ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2023, Vol. 84, No. 12, pp. 1411–1427.

THEMATIC ISSUE

Structural Spectral Methods for Solving Continuous Lyapunov Equations

I. B. Yadykin[*,a] **and I. A. Galyaev**[*,b]

[*] *Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia*
e-mail: [a] *Jad@ipu.ru,* [b] *ivan.galyaev@yandex.ru*

Received May 31, 2023
Revised September 10, 2023
Accepted September 30, 2023

Abstract—For linear multivariable continuous stationary stable control systems with a simple spectrum, presented in the form of a canonical diagonal form, controllability and observability forms, a method was developed and analytical formulas for spectral decompositions of gramians in the form of various Xiao matrices were obtained. A method and algorithm for calculating generalized Xiao matrices in the form of the Hadamard product for multiply connected continuous linear systems with many inputs and many outputs have been developed. This allows us to calculate the elements of the corresponding controllability and observability gramians in the form of products of the corresponding elements of the multiplier matrices and a matrix that is the sum of all possible products of the numerator matrices of the matrix transfer function of the system. New results are obtained in the form of spectral and singular decompositions of the inverse gramians of controllability and observability. This makes it possible to obtain invariant decompositions of energy functionals and formulate new criteria for the stability of linear systems taking into account the nonlinear effects of mode interaction.

Keywords: : spectral decompositions of gramians, singular numbers, inverse gramian matrix, stability that takes into account the interaction of modes, Xiao matrices, Lyapunov equation

DOI: 10.25728/arcRAS.2024.59.12.001

1. INTRODUCTION

Monitoring the state of control objects and controlling the damping of dangerous vibrations are important areas of research in various fields of industry (energy, mechanical engineering, aviation and astronautics, robotics). New modeling technologies require the development of tools for approximating mathematical models of complex systems of various natures [1–3]. An important role is played by the methods of calculating the Lyapunov and Sylvester matrix equations and the study of the structural properties of solutions to these equations [4–11]. The fundamental properties of linear dynamic systems associated with solutions to these equations are controllability, observability and stability. Important results were obtained in the field of computing gramians for systems which models are presented in the canonical forms of controllability and observability. In [12], methods for calculating gramians based on the use of matrices of periodic structure were first proposed for linear systems specified by equations in the forms of controllability and observability. A new approach was developed in terms of use the properties of the impulse transition function and gramian matrices in the form of the zero-plaid structure of the controllability gramian in [13, 14]. In [15], the approach was developed to compute spectral decompositions of a more general class of linear time-invariant (LTI) multiple-input multiple-output (MIMO) systems. Using this approach, a method for optimal selection of locations for sensors and actuators on the graph of a distributed control system was developed in [16]. The paper shows that for a diagonalized system the controllability gramian can be represented as the Hadamard product of two positive semidefinite matrices. In [17], the problem of optimizing the capacity of an urban transport network was solved based on minimizing the trace of the gramian controllability matrix taking into account restrictions. Various problems related to the usage of controllability, observability and cross-gramians for calculating system invariants and energy stability indices can be found in [18, 19].

The goal of this work is to develop structural methods for solving matrix Lyapunov equations and obtain spectral and singular decompositions of controllability and observability gramians, based on reducing the equations of state of a linear stationary system to the following canonical forms: diagonal, controllability and observability.

2. FORMULATION OF THE PROBLEM

We consider a stable continuous MIMO LTI dynamic system of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0, \quad y(t) = Cx(t), \tag{2.1}$$

where $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^m$. We will consider real matrices of corresponding sizes A, B, C. Let us assume that the system (2.1) is completely controllable and observable and all eigenvalues of matrix A are different. In this case, the implementation of (2.1) is minimal and there is the only one transfer function $W(s)$ in the form

$$W\left(s\right) = \sum_{i=1}^{n} M_i s^i N^{-1}\left(s\right),$$

where $N(s)$ is characteristic polynomial of matrix A, M_i is a matrix of the form

$$M_i = \sum_{i=0}^{n-1} A_i B.$$

Above, A_i denotes the "i"th Faddeev matrix in the decomposition of the resolvent of matrix A in the Faddeev–Le Verrier series [6, 7]. In accordance with [20], we write a general formula for calculating the controllability gramian from the pair spectrum of the system (2.1)

$$P^c = -\sum_{j=1}^{n}\sum_{\rho=1}^{n} \frac{1}{s_j + s_\rho} Res\left[\left(Is - A\right)^{-1}, s_j\right] BB^* Res\left[\left(Is - A^*\right)^{-1}, s_\rho\right]. \tag{2.2}$$

We consider a continuous dynamic MISO LTI system of the form

$$\begin{gathered}\dot{x}\left(t\right) = Ax\left(t\right) + b_\gamma u_\gamma\left(t\right), \quad x\left(0\right) = 0, \\ y\left(t\right) = cx\left(t\right),\end{gathered} \tag{2.3}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^1$, $u_\gamma\left(t\right) \in \mathbb{R}^m$, $\gamma = 1, \ldots m$, b_γ is column of matrix B.

We consider the transformation of the equation (2.1) of a general system to equations of state in canonical forms: diagonal, controllability and observability.

If all eigenvalues s_r of matrix A are different, then the linear system can be reduced to diagonal form using a non-degenerate coordinate transformation

$$\begin{gathered}x_d = Tx, \quad \dot{x}_d = A_d x_d + B_d u, \quad y_d = C_d x_d, \\ A_d = TAT^{-1}, \quad B_d = TB, \quad C_d = CT^{-1}, \quad Q_d = TBB^{\mathrm{T}}T^{\mathrm{T}},\end{gathered}$$

or

$$\mathrm{A} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & s_n \end{bmatrix} \begin{bmatrix} \nu_1^* \\ \nu_2^* \\ \vdots \\ \nu_n^* \end{bmatrix} = T\Lambda T^{-1},$$

where matrix T is composed of right eigenvectors u_i, and matrix T^{-1} is composed of left eigenvectors ν_i^* corresponding to the eigenvalue s_i. The gramian of the diagonalized linear part is a solution to the Lyapunov equation, which is determined from the formula [15]

$$P_d^c = -\sum_{j=1}^{n}\sum_{\rho=1}^{n} \frac{1}{s_j + s_\rho} Res\left[(Is - A_d)^{-1}, s_j\right] B_d B_d^* Res\left[(Is - A_d)^{-1}, s_\rho\right].$$

