ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2023, Vol. 84, No. 12, pp. 1449–1456.

THEMATIC ISSUE

Optimization of Group Incentive Schemes

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Received May 30, 2023
Revised September 10, 2023
Accepted September 30, 2023

Abstract—This paper considers the problem of motivating the reduction of project duration. The duration cuts of project works and the corresponding costs are given. A group incentive scheme is used to compensate for the costs. In this scheme, all works are partitioned into groups and a unified incentive scheme is applied for each group. Two types of unified incentive schemes are studied for groups, namely, linear and jump ones. The problem is to partition all project works into groups and choose an appropriate incentive scheme for each group by minimizing the total incentive fund. Solution algorithms are proposed based on determining the shortest path in the network. Special cases are also analyzed (partition with the minimum number of groups and partition with the maximum number of groups).

Keywords: group incentive scheme, linear unified incentive scheme, jump unified incentive scheme, shortest path in the network

DOI: 10.25728/arcRAS.2024.15.74.001

1. INTRODUCTION

Consider a project consisting of n works. For each work, the duration cut and the costs of this cut are given. An incentive scheme is defined to compensate for the costs.

The design problems of optimal incentive schemes have been considered by many researchers; for example, see the books [1–3] and references therein. As a rule, two types of such schemes are considered, namely, individual incentive schemes and unified incentive schemes. In individual incentive schemes, a particular scheme is defined for each group from a given class (linear, jump, rank, etc [1]). In unified incentive schemes, the same scheme is defined for all agents. Compared to unified incentive counterparts, the advantage of individual incentive schemes is an appreciably smaller incentive fund (in several cases), and the drawbacks are disinterest in reducing costs and rather high opportunities for manipulation (strategic behavior). The benefits of unified incentive schemes are smaller opportunities for manipulation and significantly greater interest in reducing costs, and the disadvantage is an appreciably larger incentive fund (in many cases). Group incentive schemes occupy an intermediate position. In such schemes, the set of all agents is partitioned into groups and a unified incentive scheme is applied for each group. Group incentive systems retain to some extent the advantages of unified and individual incentive systems and, at the same time, diminish their drawbacks.

In this paper, design problems are formulated for optimal group incentive schemes and methods for solving them are considered.

2. PROBLEM STATEMENT

Consider a project consisting of n works. There is a given plan for reducing project duration: according to this plan, the duration cut of work i is specified by a value y_i. The costs of project contractors to reduce the duration, z_i, $i = \overline{1,n}$, are also given.

To compensate for the costs, it is necessary to determine a group incentive scheme (GIS). Consider a GIS in which all works are partitioned into $1 < m < n$ groups and a certain unified incentive scheme (UIS) is defined for each group. Further analysis deals with two classes of UISs, namely, linear incentive schemes (LISs) and jump incentive schemes (JISs). We denote by R_j the set of works belonging to group j :

$$\bigcup_j R_j = R, \quad R_i \cap R_j = \varnothing \tag{1}$$

for all i and j, where R is the set of all works. If an LIS is chosen for group j, all contractors of this group will be compensated for their costs using the minimum incentive fund

$$S_j = a_j T_j, \tag{2}$$

where

$$a_j = \max_{i \in R_j} k_i, \quad T_j = \sum_{i \in R_j} y_i, \quad k_i = \frac{z_i}{y_i}, \quad i = \overline{1,n}.$$

If a JIS is chosen for group j, all contractors of this group will be compensated for their costs using the minimum incentive fund

$$S_j = n_j \max_{i \in R_j} z_i, \tag{3}$$

where n_j stands for the number of works in group j.

Problem 1. Find a partition R_j, $j = \overline{1,m}$, and choose an appropriate incentive scheme for each group by minimizing the incentive fund. This problem will be considered in three modifications as follows: in the first, only LISs are used for all groups; in the second, only JISs; in the third, both classes of the incentive schemes mentioned (further called mixed incentive schemes, MISs).

