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Exponentially Stable Adaptive Control. Part III. Time-Varying Plants

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Abstract—A state-feedback adaptive control system is proposed for a class of linear systems in the controllable canonical form with time-varying unknown parameters described by known nonstationary exosystems with unknown initial conditions. The solution ensures global exponential stability of the closed-loop system in case a regressor is finitely exciting, and also does not require *a priori* information about the sign of the high-frequency gain (control direction). The obtained theoretical results are validated via mathematical modeling.

Keywords: adaptive control, time-varying parameters, parametric error, finite excitation, identification, exponential stability

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1. INTRODUCTION

Considering plants with time-invariant parameters, classical algorithms of the model reference adaptive control guarantee asymptotic stability of the tracking error (the difference between the plant and reference model state vectors) [1]. However, as far as practical scenarios are concerned, real physical systems are often described by models with time-varying parameters. Under these conditions, conventional solutions face difficulties caused by the need to compensate for a term in the derivative of the Lyapunov function that is proportional to the unknown parameters change rate [1, p. 552].

If the unknown time-varying parameters converge exponentially to a constant value, then asymptotic stability of the tracking error is still ensured [2, p. 339]. In case the unknown parameters change their values arbitrarily, and the restrictive condition of the regressor persistent excitation is met, then the baseline solutions guarantee the boundedness of all signals and the convergence of the tracking error to a compact set. If the above-mentioned condition is not satisfied, then the application of robust modifications of the basic adaptive laws allows one to ensure similar properties for the closed-loop system [1, 2].

Further the existing methods to improve the properties of the baseline solutions are reviewed.

In [3, 4] the method of congelation of variables is proposed that is capable of reduction of the time-varying parameters estimation problem to the one of their mean identification. The upper bound of the parameters variance is assumed to be known, and the control law is designed using static nonlinear damping. In this scheme, the adaptive law compensates for the unknown parameters mean, whereas the nonlinear damping is responsible for their variance rejection. Such a solution guarantees asymptotic stability of the tracking error and boundedness of all signals of the closed-loop system. The disadvantage of the above-described method is that the power functions from the plant state (regressor) are used to form the control signal [5, pp. 222–223]. As it is highlighted in [6], another disadvantage of such method is that the parametric uncertainty represented as $\theta(t)\varphi\left(x(t)\right)$ with $\varphi\left(0\right) \neq 0$ can not be compensated.

In [7–9] an approach based on the application of the composite adaptive laws is proposed, in which the parameters are adjusted using both tracking and prediction errors. Compared to the basic solutions [1, 2], under the same assumptions (the regressor is persistently exciting and the robust modifications are used) the composite laws ensure convergence of the tracking error to a compact set of a smaller size. In contrast to the method of congelation of variables [3, 4], the asymptotic stability is not guaranteed. A detailed review of some composite laws for the case of the time-varying parameters is given in the introduction of [10].

In [11] a simple scheme to adjust the geometric mean pole of a closed-loop system is proposed, which guarantees asymptotic stability of the tracking error. The disadvantage of the solution is the need to know the plant input matrix. In [12] the adaptive law with astatism of the first order is developed, which extends the applicability of the basic adaptive laws to a class of systems with linearly varying unknown parameters. In [13–15] a scheme of robust control of nonstationary linear systems represented as transfer functions is proposed. A disadvantage of the approach is the need to know the plant control direction and upper bounds for all unknown time-varying parameters of the system.

The above discussed approaches [3–15] do not consider *a priori* information about the structure of a function according to which the system changes its parameters. However, as is known [16], taking such information into account can significantly improve the properties of a closed-loop system. Recently, in [17, 18] a state observer of time-varying systems based on a parametric identification has been proposed. The system parameters are described by known nonstationary exosystems with unknown initial conditions. The problem of the system state reconstruction is reduced to the identification of the initial conditions of both plant and exosystems. If the regressor finite excitation condition (observability of the system over a finite time interval) is satisfied, then exponential or finite-time convergence of the parametric and state observation errors is ensured.

Based on the results of [17, 18], in this study it is proposed to reduce the control problem to the estimation of the initial conditions of the exosystems used to generate the system parameters.