The controllability gramian P_d^c is related to the gramian P^c by a relation of the form

$$P^c = T P_d^c T^{\mathrm{T}}.$$

From (2.2) results the following separable spectral decomposition of the gramian controllability of a system transformed into a diagonal canonical form [21]

$$P_d^c = \sum_{j=1}^{n}\sum_{\rho=1}^{n} \frac{-b_{j\rho}}{s_j + s_\rho} \mathbb{1}_{j\rho}, \quad b_{j\rho} = [B_d B_d^*]_{j\rho},$$

where the designation is introduced

$$\mathbb{1}_{j\rho} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1_{j\rho} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Further we consider the channel "γ" of the MISO LTI system in the canonical form of controllability [1, 21]

$$\begin{gathered} x(t) = \sum_{\gamma=1}^{m} R_{c\gamma}^F x_{c\gamma}(t). \\ \dot{x}_c(t) = A_c^F x_{c\gamma}(t) + b_\gamma^F u_\gamma(t), \quad x_c(0) = 0, \\ y_c^F(t) = c_\gamma^F x_c(t), \quad \gamma = 0, 1 \dots, m. \end{gathered} \tag{2.4}$$

$$A_c^F = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, \quad b_\gamma^F = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}^{\mathrm{T}},$$

$$a = \begin{bmatrix} -a_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \quad c_\gamma^F = \begin{bmatrix} \xi_0 & \xi_1 & \dots & \xi_{n-2} & \xi_{n-1} \end{bmatrix}.$$

If we use a non-degenerate transformation of variables with the matrix R_c^F, we can consider the MISO LTI system in the canonical form of controllability. The vector B_γ of the MISO system has the form

$$B_\gamma = \begin{bmatrix} 0 & \dots & b_\gamma & \dots & 0 \end{bmatrix}^T.$$

The following relations are valid [14]:

$$\left(R_{c\gamma}^F\right)^{-1} A R_{c\gamma}^F = A_c^F,\ \left(R_{c\gamma}^F\right)^{-1} B_\gamma = b_\gamma^F,\ C R_{c\gamma}^F = c_\gamma^F,$$

$$P^c = \sum_{\gamma=1}^{m} R_\gamma^{cF} P_\gamma^{cF} (R_\gamma^{cF})^T.$$

In relation to the systems (2.1) and (2.3) we will assume that various structural conditions of stability, controllability, observability and properties of the spectrum of the dynamics matrix are fulfilled. The following spectral decomposition of the controllability gramian was obtained in [15]:

$$P_\gamma^{cF} = \sum_{k=1}^{n} \sum_{\eta=0}^{n-1} \sum_{j=0}^{n-1} \frac{s_k^j (-s_k)^\eta}{\dot{N}(s_k) N(-s_k)} \mathbb{1}_{j+1\eta+1}.$$

Next, we consider the "γ" SIMO LTI channel of a linear system in the canonical form of observability [15]. In this case, the formulas are valid

$$x_o(t) = \sum_{\gamma=1}^{m} R_{o\gamma}^F x_{o\gamma}(t),$$

$$\dot{x}_{o\gamma}(t) = A_c^F x_{o\gamma}(t) + b_{o\gamma}^F u_\gamma(t), \quad x_o(0) = 0,$$

$$y_{o\gamma}^F(t) = c_{o\gamma}^F x_{o\gamma}(t), \quad \gamma = 0, 1 \ldots m.$$

$$A_o^F = \begin{bmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & 0 & -a_{-1} \\ 0 & 1 & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & -a_{n-2} \\ 0 & 0 & \ldots & 1 & -a_{n-1} \end{bmatrix}, \quad b_{o\gamma}^F = \begin{bmatrix} \xi_0 & \xi_1 & \ldots & \xi_{n-2} & \xi_{n-1} \end{bmatrix}^{\mathrm{T}},$$

$$c_{o\gamma}^F = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \end{bmatrix}.$$

We obtain following expressions in accordance with the principle of duality [14]

$$P_{o\gamma}^F = \sum_{k=1}^{n} \sum_{\eta=0}^{n-1} \sum_{j=0}^{n-1} \frac{s_k^j (-s_k)^\eta}{\dot{N}(s_k) N(-s_k)} \mathbb{1}_{j+1\eta+1},$$

$$P^o = \sum_{\gamma=1}^{m} R_{o\gamma}^F P_\gamma^{oF} (R_{o\gamma}^F)^T.$$

Definition 1. We call the Xiao matrix (Zero plaid structure) a matrix of the form [12, 13]

$$Y = \begin{bmatrix} y_1 & 0 & -y_2 & 0 & y_3 \\ 0 & y_2 & 0 & -y_3 & 0 \\ -y_2 & 0 & y_3 & 0 & \ldots \\ 0 & -y_3 & 0 & \ldots & 0 \\ y_3 & 0 & \ldots & 0 & y_n \end{bmatrix}.$$

The matrix elements are calculated using the formulas

$$y_{j\eta} = \begin{cases} 0, \quad \text{if} \ \ j+\eta = 2k+1, \quad k = 1, \ldots, n; \\ y_n = \dfrac{1}{2Y_{n,1}}, \\ y_{n-l} = \dfrac{-\sum\limits_{i=1}^{m-1} (-1)^i Y_{n-l,i+1} y_{n-l+i}}{Y_{n-l,1}}, \quad \text{if} \ \ j+\eta = 2k, \quad k = 1, \ldots, n, \quad l = \overline{1, n-1}, \end{cases}$$

where $Y_{i,j}$ is the element of the Routh table for the system located at the intersection of i row and j column.

3. MAIN RESULTS

3.1. Identities for One Class of Stable Polynomials Which Roots are Different over the Field of Complex Numbers

We consider the spectral decomposition of the controllability gramian in simple and pair spectrum (2.2).

$$\sum_{k=1}^{n}\sum_{j=0}^{n}\sum_{\eta=0}^{n} \frac{s_k^j(-s_k)^\eta}{\dot{N}(s_k)\,N(-s_k)} \equiv \sum_{k=1}^{n}\sum_{\rho=1}^{n}\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1} \frac{-1}{s_k+s_\rho}\frac{s_k^j s_\rho^\eta}{\dot{N}(s_k)\,\dot{N}(s_\rho)}, \quad s_k+s_\rho \neq 0. \tag{3.1}$$

We introduce the notation

$$\omega(n,s_k,j,\eta) = \sum_{k=1}^{n} \frac{s_k^j(-s_k)^\eta}{\dot{N}(s_k)\,N(-s_k)},$$
$$\omega(n,s_k,s_\rho,j,\eta) = \sum_{k=1}^{n}\sum_{\rho=1}^{n} \frac{-1}{s_k+s_\rho}\,\frac{s_k^j s_\rho^\eta}{\dot{N}(s_k)\,\dot{N}(s_\rho)}.$$

Taking into account the introduced notation, the identity (3.1) takes the form

$$\omega(n,s_k,j,\eta) \equiv \omega(n,s_k,s_\rho,j,\eta) \ \text{ for } \forall s_k, s_\rho \in \mathbb{C}^-, \ s_k+s_\rho \neq 0.$$

The proof follows from the decomposition of the fractional rational function $\frac{s_k^j(-s_k)^\eta}{\dot{N}(s_k)\dot{N}(-s_k)}$ by the roots of the characteristic equation $N(-s_k)=0$.