Now we describe the formal problem statement. Let $x_{ij} = 1$ if work i belongs to group j, and $x_{ij} = 0$ otherwise. In the case of LISs, the incentive fund of group j is given by

$$S_{1j} = \left(\sum_i x_{ij} y_i \right) \max_i k_i x_{ij}. \tag{4}$$

In the case of JISs, the incentive fund of group j is given by

$$S_{2j} = \left(\sum_i x_{ij} \right) \max_i k_i y_i x_{ij}. \tag{5}$$

Finally, in the case of MISs, the incentive fund of group j is given by

$$S_{3j} = \min\left(S_{1j}, S_{2j} \right). \tag{6}$$

Consequently, the total incentive fund constitutes

$$S_k = \sum_j S_{kj}, \quad k = \overline{1,3}, \tag{7}$$

depending on the chosen incentive scheme k.

Problem constraints may have different forms. For example, given n_j works in each group j, the constraints are

$$\sum_i x_{ij} = n_j, \quad j = \overline{1,m}. \tag{8}$$

If the number of works in a group must be within given bounds, the constraints take the form

$$l_1 \leqslant \sum_i x_{ij} \leqslant l_2. \tag{9}$$

Other constraints are possible as well.

Problem 2. Find (x_{ij}), $i = \overline{1,n}$, $j = \overline{1,m}$, to minimize (7) subject to the constraints (8), (9) or others.

Methods for solving these problems are presented below.

3. LINEAR INCENTIVE SCHEMES WITH $y_i = y$

In this section, we investigate the case $y_i = y$, $i = \overline{1,n}$. Let all works be numbered in ascending (nondescending) order of k_i, i.e.,

$$k_1 \leqslant k_2 \leqslant \ldots \leqslant k_n.$$

This sequence will be called original.

Definition 1. A fragment of the original sequence is its part between some works i and $j > i$.

Theorem 1. *The optimal partition of works into groups is the set of fragments of original sequences.*

Proof. Assume first that all values k_i differ. Let P be an optimal partition. Consider the group with the maximum value k_n. If this group is not a fragment, then there exists a maximum number s of the original sequence such that the corresponding work is absent from the group with k_n but present in another group, where it has the maximum value k_s. Let us swap work s with any work from the group with k_n that does not belong to the fragment. Obviously, the incentive costs in the group with the maximum value k_n will not change; at the same time, the incentive costs in the group with i_s will decrease because the maximum value k_i in the group with i_s is smaller than k_s, which contradicts the optimality of the partition P. Thus, the group with work n is a fragment. The next group with the maximum value k_i is considered by analogy, and the procedure continues for all groups.

To proceed, we reject the assumption that all k_i are different. In this case, there exist several original sequences; for any optimal partition, however, it is possible to find an original sequence such that the partition will form the set of fragments of this sequence. The proof of Theorem 1 is complete.

Let all works be numbered in ascending (nondescending) order of k_i, i.e.,

$$k_1 \leqslant k_2 \leqslant \ldots \leqslant k_n.$$

We construct a network of admissible partitions (NAP) of works into groups. This network consists of an input, an output, and $(m-1)$ layers. Each vertex i of layer p shows the total number of works Q_{ip} in the first p groups. Note that the minimum number of works is 2 and the maximum number is $n - 2\,(m-p)$ since each group includes at least two works. Therefore, layer p contains

$$a = n - 2\,(m-p) - 2p + 1 = n - 2m + 1$$

vertices, and this number is independent of p.

