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An Attracting Cycle in a Coupled Mechanical System with Phase Shifts in Subsystem Oscillations

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Abstract—This paper considers the set of reversible mechanical systems with single-period oscillations and individual phase shifts in them. The problem of aggregating a coupled system with an attracting cycle is solved. The approach developed below is to choose a leader (control) system that acts on the other (follower) systems through one-way coupling control: in an aggregated system, there are no links between follower systems. Universal coupling controls are used. Particular attention is paid to conservative systems. Possible scenarios for the operation of the aggregated system are presented.

Keywords: reversible mechanical system, symmetric periodic motions, coupling controls, leader system, follower system, attracting cycle, stabilization

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1. PRELIMINARIES

Models containing coupled subsystems are studied in various fields of knowledge. In mechanics, A. Sommerfeld's sympathetic pendulums have become such a (classical) model. Other examples were given, e.g., in [1–5].

Aggregation consists in constructing a coupled system from a given set of systems so that the resulting whole will possess a desired dynamic property. In oscillation stabilization, this property is achieved, in particular, in an attracting cycle of the system. Aggregation occurs by finding appropriate coupling controls between the systems.

In the paper [6], the aggregation problem was solved for a set of conservative systems. According to [6, Lemma 1], there exists a cycle in the system only if all mechanical systems, possibly except one system with a degenerate family of oscillations, contain nondegenerate families of oscillations. Aggregation was carried out for systems containing nondegenerate families of oscillations that also form a nondegenerate family in an uncoupled system as a whole. The case of phase-synchronized oscillations in systems was considered. The universal control from [7] was applied.

At the same time, it is of definite interest to study oscillation modes in which, e.g., the phases in system oscillations are equidistant from each other or neighbor systems oscillate in antiphase. Therefore, a common problem is to find coupling controls for implementing an attracting cycle of the coupled mechanical system with phase shifts in the oscillations of its constituent systems. Also, it seems interesting to aggregate a coupled system containing one or more mechanical systems with degenerate families of oscillations. Thus, we arrive at the following general problem statement: aggregate a coupled system with an attracting cycle on a set of mechanical systems admitting oscillations.

Note that some approaches to aggregating a general-form autonomous system with an attracting cycle were proposed in [8]; Lyapunov method-based procedures for aggregating a complex system were described in [9].

2. PROBLEM STATEMENT

Consider the set Ξ of n smooth reversible mechanical systems with one degree of freedom

$$\ddot{q}_s + f_s(q_s, \dot{q}_s) = 0, \quad f_s(q_s, -\dot{q}_s) = f_s(q_s, \dot{q}_s), \quad s = 1, \ldots, n. \tag{1}$$

The phase portrait of the sth system is symmetric with respect to the fixed set $M_s = \{q_s, \dot{q}_s : \dot{q}_s = 0\}$, where q_s denotes the generalized coordinate. By assumption, each system of the set Ξ admits a single-frequency oscillation. It will be symmetric with respect to the set M_s and represents a symmetric periodic motion (SPM). The SPM is described by the formula

$$q_s = \varphi_s(h_s, t + \gamma_s), \quad s = 1, \ldots, n,$$

where the period $T_s(h_s)$ depends on the parameter h_s and the parameter γ_s specifies the time shift of the initial point: for $\gamma_s = 0$, the initial point belongs to the fixed set M_s. In this case, the SPM is described by a function even in the variable t. SPMs always form families. In a conservative system, the function f_s does not depend on the velocity $\dot{q}_s$.

Further considerations involve a definition from [6].

Definition 1. A family of SPMs in the parameter h is said to be nondegenerate if the derivative of the period $T(h)$ with respect to the variable h differs from zero on this family. An SPM of a nondegenerate family is called nondegenerate as well.

The period $T(h)$ on the family of nondegenerate SPMs can increase or decrease. For example, the period of oscillations of a mathematical pendulum monotonically increases with the energy of the pendulum, and the oscillations are nondegenerate. The solutions of the equation $\ddot{x} + x^3 = 0$ belong to the family of SPMs with a decreasing period.

The oscillations of a linear oscillator are isochronous and form a degenerate family of SPMs. As a rule, a degenerate SPM of a nonlinear system is on the boundary of the family of its nondegenerate SPMs. In a conservative system, the parameter h is usually the constant of the energy integral.