Lemma 1. *Consider the polynomial* $\boldsymbol{\gamma}(n,s_k,-s_k)$ *over the field of complex numbers of the following form:*

$$\gamma(n,s_k,-s_k) = \sum_{i=0}^{n-1}\sum_{\mu=0}^{n-1} s_k^i(-s_k)^\mu, \ \forall k=1,\ldots,n,$$

where s_k are roots of the characteristic equation of the system (2.1), and $-s_k$ are roots of the characteristic equation of its antistable conjugate system. We assume that all eigenvalues of the systems are prime, non-zero complex numbers. Then the polynomial $\gamma(n,s_k,-s_k)$ contains only all even powers of the numbers s_k and does not contain their odd powers.

$$\gamma(n,s_k,-s_k) = \gamma\left(n,\ s_k^0, s_k^2,\ldots,s_k^{2m}\right), \quad n=2m,$$
$$\gamma(n,s_k,-s_k) = \gamma\left(n,\ s_k^1, s_k^3,\ldots,s_k^{2m-1}\right) \equiv 0, \quad n=2m-1.$$

Proof. It is easy to verify that the result of the lemma is valid for $n=1,2,3$:

$$\gamma(1,s_k,-s_k)=1,$$
$$\gamma(2,s_k,-s_k)=1-s_k^2,$$
$$\gamma(3,s_k,-s_k)=s_k^4-s_k^2+1.$$

Further, we apply the method of mathematical induction. We assume that the result of the lemma is true for the polynomial $\gamma(n,s_k,-s_k)$:

$$\gamma(n,s_k,-s_k) = \begin{cases} \gamma\left(2m, s_k^0,\ldots,s_k^{2m}\right) & \text{for even } n=2m, \\ \gamma\left(2m-1, s_k^0,\ldots,s_k^{2m}\right) & \text{for odd } n=2m-1. \end{cases}$$

We show that it is also valid for the polynomial $\gamma(n+1, s_k, -s_k)$. As n increases by one, the polynomial $\gamma(n, s_k, -s_k)$ takes the form

$$\gamma(n+1, s_k, -s_k) = \begin{cases} \gamma(2m+1, s_k, -s_k) & \text{for even } n = 2m, \\ \gamma(2m, s_k, -s_k) & \text{for odd } n = 2m-1. \end{cases}$$

We first consider the case of even n.

$$\gamma(n+1, s_k, -s_k) = \gamma\left(2m, s_k^0, \ldots, s_k^{2m}\right) + \gamma\left(2m, s_k^0, \ldots, s_k^{2m}\right) s_k^{2m+1}$$
$$- \gamma\left(2m, s_k^0, \ldots, s_k^{2m}\right) s_k^{2m+1} + s_k^{2m+1}(-s_k)^{2m+1} = \gamma\left(2m, s_k^0, \ldots, s_k^{2m}\right)^2 - s_k^{2(2m+1)}.$$

For the case of odd n, we similarly obtain

$$\gamma(n+1, s_k, -s_k) = \gamma\left(2m-1, s_k^0, \ldots, s_k^{2m}\right) + \gamma\left(2m-1, s_k^0, \ldots, s_k^{2m+1}\right) s_k^{2(2m+1)}$$
$$- \gamma\left(2m-1, s_k^0, \ldots, s_k^{2m+1}\right) s_k^{2(m+1)} + s_k^{2(2m+1)} = \gamma\left(2m-1, s_k^0, \ldots, s_k^{2m}\right) + s_k^{2(2m+1)},$$

where the first three terms contain even powers of s_k by assumption.

Corollary 1. *We consider the multiplier $\omega(n, s_k, j, \eta)$ in the spectral decomposition of the controllability gramian in the simple spectrum* (2.2). *The identities are valid:*

$$\omega(n, s_k, j, \eta) \equiv 0, \quad \textit{if} \quad j + \eta = 2m - 1, \tag{3.2}$$

$$\omega(n, s_k, j, \eta) \equiv \sum_{k=1}^{n} \frac{s_k^j(-s_k)^\eta}{\dot{N}(s_k) N(-s_k)}, \quad \textit{if} \quad j + \eta = 2m. \tag{3.3}$$

Proof. We express the multiplier through a polynomial $\gamma(n, s_k, -s_k)$

$$\omega(n, s_k, j, \eta) \equiv \sum_{k=1}^{n} \frac{s_k^j(-s_k)^\eta}{\dot{N}(s_k) N(-s_k)} = \sum_{k=1}^{n} \frac{\gamma(n, s_k, -s_k, j, \eta)}{\dot{N}(s_k) N(-s_k)}$$

and apply the lemma.

Corollary 2. *Let us consider the multiplier $\omega(n, s_k, s_\rho, j, \eta)$ in the spectral decomposition of the controllability gramian in the pair spectrum* (2.2). *The identities are valid:*

$$\omega(n, s_k, s_\rho, j, \eta) \equiv 0, \quad \textit{if} \quad j + \eta = 2m - 1, \tag{3.4}$$

$$\omega(n, s_k, s_\rho, j, \eta) \equiv \sum_{k=1}^{n} \sum_{\rho=1}^{n} \frac{-1}{s_k + s_\rho} \frac{s_k^j s_\rho^\eta}{\dot{N}(s_k) \dot{N}(s_\rho)}, \quad \textit{if} \quad j + \eta = 2m. \tag{3.5}$$

Proof. We express the multiplier through a polynomial $\gamma(n, s_k, -s_k)$

$$\omega(n, s_k, j, \eta) \equiv \omega(n, s_k, s_\rho, j, \eta) \ \text{ for } \forall s_k, s_\rho \in \mathbb{C}^-, \ s_k + s_\rho \neq 0$$

and apply the lemma.

Corollaries 1 and 2 prove that for all continuous stable MIMO LTI systems with a simple spectrum, reduced to the canonical forms of controllability and observability, exist spectral decompositions in the form of Xiao matrices. For systems represented in the canonical forms of controllability and observability, this allows to calculate only n diagonal elements using the formulas (3.2)–(3.5), instead of calculating n^2 matrix elements.