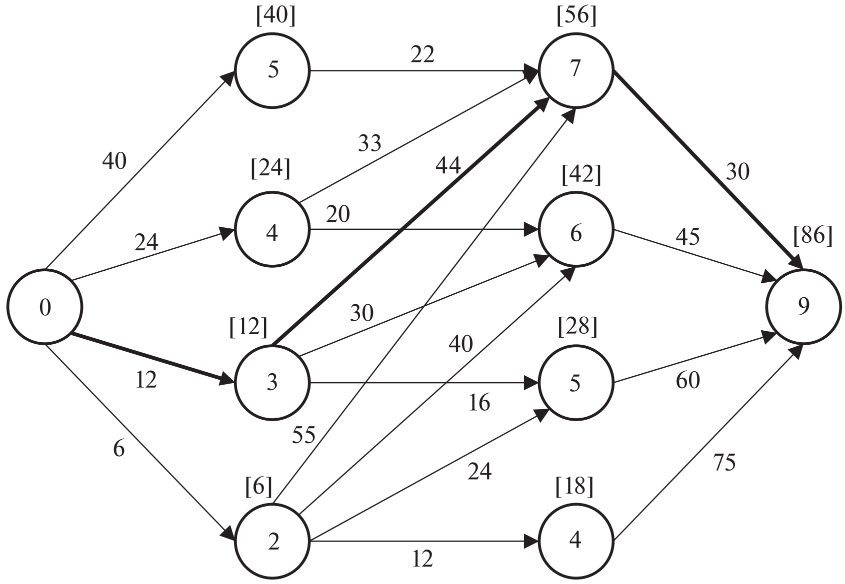

Fig. 1.

We connect vertex i of layer p to vertex j of layer $(p+1)$ by an arc if

$$Q_{jp+1} - Q_{ip} \geqslant 2.$$

Also, we connect the network input 0 to each vertex of layer 1 and each vertex of layer $(m-1)$ to the network output by an arc.

Theorem 2. *A unique path in the NAP corresponds to each admissible partition of works into groups and, conversely, a unique partition of works into groups corresponds to each path in the NAP.*

Proof. For each admissible partition $(n_1, \ldots, n_m)$, there is a sequence of values Q_{ip}, $p = \overline{1, m-1}$, such that the difference of the values of neighbor layers exceeds or equal 2. By the construction of the NAP, an arc connects the corresponding vertices. Conversely, for each path in the NAP, there is a sequence $(n_1, \ldots, n_m)$, where n_k equals the difference $(Q_{jk+1} - Q_{ik})$ of the corresponding adjacent vertices. This sequence defines a unique partition of works into groups. The proof of Theorem 2 is complete.

Example 1. Consider a project of nine works and let $m = 3$. Then we have

$$q = 9 - 6 + 1 = 4.$$

The corresponding network is demonstrated in Fig. 1. Table 1 provides the data of works. Assume that $y_i = 1$ for all i, i.e., $z_i = k_i$. The lengths of all arcs are specified in Fig. 1. The shortest path (0,3,7,9) has a length of 86. The optimal partition into three groups is given by $R_1 = (1, 2, 3)$, $R_2 = (4, 5, 6, 7)$, and $R_3 = (8, 9)$.

Table 1

i	1	2	3	4	5	6	7	8	9
k_i	1	3	4	6	8	10	11	12	15
z_i	1	15	12	12	8	40	33	24	15

Next, we solve the problem with the maximum number of groups $m = [n/2] = 4$. The corresponding NAP is presented in Fig. 2 ($q = 2$).

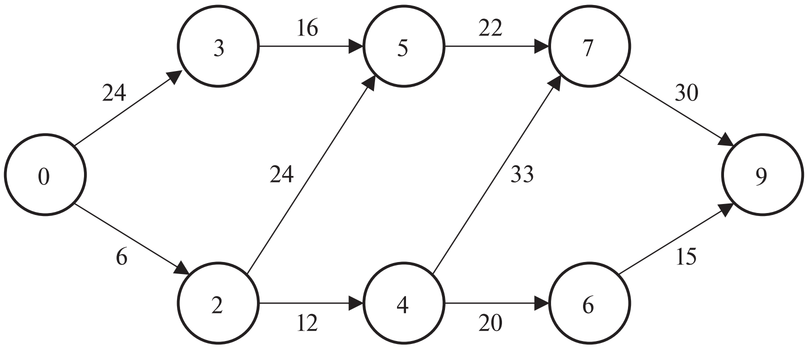

Fig. 2.