The general problem statement involves the set Ξ of reversible mechanical systems containing nondegenerate (and/or degenerate) families of SPMs with increasing (and/or decreasing) periods on the family. In this case, if the set Ξ simultaneously includes a system with an increasing period ($dT_1/dh_1 > 0$) and a system with a decreasing period ($dT_2/dh_2 < 0$), the period curves will intersect at one point where $T_1(h_1^*) = T_2(h_2^*) = T^*$. The phases of oscillations generally differ. If the set Ξ also includes a system with a degenerate family, the period on it will equal T^* as well. The set of three equations in Ξ leads in the coupled system to two arbitrary phases of oscillations in the systems. Given an arbitrary number n of equations in Ξ, it is assumed that $\gamma_s = \gamma_1 + \delta_s$, $s = 2, \ldots, n$. Therefore, we pose the problem of aggregating a coupled system with an attracting cycle for all possible vectors $\delta = (\delta_2, \ldots, \delta_n)$.

Further analysis focuses on an autonomous coupled mechanical system of the form

$$\ddot{q}_s + f_s(q_s, \dot{q}_s) = \varepsilon \sigma_s u_s(q, \dot{q}), \quad s = 1, \ldots, n, \tag{2}$$

where the coupling control

$$u(q, \dot{q}) = (u_1(q, \dot{q}), \ldots, u_n(q, \dot{q})) \tag{3}$$

acts with a small gain ε: the switches σ_s are $+1$ or (-1). By assumption, for $\varepsilon = 0$, system (2), treated as a whole, admits a T^*-periodic SPM. The problem is to find the coupling control (3) ensuring the existence of an attracting cycle with the period T^* in system (2).

This problem covers the following special cases:

1) All reversible mechanical systems in the set Ξ admit a family of nondegenerate SPMs with an increasing (decreasing) period.

2) The monotonicity of the period in the systems differs by character.

3) The set of mechanical systems contains nondegenerate and degenerate families of SPMs.

In [6], case 1) was investigated for conservative systems under the additional assumption that the set of uncoupled systems as a whole admits a nondegenerate family of SPMs.

3. UNIVERSAL COUPLING CONTROLS UNDER $\delta \neq 0$

For the vector $\delta \neq 0$, we find universal coupling controls ensuring the existence and orbital asymptotic stability of the cycle of system (2). Such coupling controls can be treated as a generalization of the couplings from [7].

The proposed coupling controls have the form

$$\begin{aligned} u_1 &= [1 - K_1(h_1)q_1^2]\dot{q}_1, \\ u_j &= [1 - K_j(h_j, \delta_j)q_1^2]\dot{q}_j, \quad j = 2, \ldots, n. \end{aligned} \tag{4}$$

The functions $K_1(h_1)$ and $K_j(h_j, \delta_j)$ are calculated below.

By assumption, for $\varepsilon = 0$ system (2) admits a T^*-periodic SPM and the corresponding values in the subsystems are $h_s = h_s^*$, $s = 1, \ldots, n$. According to formulas (4), the equations in (2) become unequal: we construct a controlled coupled system in which the system with number $s = 1$ is the leader and the other systems are followers. Another feature of the controls (4) is that the subsystems with numbers $s = 2, \ldots, n$ have no direct influence on each other. Due to these remarks, we analyze $(n-1)$ independent subsystems of the same type:

$$\begin{aligned} \ddot{q}_1 + f_1(q_1, \dot{q}_1) &= \varepsilon\sigma_1[1 - K_1(h_1^*)q_1^2]\dot{q}_1, \\ \ddot{q}_j + f_j(q_j, \dot{q}_j) &= \varepsilon\sigma_j[1 - K_j(h_j^*, \delta_j)q_1^2]\dot{q}_j, \quad j = 2, \ldots, n. \end{aligned} \tag{5}$$

For the subsystem with number j in (5), we solve the cycle problem under the condition $\varepsilon \neq 0$. Then, applying the obtained result to all subsystems with numbers $j = 2, \ldots, n$, we come to the solution of the cycle problem for the coupled system. In system (5), $K_1(h_1^*)$ and $K_j(h_j^*)$ denote values. In addition, h_1^* and h_j^* mean that the controls are chosen for the SPM with the period T^* and the corresponding values $h_1 = h_1^*$ and $h_j = h_j^*$. On the other hand, when solving the control problem in (5) for another pair (h_1, h_j), a different pair of the coefficients $(K_1(h_1), K_j(h_j, \delta))$ is chosen: in (6), the control is applied with some changed coefficients K_1 and K_j. Hence, we design an adaptive control system in (6).