Remark. The multiplier $\omega\left(n, s_k, s_\rho, j, \eta\right)$ should be used with caution in the spectral decomposition of the controllability gramian in the pair spectrum (2.2). For example, in the case of a MIMO LTI system reduced to a diagonal canonical form, the spectral decomposition of the controllability gramian has a simple form

$$P_d^c = \sum_{j=1}^{n}\sum_{\rho=1}^{n} \frac{-b_{j\rho}}{s_j + s_\rho}\mathbb{1}_{j\rho}\,, \qquad b_{j\rho} = [B_d B_d^*]_{j\rho}. \tag{3.6}$$

On the other hand, we have

$$P_d^c = \sum_{j=0}^{n-1}\sum_{\rho=0}^{n-1}\omega(n, s_j, j, \rho)A_j B_d B_d^* A_\rho^*, \omega\left(n, s_k, j, \rho\right) = \sum_{k=1}^{n}\frac{s_k^j(-s_k)^\rho}{\dot{N}\left(s_k\right)N\left(-s_k\right)}. \tag{3.7}$$

We note that both formulas (3.6), (3.7) give the same numerical result, which corresponds to *different* spectral decompositions. Let's give an example.

Illustrative example 1

We consider the problem of controlling a two-zone furnace. The model of the control object of the heating furnace can be described by equations of state of the form

$$\Sigma_1\colon \begin{cases} \dfrac{dx}{dt} = Ax\left(t\right) + Bu\left(t\right), \quad x\left(0\right) = 0, \\ y\left(t\right) = Cx\left(t\right). \end{cases}$$

$$A = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In this case, you can evaluate the expressions

$$N\left(s\right) = s^2 + 1.5s + 0.5, \quad \dot{N}\left(s\right) = 2s + 1.5,$$

$$\left(Is - A\right)^{-1} = \begin{bmatrix} s+1 & 0 \\ 0 & s+0.5 \end{bmatrix}\left(s^2 + 1.5s + 0.5\right)^{-1},$$

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \; BB^T = \begin{bmatrix} 1.25 & 1.5 \\ 1.5 & 4.25 \end{bmatrix}.$$

The controllability gramian calculated using the formula (3.6), is equal to

$$P^c = \begin{bmatrix} 1.25 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2.125 \end{bmatrix}.$$

The expression for the decomposition of the controllability gramian has the form

$$P^c = \sum_{j=0}^{n-1}\sum_{\rho=0}^{n-1}\sum_{k=1}^{2}\frac{s_k^j(-s_k)^\rho}{\dot{N}\left(s_k\right)N\left(-s_k\right)}A_j BB^{\mathrm{T}}A_\rho^{\mathrm{T}},$$

$$P^c = \sum_{j=0}^{n-1}\sum_{\rho=0}^{n-1}\sum_{k=1}^{2}\frac{s_k^j(-s_k)^\rho}{\dot{N}\left(s_k\right)N\left(-s_k\right)}\frac{-b_{j\rho}}{s_j + s_\rho}\mathbb{1}_{j\rho},$$

where A_j is the Faddeev matrix, constructed for the matrix A using the Faddeev–Le Verrier algorithm [6, 7]. Let's calculate the matrices $A_j BB^{\mathrm{T}}A_\rho^{\mathrm{T}}$:

$$A_0 BB^T A_0^T = \begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 1.0625 \end{bmatrix}, \; A_0 BB^{\mathrm{T}}A_1^{\mathrm{T}} = \begin{bmatrix} 1.25 & 0.75 \\ 1.5 & 2.125 \end{bmatrix},$$

$$A_1 BB^{\mathrm{T}}A_0^{\mathrm{T}} = \begin{bmatrix} 1.25 & 1.5 \\ 0.75 & 2,125 \end{bmatrix}, \; A_1 BB^{\mathrm{T}}A_1^{\mathrm{T}} = \begin{bmatrix} 1.25 & 1.5 \\ 1.25 & 4.25 \end{bmatrix}.$$

Substituting these expressions into (3.7), we obtain the spectral decomposition:

$$P^c = \begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 1.0625 \end{bmatrix} \frac{2}{3} + \begin{bmatrix} 1.25 & 1.5 \\ 1.25 & 4.25 \end{bmatrix} \frac{1}{3} = \begin{bmatrix} 1.25 & 1 \\ 1 & 2.125 \end{bmatrix}.$$

Matrices of infinite sub-gramians are symmetric and positive definite, and so is their sum. We verify that the calculated controllability gramian is a solution to the Lyapunov equation by direct substitution. The separable spectral decomposition of the controllability gramian, calculated using the formula (3.6), has the form

$$P^c = \begin{bmatrix} 1.25 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2.125 \end{bmatrix}.$$

The matrices of infinite sub-gramians in this decomposition are not symmetric and positive definite, although their sum is. The example shows that the same gramian can have several different spectral decompositions.

3.2. Decomposition of Gramians in the Form of Hadamard Products [3]

We introduce the matrices of the gramian controllability multiplier of a continuous MIMO LTI system in the form

$$\Omega_{\mathrm{c}} = [\omega_{\mathrm{c},j\eta}]_{n\times n}$$

and its observability gramian in the form

$$\Omega_o = [\omega_{o,j\eta}]_{n\times n},$$

where j is the row index, and η is the column index of the multiplier matrices.

We introduce matrices Ψ_{c} and Ψ_o in the form

$$\Psi_{\mathrm{c}} = \sum_{i=0}^{n-1}\sum_{\mu=0}^{n-1} A_i BB^{\mathrm{T}} A_\mu^{\mathrm{T}},$$
$$\Psi_o = \sum_{i=0}^{n-1}\sum_{\mu=0}^{n-1} A_i^{\mathrm{T}} C^{\mathrm{T}} C A_\mu.$$

We introduce an element-wise representation of these matrices in the form

$$\psi_{\mathrm{c},j\eta} = e_j^{\mathrm{T}} \Psi_{\mathrm{c}}\, e_\eta,$$
$$\psi_{o,j\eta} = e_j^{\mathrm{T}} \Psi_o\, e_\eta.$$

Theorem 1 [15]. *We consider a stable continuous dynamic MIMO LTI system with a simple spectrum*

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0,$$
$$y(t) = Cx(t),$$

where $x(t) \in R^n,\ u(t) \in R^m,\ y(t) \in R^m$.

