetilde{T}(\mathcal{X}),$$

where $B = A^{-M}$, $\mathcal{C} = \sum\limits_{k=1}^{M}(-A^{-k}\mathcal{U})$.

Since all eigenvalues of the matrix A are strictly greater than 1 in absolute value, then all eigenvalues of the matrix A^{-1} are strictly less than 1 in absolute value. Then according to the [24, Theorem 5.6.12] $\|A^{-k}\| \overset{k\to\infty}{\longrightarrow} 0$. Then by definition of the limit for $\alpha \in [0; 1)$ there exists $M \in \mathbb{N}$ such, that $\|A^{-M}\| < \alpha$. As the following inequality

$$\|A^{-M}(x-y)\| \leqslant \|A^{-M}\| \cdot \|x-y\| < \alpha\|x-y\|,$$

is fair, then A^{-M} is a contraction mapping with the compression ratio $\alpha \in [0; 1)$. Then $T^M : \mathbb{K}_n \longrightarrow \mathbb{K}_n$ is also a contraction mapping with the compression ratio α.

By virtue of Lemma 7 for all $N \in \mathbb{N}$ it is right, that $\mathcal{X}(N) \subset \mathcal{X}(N+1)$, in addition, $\mathcal{X}(N)$ is a compact set. Then according to the [27, Corollary A.3.4]

$$\rho_H(\mathcal{X}_\infty, \mathcal{X}(N)) \overset{N\to\infty}{\longrightarrow} 0. \tag{A.5}$$

In contrast, by virtue of [27, Theorem A.3.9] the metric space $(\mathbb{K}_n, \rho_H)$ is complete. Then the contraction mapping $\widetilde{T}$ has a unique fixed point $\mathcal{X}^* \in \mathbb{K}_n$, which can be computed by fixed point iteration method:

$$\mathcal{X}^* = \lim_{N\to\infty} \underbrace{(\widetilde{T} \circ \ldots \circ \widetilde{T})}_{N}(\mathcal{X}), \tag{A.6}$$

where $\mathcal{X} \in \mathbb{K}_n$ is arbitrary. Let us take $\mathcal{X} = \{0\}$. Then by virtue of Lemma 7

$$\underbrace{(\widetilde{T} \circ \ldots \circ \widetilde{T})}_{N}(\{0\}) = -\sum_{k=1}^{MN} A^{-k}\mathcal{U} = \mathcal{X}(NM).$$

According to the uniqueness of the limit point and formulae (A.5) and (A.6)

$$\overline{\mathcal{X}_\infty} = \overline{\bigcup_{N=0}^{\infty} \mathcal{X}(N)} = \mathcal{X}^*.$$

The error in the fixed point iteration method can be estimated by the following formula [28]:

$$\rho_H(\overline{\mathcal{X}_\infty}, \mathcal{X}(NM)) \leqslant \frac{\alpha^N}{1-\alpha}\rho_H(\mathcal{X}(M), \{0\}).$$

The Theorem 2 is proved.

Proof of Theorem 3. By virtue of the point 3 of the Theorem 2

$$\rho_H(\overline{\mathcal{X}_\infty}, \mathcal{X}(NM)) \leqslant \frac{\alpha_p^N}{1-\alpha_p^N}\rho_H(\mathcal{X}(M), \{0\}) = R_p,\ p \in \{1, 2, \infty\}.$$

Then according to the definition of the Hausdorff distance

$$\mathcal{X}_\infty \subset \overline{\mathcal{X}_\infty} \subset \mathcal{X}(NM) + B_{R_p}(0),$$

where

$$B_{R_1}(0) = \operatorname{conv}\left\{(\underbrace{0,\ldots,0}_{i}, r, 0, \ldots, 0)^{\mathrm{T}} : r \in \{-R_1, R_1\},\ i = \overline{0, n-1}\right\},$$

$$B_{R_2}(0) = \left\{x \in \mathbb{R}^n\colon \sqrt{\sum_{i=1}^{n} |x_i|^2} \leqslant R_2\right\},$$

$$B_{R_\infty}(0) = \left\{x \in \mathbb{R}^n\colon \max_{i=\overline{1,n}} |x_i| \leqslant R_\infty\right\}.$$

The Theorem 3 is proved.

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