Thus, for the adaptive control system (6), it is required to find the relationships $K_1(h_1)$ and $K_j(h_j, \delta)$ (the second with the parameter δ) ensuring the existence of an attracting cycle.

For the subsystem with number j, we write the system of amplitude equations

$$\begin{aligned} I_1(h_1) &\equiv \int_0^{T^*} [1 - K_1(h_1^*)\varphi_1^2(h_1, t)]\dot{\varphi}_1(h_1, t)\psi_1(h_1, t)dt = 0, \\ I_j(h_1, h_j, \delta_j) &\equiv \int_0^{T^*} [1 - K_j(h_j^*, \delta_j)\varphi_1^2(h_1, t)]\dot{\varphi}_j(h_j, t + \delta_j)\psi_j(h_j, t + \delta_j)dt = 0. \end{aligned} \tag{6}$$

These equations are used to find $h_1 = h_1^*$ and $h_j = h_j^*$ that meet the necessary conditions for the existence of a cycle with the period T^* in the controlled system (5). In (6), $(\psi_1(h_1, t), \psi_j(h_j, t + \delta_j))$

denotes the solution of the adjoint equation for $q_1 = \varphi_1(h_1, t)$ and $q_j = \varphi_j(h_j, t + \delta_j)$. This solution is calculated in the Appendix.

The first equation in system (6) is the same for all numbers j. It can be analyzed independently of the second one.

We begin with the first equation of (5). The necessary conditions for the existence of a cycle must hold for all values of the parameter h_1 and the corresponding values of the period $T_1(h_1)$. Therefore,

$$\int\limits_0^{T_1(h_1)} [1 - K_1(h_1)\varphi_1^2(h_1, t)]\dot{\varphi}_1(h_1, t)\psi_1(h_1, t)dt \equiv 0, \tag{7}$$

which gives

$$K_1(h_1) = \frac{\int\limits_0^{T_1(h_1)} \dot{\varphi}_1(h_1, t)\psi_1(h_1, t)dt}{\int\limits_0^{T_1(h_1)} \varphi_1^2(h_1, t)\dot{\varphi}_1(h_1, t)\psi_1(h_1, t)dt}.$$

The denominator of this expression does not vanish; for the case of a conservative system, see Section 4. In the general case of a reversible mechanical system, this result follows from the solution of the adjoint equation calculated in the Appendix.

In view of the odd function $\dot{\varphi}_1(h_1, t)$ and the equality $T_1(h_1^*) = T^*$, we determine the derivative of the function $I_1(h_1)$ at the point $h_1 = h_1^*$ from identity (7):

$$\frac{dI_1(h_1^*)}{dh_1} = \chi_1\nu_1,$$

$$\chi_1 = \frac{dK_1(h_1^*)}{dh_1}, \quad \nu_1 = \int\limits_0^{T^*} \varphi_1(h_1^*, t)^2\dot{\varphi}_1(h_1^*, t)\psi_1(h_1^*, t)dt.$$

The equality $I(h_1^*) = 0$ means that the necessary condition for the existence of a T^*-periodic solution holds in the first equation of system (5). Due to the inequality $\chi_1\nu_1 \neq 0$, this solution is a cycle, which becomes attracting under an appropriately chosen sign of σ_1 (see [7]).

The second equation of system (6) is considered by analogy. We define the function

$$K_j(h_j, \delta_j) = \frac{\int\limits_0^{T_j(h_j)} \dot{\varphi}_j(h_j, t + \delta_j)\psi_j(h_j, t + \delta_j)dt}{\int\limits_0^{T_j(h_j)} \varphi_1^2(h_1^*, t)\dot{\varphi}_j(h_j, t + \delta_j)\psi_j(h_j, t + \delta_j)dt}$$

and calculate the derivative

$$\frac{dI_j(h_1^*, h_j^*, \delta_j)}{dh_j} = \chi_j\nu_j,$$

$$\chi_j = \frac{dK_j(h_j^*, \delta_j)}{dh_j}, \quad \nu_j = \int\limits_0^{T^*} \varphi_1(h_1^*, t)^2\dot{\varphi}_j(h_j^*, t + \delta_j)\psi_j(h_j^*, t + \delta_j)dt$$

for $h_j = h_j^*$ $(h_1 = h_1^*)$.