Then the controllability subramian P^c *is a matrix of the form* (2.2), *and in accordance with* [7] *formulas* (2.1), (2.2) *is defined as*

$$P^c = \sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1} P^c_{j,\eta}, \quad P^c_{j,\eta} = \omega(n, s_k, s_\rho, j, \eta) A_j BB^{\mathrm{T}} A_\eta^{\mathrm{T}}, \tag{3.8}$$

where

$$\omega\left(n, s_k, s_\rho, j, \eta\right)=\begin{cases} 0, \ \text{if index } j+\eta \text{ is odd}, \\ \displaystyle\sum_{k=1}^{n}\sum_{\rho=1}^{n}\frac{-1}{s_\rho+s_k}\frac{s_k^j s_\rho^\eta}{\dot N\left(s_k\right)\dot N\left(s_\rho\right)}, \quad \text{if index } j+\eta \text{ is even.}\end{cases}$$

Proof of Theorem 1. As is known, the spectral decomposition of the controllability gramian under the conditions of Theorem 1 has the form [7, 20]

$$P^c=\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}\sum_{k=1}^{n}\sum_{\rho=1}^{n}\frac{-1}{s_\rho+s_k}\frac{s_k^j s_\rho^\eta}{\dot N\left(s_k\right)\dot N\left(s_\rho\right)}A_jBB^{\mathrm T}A_\eta^{\mathrm T}.$$

We substitute the newly introduced scalar function $\omega\left(n, s_k, s_\rho, j, \eta\right)$ into this formula and obtain the formula (3.8).

Theorem 2 [13]. *We consider a continuous MIMO LTI system of the form* (2.1). *We assume that the system is stable and all the roots of its characteristic equation are different. Then its controllability and observability gramians have the form of the generalized Xiao matrices of the following form:*

$$P^c=\Omega_c\circ\Psi_c=\left[p_{j\eta}^c\right]_{n\times n}, \qquad j,\eta=1,\dots,n. \tag{3.9}$$

$$\Psi_c=\left[\psi_{c,j\eta}\right]_{n\times n}, \ j,\eta=1,\dots,n. \qquad \Psi_c=\sum_{i=0}^{n-1}\sum_{\mu=0}^{n-1}\Psi_{c,i\mu}, \ \Psi_{c,i\mu}=M_iM_\mu^*,$$

$$M_i=A_iB, \qquad \Omega_c=\left[\omega_c(n,j,\eta)\right]_{n\times n}, \qquad j,\eta=1,\dots,n.$$

$$p_{j\eta}^c=\omega_c(n,j,\eta)\times\psi_{c,j\eta}.$$

Proof of Theorem 2. We use the spectral decomposition of the controllability gramian (2.2). We introduce a representation of gramians in the form of Hadamard products

$$P^c=\Omega_{\mathrm c}\circ\Psi_{\mathrm c}, \tag{3.10}$$

$$P^o=\Omega_o\circ\Psi_o. \tag{3.11}$$

This representation allows us to write simple formulas for calculating the elements of the controllability and observability gramians of MIMO LTI systems P^c and P^o in the form [13]

$$p_{j\eta}^c=\omega_c(n,j,\eta)\times\psi_{c,j\eta}, \tag{3.12}$$

$$p_{j\eta}^o=\omega_c(n,j,\eta)\times\psi_{o,j\eta}. \tag{3.13}$$

Next, we use identities for one class of stable polynomials whose roots are different over the field of complex numbers (Lemma). Formulas (3.10)–(3.13) express algorithms for calculating elements of generalized Xiao matrices in the form of products of elements of multiplier matrices and elements of sums of all possible products of matrices $A_jBB^{\mathrm T}A_\eta^{\mathrm T}$, written in the form of products of Hadamard matrices

$$\Omega_{\mathrm c}\circ\Psi_{\mathrm c}.$$

Corollary 3. *We consider an important special case of continuous linear SISO systems represented by equations of state in the canonical forms of controllability and observability. In this case, the controllability and observability gramians are determined by the formulas* [15]

$$P^{cF}=\sum_{k=1}^{n}\sum_{\eta=0}^{n-1}\sum_{j=0}^{n-1}\frac{s_k^j(-s_k)^\eta}{\dot N\left(s_k\right)N\left(-s_k\right)}\mathbb{1}_{j+1\eta+1}, \tag{3.14}$$

$$P^{oF}=\sum_{k=1}^{n}\sum_{\eta=0}^{n-1}\sum_{j=0}^{n-1}\frac{s_k^j(-s_k)^\eta}{\dot N\left(s_k\right)N\left(-s_k\right)}\mathbb{1}_{j+1\eta+1}.$$

The representation of gramians in Hadamard form according to (3.10)–(3.11) *takes the form*

$$P^{cF} = \Omega_{cF} \circ \Psi_c, \quad \Psi_c = \sum_{\eta=0}^{n-1} \sum_{j=0}^{n-1} \mathbb{1}_{j+1\eta+1},$$
$$P^{oF} = \Omega_{oF} \circ \Psi_o, \quad \Psi_o = \sum_{\eta=0}^{n-1} \sum_{j=0}^{n-1} \mathbb{1}_{j+1\eta+1.}$$

This implies the identities

$$P^{cF} \equiv \Omega_{cF}, \tag{3.15}$$
$$P^{oF} \equiv \Omega_{oF}. \tag{3.16}$$

This means that the controllability gramian in the canonical form of controllability coincides with the multiplier matrix for this gramian, which allows us to apply the formulas (3.15), (3.16) *to calculate all elements of the gramian and establishes that the gramian belongs to the class of Xiao matrices. A similar result holds for the gramian observability in the canonical form of observability.*

Multiplier matrices in different canonical forms have the form

$$\Omega_{cF} \equiv \Omega_{oF} = \left[\omega\left(n, s_k, s_\rho, j, \eta\right)\right]_{n\times n} = \left[\omega\left(n, s_k, j, \eta\right)\right]_{n\times n}.$$

3.3. Spectral and Singular Decompositions of Inverse Matrices of Gramians

General formulas for calculating inverse gramian matrices (hereinafter inverse gramians) for continuous MIMO LTI systems have the form [1]

$$(P^c)^{-1} = \frac{-1}{\gamma_0}\left[(P^c)^{n-1} + \gamma_{n-1}(P^c)^{n-2} + \cdots + \gamma_2 P^c + \gamma_1 I\right];$$
$$(P^o)^{-1} = \frac{-1}{\gamma_0}\left[(P^o)^{n-1} + \gamma_{n-1}(P^o)^{n-2} + \cdots + \gamma_2 P^o + \gamma_1 I\right].$$