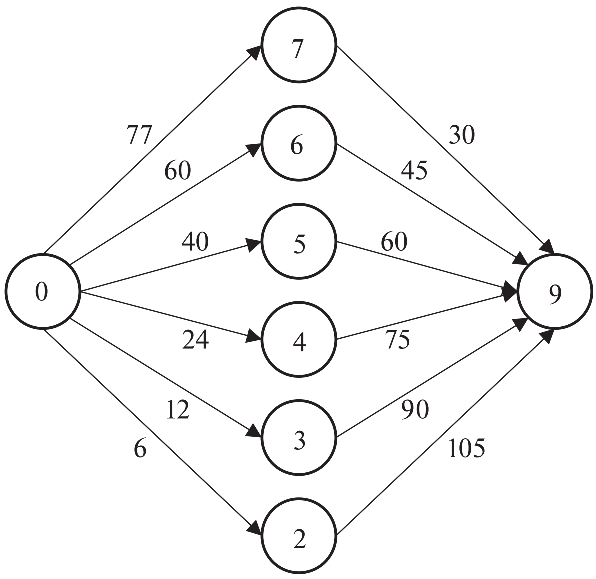

Fig. 3.

We calculate the vertices λ_{ip}:

$$\lambda_{\text{in}} = 0, \quad \lambda_{11} = 6, \quad \lambda_{21} = 24,$$
$$\lambda_{12} = \lambda_{11} + 12 = 18,$$
$$\lambda_{22} = \min\left[\lambda_{11} + 24, \lambda_{21} + 16\right] = 30,$$
$$\lambda_{13} = \lambda_{12} + 20 = 38,$$
$$\lambda_{23} = \min\left[\lambda_{12} + 33, \lambda_{22} + 22\right] = 51,$$
$$\lambda_{\text{out}} = \min\left[\lambda_{13} + 45, \lambda_{23} + 30\right] = 81.$$

The optimal partition is the one with four groups: (1, 2), (3, 4), (5, 6, 7), and (8, 9).

Finally, we solve the problem with the minimum number of groups $m = 2$, $q = 6$. The corresponding NAP is presented in Fig. 3.

We calculate the vertices:

$$\lambda_{\text{in}} = 0, \quad \lambda_{11} = 6, \quad \lambda_{21} = 12,$$
$$\lambda_{31} = 24, \quad \lambda_{41} = 40, \quad \lambda_{51} = 60, \quad \lambda_{61} = 77,$$
$$\lambda_{\text{out}} = \min\left[\lambda_{11} + 105, \lambda_{21} + 90, \lambda_{31} + 75, \lambda_{41} + 60, \lambda_{51} + 45, \lambda_{61} + 30\right] = 99.$$

The optimal partition is given by (1, 2, 3, 4) and (5, 6, 7, 8, 9).

Note that incentive costs decrease as the number of groups grows, which is fairly obvious.

4. AN HEURISTIC ALGORITHM

The algorithm described above can be applied to the general case of different values y_i as a heuristic. It can be justified as follows. If there exists a partition into groups such that the coefficients k_i are the same for each group, then this partition is optimal. Therefore, a reasonable assumption is that the closer the coefficients k_i in the groups are, the closer the partition will be to the optimal one.

Example 2. Consider the problem with three groups and the data of Table 1. The corresponding network with the arc lengths (5) is shown in Fig. 4. Its structure coincides with that of the network in Fig. 1.

We calculate the vertices:

$$\lambda_{\text{in}} = 0, \quad \lambda_{11} = 18, \quad \lambda_{21} = 36, \quad \lambda_{31} = 66, \quad \lambda_{41} = 96,$$
$$\lambda_{12} = 48, \quad \lambda_{22} = \min[18 + 48,\ 36 + 24] = 60, \quad \lambda_{32} = 106, \quad \lambda_{42} = 146,$$
$$\lambda_{\text{out}} = \min[48 + 165,\ 60 + 150,\ 106 + 90,\ 146 + 45] = 191.$$

The optimal partition is given by (1, 2, 3), (4, 5, 6, 7), and (8, 9).