The conditions $\chi_1\nu_1 \neq 0$ and $\chi_j\nu_j \neq 0$ are now sufficient for the existence of a simple root (h_1^*, h_j^*) of the system of amplitude equations (6) with a fixed number j. Then the simplicity of this root ensures the existence of a cycle in system (5) with a fixed number j. The cycle will be attracting if the switches are chosen from the conditions $\sigma_1\chi_1\nu_1 < 0$ and $\sigma_j\chi_j\nu_j < 0$.

Consider the systems of amplitude equations (6) for all numbers $j = 2, \ldots, n$. Then, under the inequalities $\chi_s\nu_s \neq 0,\ s = 1, \ldots, n$, a cycle is implemented in the coupled system (5). Given the additional condition $\sigma_s\chi_s\nu_s < 0,\ s = 1, \ldots, n$, the cycle becomes attracting.

Thus, the following result is true.

Theorem 1. *Assume that the set of reversible mechanical systems with one degree of freedom admits a T^*-periodic motion. Then the coupled mechanical system* (5)*, where $j = 2, \ldots, n$, has a unique cycle of the period T^* if $\chi_s\nu_s \neq 0,\ s = 1, \ldots, n$. Under the additional conditions $\sigma_s\chi_s\nu_s < 0$, $s = 1, \ldots, n$, the cycle becomes attracting.*

Remark 1. The cycle of the coupled system (5) is determined within an arbitrary shift on the trajectory. The cycle-generating oscillations have the phase shifts $\delta_2, \ldots, \delta_n$ with respect to the phase of the oscillation in the first equation of system (5).

Remark 2. In system (6), the integral

$$\kappa_j = \int_0^{T^*} \dot{\varphi}_j(h_j^*, \tau + \delta_j)\psi_j(h_j^*, \tau + \delta_j)d\tau$$

does not depend on δ_j on the period. Therefore, for $\kappa_j \neq 0$, we define the function

$$K_j(h_j^*, \delta_j) = \frac{\kappa_j}{\int_0^{T^*} \varphi_1^2(h_1^*, \tau - \delta_j)\dot{\varphi}_j(h_j^*, \tau)\psi_j(h_j^*, \tau)d\tau}, \tag{8}$$

which will be T^*-periodic in δ_j.

Remark 3. In formula (8), the nonzero denominator defines the admissible range of the phase shift δ_j in the jth subsystem of system (5).

Remark 4. Theorem 1 designs the piecewise continuous system (5). Since the amplitude equations (6) are independent of σ_j, there exists a cycle in every smooth switchless system. The attraction conditions ($\chi_s\nu_s \neq 0$) must hold in the subsystem on the trajectories with both $h_s > h_s^*$ and $h_s < h_s^*$. Therefore, the signs of σ_s for these trajectories usually differ. An example of a switch control law was provided in [10].

4. CONSERVATIVE SYSTEMS

For the set of conservative systems, the functions f_s in (1) are independent of the velocities $\dot{q}_s$, and each system admits an energy integral under $\varepsilon = 0$. The variational equations for SPMs contain a symmetric matrix; therefore, the one-degree-of-freedom system under consideration satisfies the equations

$$\psi_s(h_s^*, \tau + \delta_s) = -\dot{\varphi}_s(h_s^*, \tau + \delta_s), \quad s = 1, \ldots, n \quad (\delta_1 = 0).$$

Consequently, $\nu_s > 0, \kappa_s < 0,\ s = 1, \ldots, n$.

The integrand in (8) is $(T^*/2)$-periodic on δ and two points symmetric with respect to the fixed set correspond to each value $K_j(h_j^*, \delta^*)$. In turn, these points implement one cycle.

Thus, we arrive at the following result.

Theorem 2. *For the set of conservative systems with one degree of freedom that admits a T^*-periodic SPM, the coupled system* (5) *has a unique attracting cycle under the conditions $\sigma_s\chi_s < 0$, $s = 1, \ldots, n$.*

Example 1. In the coupled system

$$\begin{aligned} &\ddot{x} + \sin x = \varepsilon(1 - K_x(h_x)x^2)\dot{x}, \\ &\ddot{y} + y^3/4 = \sigma\varepsilon(1 - K_y(h_y, \delta)x^2)\dot{y} \end{aligned} \tag{9}$$

with $\varepsilon = 0$, the first equation describes a mathematical pendulum. Starting at 2π, the period $T_x(h_x)$ grows monotonically with the pendulum energy h_x on the family of oscillations, and the function $K_x(h_x)$ is monotonically decreasing (see [11]). The solutions of the second equation form a family of oscillations with the period $T_y(h_y)$ representing a decreasing function of the constant energy h_y.