In the case of continuous SISO LTI systems, these formulas, in accordance with (3.15), (3.16), take the form

$$\left[P^{cF}(\omega(n, s_k, j, \eta))\right]^{-1} = \frac{-1}{\gamma_0}\left[(\Omega_{cF})^{n-1} + \gamma_{n-1}(\Omega_{cF})^{n-2} + \cdots + \gamma_2\Omega_{cF} + \gamma_1 I\right];$$
$$\left[P^{oF}(\omega(n, s_k, j, \eta))\right]^{-1} = \frac{-1}{\gamma_0}\left[(\Omega_{oF})^{n-1} + \gamma_{n-1}(\Omega_{oF})^{n-2} + \cdots + \gamma_2\Omega_{oF} + \gamma_1 I\right].$$

The presence of powers of the multiplier matrices on the right side of the formulas leads to the appearance of complex fractional rational functions of eigenvalues s_k, which limits the scope of application of the formulas for spectral decompositions of inverse gramians to systems of small and medium dimensions. We return to stable continuous MIMO LTI systems with a simple spectrum and note that the controllability and observability gramians are symmetric complex-valued matrices. In this case, there are their singular decompositions of the form [1]

$$P^c = P^{c*} = V_c \Lambda V_c^*,$$
$$P^o = P^{o*} = V_o \Lambda V_o^*,$$

where the matrix V_c is formed by the right singular vectors of the matrix P^c, the matrix V_c^* is formed by the left singular vectors of the matrix P^c, and the matrix Λ is a diagonal matrix of the form

$$\Lambda = diag\left\{|\lambda_1|\,|\lambda_2| \ldots |\lambda_n|\right\}.$$

We define matrices S and U in the form

$$S = \ diag\left\{sgn\lambda_1 \ sgn\lambda_2 \ldots sgn\lambda_n \ \right\}, \ U_c = V_c S,$$
$$sgn\lambda = \begin{cases} +1, & \text{if } \lambda \geqslant 0, \\ -1, & \text{if } \lambda < 0. \end{cases}$$

Then

$$P^c = U_c \Lambda V_c^*,$$
$$P^o = U_o \Lambda V_o^*,$$

where the matrix U_c is formed by the left singular vectors of the matrix P^c. Since Λ, U_c, V_c are nonsingular matrices, then

$$(P^c)^{-1} = (U_c)^{-1}\Lambda^{-1}(V_c^*)^{-1} = V_c^*\Lambda^{-1}U_c. \tag{3.17}$$

In a similar way we get

$$(P^o)^{-1} = (U_o)^{-1}\Lambda^{-1}(V_o^*)^{-1} = V_o^*\Lambda^{-1}U_o. \tag{3.18}$$

Since the matrix Λ is diagonal, its inverse matrix can be written as

$$\Lambda^{-1} = \left[|\lambda_1|^{-1}\mathbb{1}_{11} + |\lambda_2|^{-1}\mathbb{1}_{22} + \cdots + |\lambda_n|^{-1}\mathbb{1}_{nn}\right]. \tag{3.19}$$

Substituting (3.19) into (3.17), (3.18), we obtain the following singular decompositions of the inverse gramians of controllability and observability in terms of their singular spectrum:

$$(P^c)^{-1} = V_c^*\left[|\lambda_1|^{-1}\mathbb{1}_{11} + |\lambda_2|^{-1}\mathbb{1}_{22} + \cdots + |\lambda_n|^{-1}\mathbb{1}_{nn}\right]U_c;$$
$$(P^o)^{-1} = V_o^*\left[|\lambda_1|^{-1}\mathbb{1}_{11} + |\lambda_2|^{-1}\mathbb{1}_{22} + \cdots + |\lambda_n|^{-1}\mathbb{1}_{nn}\right]U_o.$$

Theorem 3. *Consider a continuous stable and fully controllable dynamic MIMO LTI system of the form* (2.1).

The singular decompositions of its inverse controllability gramian in terms of the eigenvalues of the gramian matrix have the following form.

For a simple spectrum of the gramian matrix

$$(P^c)^{-1} = \frac{\sum_{\lambda=1}^{n}\sum_{j=0}^{n-1} P_j^c \sigma_\lambda^j}{\dot{N}_c(\sigma_\lambda)}\frac{1}{\sigma_\lambda}, \tag{3.20}$$

where P^c is the gramian controllability matrix, P_j^c is the Faddeev matrix in the decomposition of the gramian resolvent, σ_λ is the eigenvalue of the gramian matrix P^c.

For the multiple spectrum of the gramian matrix

$$(P^c)^{-1} = -\sum_{\delta=1}^{q}\sum_{\rho=1}^{m_\delta}\frac{K_{\delta\rho}}{(-\sigma_\delta)^{m_\delta - j+1}}, \tag{3.21}$$

$$K_{\delta\rho} = \frac{1}{(\rho-1)!}\left\{\frac{d^{\rho-1}}{d\sigma^{\rho-1}}\left[\frac{(\sigma-\sigma_\delta)^{m_\delta}\sum_{j=0}^{n-1}\sigma^j P_j^c}{\prod_{\delta=1}^{n}(\sigma-\sigma_\delta)^{m_\delta}}\right]\right\}\Bigg|_{s=s_\delta}, \tag{3.22}$$

where P^c is the gramian controllability matrix, P_j^c is the Faddeev matrix in the decomposition of the gramian resolvent, σ_δ is the eigenvalue of the gramian matrix P^c multiplicity m_δ, ρ is the multiplicity index of the eigenvalue σ_δ.

Proof of Theorem 3. We consider the decomposition of the resolvent of the gramian controllability matrix in the form of a segment of the Faddeev series [6]

$$(I\sigma - P^c)^{-1} = \frac{\sum_{j=0}^{n-1} P_j^c \sigma^j}{N_c(\sigma)}\ . \tag{3.23}$$

We denote: $N_c\,(\sigma) = s^n + a_{c,n-1}\sigma^{n-1} + \dots a_{c,1}\sigma + a_{c,0}$, $j = 1,\dots,n$; $N_c\,(\sigma)$ is characteristic polynomial of the resolvent of the gramian matrix, P_j^c is the Faddeev matrix in the decomposition of the resolvent in the Faddeev series, $j = 1,\dots,n$.