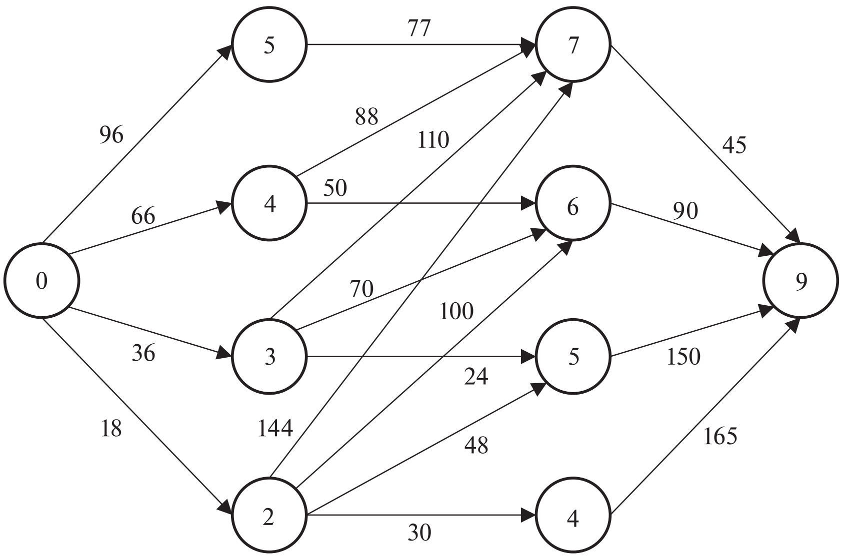

Fig. 4.

5. JUMP INCENTIVE SCHEMES

In this section, we study jump incentive schemes (JISs). For such schemes, an analog of Theorem 1 holds. Let all works be numbered in ascending (nondescending) order of z_i, i.e.,

$$z_1 \leqslant z_2 \leqslant \ldots \leqslant z_n.$$

This sequence will also be called the original sequence. The definition of its fragment is the same as the one for linear incentive schemes. (In other words, a fragment is some part of the original sequence.)

Theorem 3. *The optimal partition is the set of fragments of the original sequence (one of them if there are several original sequences).*

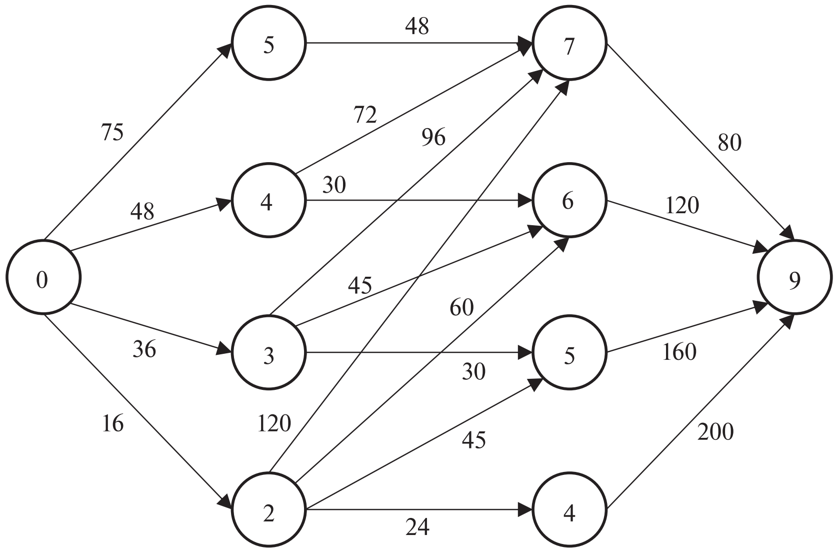

Fig. 5.

Proof. This result is established similarly to Theorem 1. In the optimal partition, we take the group with the maximum value z_n and show that it is a fragment. Assume on the contrary that it is not. We find the work with the maximum value z_s in a fragment that is absent from this group but present in another group. Let us swap work s with any work from the group with work n that does not belong to the fragment. Obviously, the incentive costs will decrease. Thus, the group with work n is a fragment. Then we eliminate the works of this fragment and consider the next group with the maximum value z. The procedure continues for all groups by analogy.