Indeed, the period $T_y(h_y)$ is given by

$$T_y(h_y) = 2\int\limits_{-y(0)}^{y(0)} \frac{dy}{\sqrt{h - y^4}},$$

where $y(0)$ denotes the initial value of the variable y. Passing to the variable $z = y/h_y^{1/4}$ yields

$$T_y(h_y) = \frac{2}{h_y^{1/4}}\int\limits_{1}^{-1} \frac{dz}{\sqrt{1 - z^4}} = \frac{a}{h_y^{1/4}}, \quad a = 4 * 1.3\ldots,$$

an explicit-form relationship between the period and the system energy.

According to the analysis above, for any h_x^* and $T_x(h_x^*) > 2\pi$, there exists a value h_y^* such that $T_x(h_x^*) = T_y(h_y^*)$ (the equality of periods in (9)) with an increasing function f, i.e., $h_y^* = f(h_x^*)$. Hence, system (9) with $\varepsilon = 0$ admits a one-parameter family of SPMs with the parameter h_x^*.

The function $K_x(h_x)$ is monotonically decreasing. Therefore, for $h_x = h_x^*$, the first equation in (9) has an attracting cycle. By Theorem 2, the additional condition $dK_y(h_y^*, \delta)/dh_y \neq 0$ leads to an attracting cycle of the coupled system (9).

Thus, for any oscillation of the mathematical pendulum corresponding to the energy value h_x^*, there exists an energy value h_y^* of the second equation in (9) such that an attracting cycle is implemented in the coupled system. Moreover, the phases in the oscillations of the equations differ by the desired value δ.

5. THE CASE OF A DEGENERATE FAMILY OF SPMS

Under identical phases in the oscillations of subsystems, a cycle in the coupled system exists only if all mechanical systems, possibly except one system with a degenerate family of SPMs, contain nondegenerate families of SPMs. This result was established in [6, Lemma 1]. In what follows, we investigate in detail the case where one of the families is degenerate. Assume that the oscillations in the systems are not phase-synchronized and $n = 2$ in system (5).

Consider the system

$$\begin{aligned} &\ddot{x} + x = \varepsilon(1 - K_x(h_x)x^2)\dot{x}, \\ &\ddot{y} + f(y) = \varepsilon\sigma(1 - K_y(h_y, \delta)x^2)\dot{y}, \end{aligned} \tag{10}$$

in which the first equation contains a degenerate family of oscillations under $\varepsilon = 0$ and the period of oscillations in the second equation monotonically depends on the energy h_y. The solution of the uncoupled system is described by the formulas $x = A_x \cos t$ and $y = \varphi(h_y, t + \delta)$. On the generating solution, we have $A_x = 2/\sqrt{K_x}$, and the period of oscillations 2π corresponds to the constant h_y^* in the second equation. Let us find the relationship between $K_y(h_y, \delta)$ and K_x.

For system (10), formula (8) is written as

$$K_y(h_y^*,\delta) = -\frac{\kappa}{\int\limits_0^{T^*} A_x^2 \cos^2 t\dot{\varphi}^2(h_y^*, t+\delta)dt}, \qquad (11)$$
$$\kappa = -\int\limits_0^{T^*} \dot{\varphi}^2(h_y^*, t)dt, \quad T^* = 2\pi.$$

The integral in the denominator can be transformed as follows:

$$\frac{1}{2}\int\limits_0^{T^*}(1+\cos 2t)\dot{\varphi}^2(h_y^*, t+\delta)dt = \frac{1}{2}\int\limits_{\delta}^{T^*+\delta}\dot{\varphi}^2(h_y^*,\tau)d\tau$$
$$+\frac{1}{2}\left(\cos 2\delta\int\limits_{\delta}^{T^*+\delta}\cos 2\tau\dot{\varphi}^2(h_y^*,\tau)d\tau + \sin 2\delta\int\limits_{\delta}^{T^*+\delta}\sin 2\tau\dot{\varphi}^2(h_y^*,\tau)d\tau\right).$$

Consider the bracketed expression above; here, the first integral of a 2π-periodic function does not depend on δ on the period, and the second integral is taken for an odd function (and vanishes accordingly). Hence, we obtain a linear function of $\cos 2\varphi$, and $K_y(h_y^*,\delta)$ is given by an even π-periodic function of δ.