We first consider the case when all singular values σ_λ of the gramian are different. In this case, the decomposition (3.23) is transformed to the form

$$(I\sigma - P^c)^{-1} = \frac{\sum_{\lambda=1}^{n}\sum_{j=0}^{n-1} P_j^c \sigma_\lambda^j}{\dot{N}_c(\sigma_\lambda)}\frac{1}{\sigma - \sigma_\lambda}. \tag{3.24}$$

Iterative algorithm for calculating Faddeev matrices and coefficients of the characteristic equation:

First step: $a_{c,n-1} = 1$, $R_n = I$,

Step “k”: $a_{c,n-k} = -\frac{1}{k} tr\left(P^c R_{n-k+1}\right)$, $\quad R_{n-k} = a_{c,n-k}I + P^c R_{n-k+1}$, $\quad k = 1,\dots,n$;

In accordance with the Faddeev–Le Verrier algorithm, the following matrix equalities are also valid:

$$\begin{gathered}
P_0^c = a_{c,1}I + a_{c,2}P^c + \dots + a_{c,n}(P^c)^{n-1},\\
P_1^c = a_{c,2}I + a_{c,3}P^c + \dots + a_{c,n}(P^c)^{n-2},\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
P_{n-2}^c = a_{c,n-1}I + a_{c,n}P^c,\\
P_{n-1}^c = a_{c,n}I.
\end{gathered}$$

The above system can be written in the form

$$P_j^c = \sum_{j=0}^{n-1} a_{c,j+1}(P^c)^j, \quad \forall j : j = 0, 1, \dots, n-1.$$

We put $\sigma = 0$ in (3.24) and get the formula (3.20):

$$(P^c)^{-1} = \frac{\sum_{\lambda=1}^{n}\sum_{j=0}^{n-1} P_j^c \sigma_\lambda^j}{\dot{N}_c(\sigma_\lambda)}\frac{1}{\sigma_\lambda}. \tag{3.25}$$

Thus, (3.20)–(3.25) in the case of a simple spectrum, the gramian matrices determine the singular decomposition of the inverse gramian of controllability. A similar approach can be applied to the case of a multiple spectrum of the gramian matrix. We assume that the characteristic equation of the gramian matrix can be represented in the form

$$N_c\,(\sigma) = \prod_{i=1}^{n}(\sigma - \sigma_i)^{m_i}, \qquad \sum_{i=1}^{n} m_i = q, \quad \sigma_i \in C^+.$$

For any square gramian matrix, its resolvent has the form of a matrix function (3.24). In accordance with [22, 23], its decomposition into simple fractions has the form

$$(I\sigma - P^c)^{-1} = \sum_{\delta=1}^{q}\sum_{\rho=1}^{m_\delta}\frac{K_{\delta\rho}}{(\sigma - \sigma_\delta)^{m_\delta - j + 1}}, \tag{3.26}$$

$$K_{\delta\rho} = \frac{1}{(\rho-1)!}\left\{\frac{d^{\rho-1}}{d\sigma^{\rho-1}}\left[\frac{(\sigma-\sigma_\delta)^{m_\delta}\sum_{j=0}^{n-1}\sigma^j P_j^c}{\prod_{\delta=1}^{n}(\sigma-\sigma_\delta)^{m_\delta}}\right]\right\}\Bigg|_{\sigma=\sigma_\delta}.$$

We set $\sigma = 0$ in (3.26) and obtain formulas (3.21)–(3.22) for the singular decomposition of the inverse gramian of controllability for the case of a multiple spectrum of the gramian matrix.

Illustrative example 2

We consider the problem of controlling an asynchronous motor. The model of the control object can be described by equations of state of the form

$$\Sigma_1\colon \begin{cases} \dfrac{dx}{dt} = Ax(t) + Bu(t), \quad x(0) = 0, \\ y(t) = Cx(t). \end{cases}$$

$$A = \begin{bmatrix} -4.67 & 3 & -1.33 & 2.33 \\ -2.17 & 2.33 & -3.83 & 5.17 \\ 1.5 & -0.33 & -1.5 & 0.17 \\ 2.17 & -3.33 & 3.83 & -6.17 \end{bmatrix}, \; B = \begin{bmatrix} 3 \\ -3 \\ -7 \\ -4 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We present the eigenvalues of the system dynamics matrix

$$\lambda_i = -4;\ -3;\ -2;\ -1.$$

To construct a singular decomposition of the inverse gramian of the controllability of the system by the singular values of the gramian matrix, we calculate the gramian of controllability using the formula (3.6)

$$P^c = \begin{bmatrix} 2.5 & 3 & 2.5 & 0.56 \\ 3 & 11.2 & 13.2 & 5.1 \\ 2.5 & 13.2 & 16.6 & 6.9 \\ 0.56 & 5.1 & 6.9 & 3 \end{bmatrix}.$$

We note that the formula (3.6) is valid not only for stable linear systems, but also for unstable systems in which the condition $s_k + s_p \neq 0$ is not violated. It is violated in the case of $s_k = 0$ or $s_k = +j\omega$, $s_{k+1} = -j\omega$ [21].

Then the singular numbers of this gramian take the form

$$\sigma_i = 30.7;\ 2.5;\ 0.17;\ 0.0002.$$

The controllability gramian of the system is represented by a symmetric matrix, therefore there is its SVD decomposition [1]

$$P^c = \begin{bmatrix} -0.13 & 0.86 & -0.48 & -0.009 \\ -0.6 & 0.28 & 0.67 & 0.35 \\ -0.73 & -0.25 & -0.25 & -0.58 \\ -0.3 & -0.32 & -0.51 & 0.74 \end{bmatrix} \times \begin{bmatrix} 31 & 0 & 0 & 0 \\ 0 & 2.5 & 0 & 0 \\ 0 & 0 & 0.17 & 0 \\ 0 & 0 & 0 & 0.0002 \end{bmatrix}$$

$$\times \begin{bmatrix} -0.13 & 0.86 & -0.48 & -0.009 \\ -0.6 & 0.28 & 0.67 & 0.35 \\ -0.73 & -0.25 & -0.25 & -0.58 \\ -0.3 & -0.32 & -0.51 & 0.74 \end{bmatrix}.$$

In accordance with the Faddeev–Le Verrier algorithm, we calculate the Faddeev matrices and the coefficients of the characteristic equation for the inverse gramian

$$P_0^c = \begin{bmatrix} -0.01 & 0.05 & -0.07 & 0.08 \\ 0.05 & -1.62 & 2.69 & -3.4 \\ -0.07 & 2.69 & -4.5 & 5.6 \\ 0.08 & -3.4 & 5.6 & -7.2 \end{bmatrix}, \; P_1^c = \begin{bmatrix} 21.8 & -23.5 & 8.4 & 17 \\ -23.5 & 44.6 & -29.5 & -5.5 \\ 8.4 & -29.5 & 33 & -25 \\ 17 & -5.5 & -25 & 65.7 \end{bmatrix},$$

$$P_2^c = \begin{bmatrix} -31 & 3 & 2.5 & 0.56 \\ 3 & -22.1 & 13.2 & 5.1 \\ 2.5 & 13.2 & -16.7 & 6.9 \\ 0.56 & 5.1 & 6.9 & -30.3 \end{bmatrix}, \; P_3^c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$a_{c,0} = 0.0031, \; a_{c,1} = -13.3, \; a_{c,2} = 82.6, \; a_{c,3} = -33.3, \; a_{c,4} = 1.$$