Example 3. Consider the data of Table 1. We renumber the works appropriately to obtain an original sequence.

Table 2

i	1	2	3	4	5	6	7	8	9
K_i	1	8	4	6	15	3	12	11	10
y_i	1	1	3	2	1	5	2	3	4
z_i	1	8	12	12	15	15	24	33	40

Let us find the optimal GIS for three groups. Note that the corresponding network will have the same structure as in Fig. 1 but with other arc lengths (see Fig. 5).

We calculate the vertices:

$$\lambda_{\text{in}} = 0, \quad \lambda_{11} = 16, \quad \lambda_{21} = 36, \quad \lambda_{31} = 48, \quad \lambda_{41} = 75,$$
$$\lambda_{12} = 40, \quad \lambda_{22} = \min[16 + 45,\ 36 + 30] = 61,$$
$$\lambda_{32} = \min[48 + 30,\ 36 + 45,\ 16 + 60] = 76,$$
$$\lambda_{42} = \min[75 + 48,\ 48 + 72,\ 36 + 96,\ 16 + 120] = 120,$$
$$\lambda_{\text{out}} = \min[40 + 200,\ 61 + 160,\ 76 + 120,\ 120 + 80] = 196.$$

The optimal partition is given by (1, 2), (3, 4, 5, 6), and (7, 8, 9).

6. TWO-GROUP PARTITION FOR LINEAR INCENTIVE SCHEMES

This section is devoted to a special case of partitions into two groups. Let the maximum coefficient k_j be given for the second group. The resulting problem is easy to solve. If $k_j < k_{n-1}$, then the first group includes all works with $k_i > k_j$ whereas the second group all works with $k_i \leqslant k_j$. Indeed, any transfer of work with $k_i \leqslant k_j$ to the first group increases the incentive fund by $(k_{n-1} - k_j)y_i > 0$. If $k_j = k_{n-1} < k_n$, we add a work to the first group for the number of works to exceed one. The matter concerns the work with the minimum value y.

Example 4. Consider the data of Table 1. We perform the calculations:

1. $k_j = 12$. We add work 1 with the minimal duration $y_1 = 1$ in the first group. The incentive fund is

$$\Phi_1 = 15 \times 2 + 12 \times 20 = 270.$$

2. $k_j = 11$. The first group contains works 8 and 9.

$$\Phi_2 = 45 + 209 = 254.$$

3. $k_j = 10$. The first group contains works 7, 8, and 9. The incentive fund is

$$\Phi_3 = 90 + 160 = 250.$$

4. $k_j = 8$. The first group contains works 6, 7, 8, and 9. The incentive fund is

$$\Phi_4 = 150 + 96 = 246.$$

5. $k_j = 6$. The first group contains works 5, 6, 7, 8, and 9. The incentive fund is

$$\Phi_5 = 165 + 66 = 231.$$

6. $k_j = 4$. The first group contains works from 4 to 9. The incentive fund is

$$\Phi_6 = 195 + 36 = 231.$$

7. $k_j = 3$. The first group contains works from 3 to 9. The incentive fund is

$$\Phi_4 = 240 + 18 = 258.$$

The optimal partition of works into groups is given by (1, 2, 3) and (4, 5, 6, 7, 8, 9).

7. CONCLUSIONS

This paper has considered the design of group incentive schemes for linear and jump incentive schemes. Note that for the two-group partition with linear incentive schemes, the heuristic algorithm yields an optimal solution in many cases. It seems interesting to justify this conclusion rigorously. Another promising line is to consider other incentive schemes (basic and combined). As for mixed incentive systems, we emphasize that any linear or jump incentive scheme can be turned into a mixed one by recalculating the arc lengths of the corresponding network using formula (6). However, generally speaking, the resulting solution will be nonoptimal. The problem of an optimal mixed incentive scheme has not been solved yet. All these problems require further research.

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This paper was recommended for publication by A.I. Mikhalskii, a member of the Editorial Board