Due to the equality $K_x = 4/A_x^2$, formula (11) reduces to

$$K_y(h_y^*,\delta) = \frac{K_x\int\limits_0^{T^*}\dot{\varphi}^2(h_y^*,t)dt}{2\int\limits_0^{T^*}(1+\cos 2\delta\cos 2t)\dot{\varphi}^2(h_y^*,t)dt}. \qquad (12)$$

The derivative of (12) is an odd π-periodic function of δ. On the interval $\delta \in (-\pi/2, \pi/2)$, this derivative vanishes for $\delta = 0$.

The cycle of the coupled system (2) can be constructed using Theorem (10). The characteristic $K_y(h_y,\delta)$ is calculated for a given function $\varphi(h_y,t)$. For a mathematical pendulum, the function $K_y(h_y,\delta)$ is monotonically decreasing under $\delta = 0$ (see [11]).

Example 2. Consider system (10) in which the function $K_y(h_y,\delta)$ in the second equation is independent of δ and coincides with K_x. Let this system be applied in the mechatronic oscillation stabilization scheme proposed in [12]. More precisely put, the amplitude A_x at the point $\delta = \delta^*$ is selected to adjust the mode of satisfying the equality $K_y(h_y^*,\delta) = K_x = 4/A_x^2$. As a result, we obtain a possible scenario for the birth of a cycle described in [10]. Note that the existence of the scenario was proved by analyzing the second equation in (10) by substituting the generating solution of the first equation. In the mechatronic stabilization scheme, the time shift δ^* between the oscillations of the van der Pol oscillator and the mechanical system is given by (12).

6. THE CASE OF TWO DEGENERATE FAMILIES

In system (10), the van der Pol oscillator is used to generate signals for a mechanical system admitting a nondegenerate family of oscillations. The system is designed to stabilize mechanical oscillations. For $K_y = K_x$, the shift δ in the solutions of the equations of system (10) is given by formula (12); see Section 5.

It seems interesting to analyze how the amplitude and phase of oscillations in the leader and follower systems are synchronized in the cycle of the coupled system. We consider this problem for equal systems in Ξ, on an example of two identical linear oscillators. Then the coupled system has the form

$$\begin{aligned} \ddot{x} + x &= \varepsilon(1 - K_x(h_x)x^2)\dot{x}, \\ \ddot{y} + y &= \varepsilon\sigma(1 - K_y(h_y,\delta)x^2)\dot{y}, \end{aligned} \tag{13}$$

where the first equation describes the van der Pol oscillator and the second equation becomes the follower for this oscillator.

Under $\varepsilon = 0$, system (13) oscillates in each coordinate with a frequency of 1: the oscillations are isochronous. The generating oscillations are given by

$$x = A_x \cos t, \quad A_x = 2/\sqrt{K_x}, \quad y = A_y \cos(t+\delta).$$

For the second equation in (13), we calculate $\kappa = -\int\limits_0^{2\pi} A_y^2 \sin^2 t dt = -\pi A_y^2$.

In the coupled system, $K_y = K_y(h_y, \delta)$; therefore, formula (8) yields

$$K_y(h_y,\delta) = -\frac{4\kappa}{A_x^2 A_y^2 \pi(2-\cos 2\delta)} = \frac{4}{A_x^2(2-\cos 2\delta)} = \frac{K_x}{2-\cos 2\delta}.$$

Hence, in the cycle of the coupled system, the amplitudes of oscillations in the leader and follower systems are synchronized ($K_y = K_x$) only under $\delta = 0$; phase synchronization also occurs under $\delta = 0$.

According to the formula $K_y(h_y,\delta) = 2/(h_y(2-\cos 2\delta))$, the conditions for the existence of a cycle in the coupled system (10) hold everywhere in δ. The control law $\sigma = 1$ is chosen for the attracting cycle.

Note that the amplitudes of oscillations in the systems of the coupled system (13) are close to linear oscillations. Therefore, the oscillations of the systems will appear to be synchronized in δ in the cycle (operating mode) of the coupled system (13) under consideration regardless of the shift δ.

7. CONCLUSIONS

This paper has proposed an approach to aggregating a coupled system with an attracting cycle on a given set n of reversible mechanical systems with oscillations. Within the approach, a leader (control) system is selected to act on the other (follower) systems through one-way coupling control: in an aggregated system, there are no links between follower systems. The coupled system oscillates as $(n-1)$ independent subsystems controlled by the leader system. In addition, the oscillation of each system may have an individual phase shift with respect to the phase of the oscillation in the leader system.