Then the inverse gramian can be calculated using the formula (3.20)

$$(P^c)^{-1} = \begin{bmatrix} 1.97 & -14.8 & 22.3 & -26.2 \\ -14.8 & -517 & -856 & 1083 \\ 22.3 & -856 & 1422 & -1803 \\ -26.2 & 1083 & -1803 & 2290 \end{bmatrix}.$$

3.4. Spectral Decompositions of Energy Functionals and New Stability Criteria

Within the framework of the SISO LTI assumptions made above, we consider a system of the form (2.4), the equations of state of which are reduced to the canonical form of controllability, and calculate the energy functional J, which is the value of the square H_2 is the norm of the transfer functions of the system and gives an assessment of the risk of loss of stability [1, 19, 22]. To do this, we use (3.12) and (3.14) and, for definiteness, choose the spectral decomposition of the controllability gramian in a simple spectrum

$$J = trC^F \Omega_c (C^F)^T$$

$$= \left(\frac{{\xi_0}^2}{\sum\limits_{k=1}^{n} \dot{N}(s_k) N(-s_k)} - \frac{{\xi_1}^2 \sum\limits_{k=1}^{n} s_k^2}{\sum\limits_{k=1}^{n} \dot{N}(s_k) N(-s_k)} + \cdots + \frac{(-1)^{n-1} {\xi_{n-1}}^2 \sum\limits_{k=1}^{n} s_k^{2n}}{\sum\limits_{k=1}^{n} \dot{N}(s_k) N(-s_k)} \right). \tag{3.27}$$

We obtained an invariant spectral decomposition of the energy functional over the simple spectrum of the dynamics matrix. This simple formula shows the advantage of using spectral decompositions in canonical form over the general decomposition (3.23). The decomposition does not depend on the choice of a non-singular matrix of linear transformations of the system coordinates. Two main factors influence the value of the buckling risk J:

1) the values of the diagonal terms of the Xiao matrix Ω_c,

2) the squares of the elements of the reduced output vector.

The expression (3.27) can be simplified by simplifying the SISO LTI system. In [14] it is shown that the Xiao matrix is the controllability gramian for a SISO LTI system with a transfer function

$$W(s) = \frac{y(s)}{u(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}. \tag{3.28}$$

The asymptotic stability of SISO LTI systems of the form (2.4) is equivalent to the asymptotic stability of this system. Moreover, we will show that the asymptotic stability of MIMO LTI systems of the form (2.1) is equivalent to its asymptotic stability. The energy functional J for the system (3.28) according to (3.27) is equal to

$$J = \left(\frac{1}{\sum_{k=1}^{n} \dot{N}(s_k)N(-s_k)} - \frac{\sum_{k=1}^{n} s_k^2}{\sum_{k=1}^{n} \dot{N}(s_k)N(-s_k)} + \ldots + \frac{(-1)^{n-1}\sum_{k=1}^{n} s_k^{2n}}{\sum_{k=1}^{n} \dot{N}(s_k)N(-s_k)} \right). \tag{3.29}$$

Theorem 4. *We consider a continuous fully controllable dynamic MIMO LTI system with a simple spectrum of the form* (2.1)*, as well as a continuous dynamic SISO LTI system with the same spectrum, the equations of state of which are reduced to the canonical form of controllability of the form* (2.4)*.*

Then a sufficient condition for the asymptotic stability of the system (2.1) *according to Lyapunov is the boundedness of the energy functional* (3.29) *for a SISO LTI system with the same spectrum and transfer function*(3.28) *is*

$$J < +\infty, \tag{3.30}$$

$$\textit{for any } s_k \textit{ belonging to } C^-, \ \ k = 1, \ldots, n. \tag{3.31}$$

Proof of Theorem 4. Let us recall that the MIMO LTI system (2.1) is completely controllable and observable, all eigenvalues of matrix A are different, the implementation of the system (2.1) is minimal and there is a single transfer function of the system. When the specified conditions are met, the boundedness of the functional $\sqrt{J}$ is a necessary and sufficient condition for the asymptotic stability of the system (2.1) according to Lyapunov [1, Theorem 5.14]. Thus, the boundedness of the functional J is a sufficient condition for the asymptotic stability of the system (3.27)

$$J < \infty.$$

But the functional J is a trace of the Xiao matrix SISO LTI system (2.4), the equations of state of which are reduced to the canonical form of controllability. This leads to the conclusion that the boundedness of the energy functional of a simple SISO LTI system (3.28) in the form of inequality (3.30) guarantees the asymptotic stability of a complex MIMO LTI system of the form (2.1). Verification of the condition (3.31) requires the use of asymptotic gramian models [22].

Thus, a new criterion for the stability of a complex stationary linear dynamic MIMO LTI system is obtained in the form of a criterion for the boundedness of the trace of the Xiao matrix Ω_c for a simple SISO LTI system (3.21), the equations of which are reduced to the canonical form of controllability. The new criterion does not contradict the well-known criterion that the eigenvalues of the dynamics matrix of a linear system belong to the left half-plane of the plane of eigenvalues, but refines it taking into account the nonlinear effects of mode interaction (multiple eigenvalues, close aperiodic and vibrational modes) [22].

4. CONCLUSION

This article is dedicated to the development of spectral methods for solving the Lyapunov equation. The main results are obtained using structural methods in developing new methods and tools that are closely related to the fundamental properties of linear dynamic systems: controllability, observability and stability. Among the solution methods, two should be mentioned first of all: determining the structure of the solution matrix in the form of the Xiao matrix and spectral decompositions of the solution in the form of Hadamard products. A method and algorithm for calculating matrices in the form of the Hadamard product for multiply connected continuous linear systems with many inputs and many outputs has been developed. This allows us to calculate the elements of the corresponding controllability and observability gramians in the form of products of the corresponding elements of the multiplier matrices and a matrix that is the sum of all possible products of the numerator matrices of the matrix transfer function of the system. When using the canonical forms of controllability or observability, the Hadamard decomposition of the corresponding gramians is reduced to a multiplier matrix, the trace of which is equal to the energy functional of the SISO LTI system. New results are obtained in the form of spectral and singular decompositions of the inverse gramians of controllability and observability. This makes it possible to obtain invariant decompositions of energy functionals and formulate new criteria for the stability of linear systems taking into account the nonlinear effects of mode interaction [20].

FUNDING

This work was supported in part by the Russian Science Foundation, project no. 19-19-00673.

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This paper was recommended for publication by A.I. Mikhalskii, a member of the Editorial Board