Different control scenarios are possible for the aggregated system. If the subsystems have no phase shift, the *simultaneous control* scenario is implemented for $(n-1)$ mechanical systems; see [6]. The *conveyor* scenario is implemented in the controlled coupled system when specifying a shift change law for $(n-1)$ mechanical systems: for example, the maximum amplitude of oscillations in the follower systems is achieved at different time instants. For $n = 2$, the *leader–follower* scenario is implemented, a common one described, e.g., in [12] for a mechatronic oscillation stabilization scheme.

The aggregation approach has been presented on an example of reversible mechanical systems in the plane. It remains valid for a set of mechanical systems of arbitrary dimension. The constructed coupled system represents one level of the hierarchy of a multilevel aggregated system with an attracting cycle (for details, see [8]).

APPENDIX

The adjoint solution can be calculated using Lemma 1.

Consider a smooth reversible mechanical system of the second order:

$$\dot{u} = U(u, v), \quad \dot{v} = V(u, v), \quad U(u, -v) = -U(u, v), \quad V(u, -v) = V(u, v).$$

Let this system admit an SPM described by the functions

$$u = \varphi(t), \quad v = \theta(t), \quad \varphi(-t) = \varphi(t), \quad \theta(-t) = -\theta(t).$$

The variational equations for the SPM have the form

$$\begin{aligned} \dot{x} &= a_-(t)x + a_+(t)y, \\ \dot{y} &= b_+(t)x + b_-(t)y, \end{aligned} \tag{A.1}$$

where $a_\pm(t), b_\pm(t)$ denote even (+) and odd (–) periodic functions. They have the solution $x = \dot{\varphi}(t), y = \dot{\theta}(t)$.

Lemma 1. *For a given SPM, the solution of the system adjoint to* (A.1) *is calculated by constructive formulas.*

Proof. Let us apply the transformation

$$x = \xi_+(t)\tilde{x}, \quad y = \eta_+(t)\tilde{y}$$

with even periodic functions $\xi_+(t)$ and $\eta_+(t)$ with nonzero means. As a result,

$$\begin{aligned} \xi_+(t)\dot{\tilde{x}} + \dot{\xi}_+(t)\tilde{x} &= a_-(t)\xi_+(t)\tilde{x} + a_+(t)\eta_+(t)\tilde{y}, \\ \eta_+(t)\dot{\tilde{y}} + \dot{\eta}_+(t)\tilde{y} &= b_+(t)\xi_+(t)\tilde{x} + b_-(t)\eta_+(t)\tilde{y}. \end{aligned}$$

The functions $\xi_+(t)$ and $\eta_+(t)$ are appropriately chosen to satisfy the equalities

$$\dot{\xi}_+ = a_-(t)\xi_+, \quad \dot{\eta}_+ = b_-(t)\eta_+.$$

Then the transformed system

$$\dot{\tilde{x}} = \tilde{a}_+(t)\tilde{y}, \quad \dot{\tilde{y}} = \tilde{b}_+(t)\tilde{x} \tag{A.2}$$

contains no odd functions of t.

The adjoint system of

$$x_1 = \xi_{1+}(t)\tilde{x}_1, \quad y_1 = \eta_{1+}(t)\tilde{y}_1$$

is transformed by analogy. We obtain

$$\dot{\tilde{x}}_1 = -\tilde{b}_+(t)\tilde{y}_1, \quad \dot{\tilde{y}}_1 = -\tilde{a}_+(t)\tilde{x}_1. \tag{A.3}$$

In the variables $\tilde{x}_1 = -\tilde{y}$ and $\tilde{y}_1 = \tilde{x}$, the resulting system (A.3) coincides with (A.2). Hence, its solution is given by $\tilde{x}_1 = -\xi_+(t)^{-1}\dot{\theta}(t)$, $\tilde{y}_1 = \eta_+(t)^{-1}\dot{\varphi}(t)$. Therefore, the solution of the adjoint system can be written as

$$x_1 = -\xi_{1+}(t)\xi_+(t)^{-1}\dot{\theta}(t), \quad y_1 = \eta_{1+}(t)\eta_+(t)^{-1}\dot{\varphi}(t).$$

The proof of Lemma 1 is complete.

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