A class of linear completely controllable systems with time-varying parameters described by known nonstationary exosystems with unknown initial conditions is considered. The difference between ideal and actual control signals is represented as a linear regression equation with an unknown regressor with respect to the unknown initial conditions of the exosystems. Then, in accordance with the results from [18] and using the measurable state and control signal, a measurable regression equation,[1] which scalar regressor is bounded away from zero, is derived with respect to the initial conditions of the exosystems. Using the results of the first part [19] of this paper series, an adaptive law is derived on the basis of such regression, which, in contrast to [3–15], guarantees exponential convergence of the tracking error to zero if the regressor finite excitation (FE) requirement is met.

In addition to FE, the proposed extension of the results from [19] to a class of systems with time-varying parameters also requires:

— the lower and upper bounds of the absolute value of the high-frequency gain to be known;

— the sign of the high-frequency gain to be constant;

— application of the projection operator, which prevents division by zero in the control law.

Compared to [3–11, 13–15], the proposed approach requires knowledge of the state and output matrices of the nonstationary exosystems, and hence, of the physical nature of the processes that cause the system parameters variation.

[1] A measurable regression equation means that its regressor and regressand are measurable or can be computed, while its parameters are unknown.

Main definitions

To prove theorem and propositions, the regressor finite excitation definition and corollary of the Kalman–Yakubovich–Popov (KYP) lemma [1, 2] are used further .

Definition 1. A regressor $\omega(t)$ is finitely exciting $\omega(t)\in\mathrm{FE}$ over $[t_r^+;\, t_e]$ if there exists $t_r^+ \geqslant 0$, $t_e > t_r^+$ and α such that the following inequality holds

$$\int\limits_{t_r^+}^{t_e} \omega(\tau)\,\omega^{\mathrm{T}}(\tau)\,d\tau \geqslant \alpha I_{n\times n}, \tag{1.1}$$

where $\alpha > 0$ is an excitation level, $I_{n\times n}$ stands for an identity matrix.

Corollary 1. *For any matrix $D > 0$, a controllable pair $(A,\, B)$ with $B\in\mathbb{R}^{n\times m}$, a Hurwitz matrix $A\in\mathbb{R}^{n\times n}$ there exist matrices $P = P^{\mathrm{T}} > 0$, $Q\in\mathbb{R}^{n\times m}$, $K\in\mathbb{R}^{m\times m}$ and a number $\mu > 0$ such that*

$$\begin{gathered} A^{\mathrm{T}}P + PA = -QQ^{\mathrm{T}} - \mu P, \quad PB = QK, \\ K^{\mathrm{T}}K = D + D^{\mathrm{T}}. \end{gathered} \tag{1.2}$$

2. PROBLEM STATEMENT

A class of continuous linear time-varying systems is considered[2]:

$$\begin{gathered} \forall t \geqslant t_0^+ \; \dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x\left(t_0^+\right) = x_0, \\ A(t) = A_0 + e_1\vartheta^{\mathrm{T}}(t), \quad B(t) = e_1\beta(t), \\ A_0 = \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ & 0_{1\times n} \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0_{(n-1)\times 1} \\ 1 \end{bmatrix}, \end{gathered} \tag{2.1}$$

where $x(t)\in\mathbb{R}^n$ is a system state with unknown initial conditions x_0, $u(t)\in\mathbb{R}$ stands for a control signal, $A_0\in\mathbb{R}^{n\times n}$ denotes a known state matrix, $B(t)\in\mathbb{R}^n$, $\vartheta(t)\in\mathbb{R}^n$ are unknown vectors, t_0^+ stands for a known initial time instant. The pair $(A(t),\, B(t))$ is completely controllable for all $t \geqslant t_0^+$ in a sense of criterion from [20].

The following assumptions are adopted for the unknown parameters of the system (2.1).

Assumption 1. The vectors $\vartheta(t)$, $B(t)$ are bounded, continuous and formed by time-varying exosystems[3]:

$$\begin{gathered} \begin{cases} \dot{x}_\vartheta(t) = \mathcal{A}_\vartheta(t)x_\vartheta(t), \quad x_\vartheta\left(t_0^+\right) = x_{\vartheta_0}, \\ \vartheta(t) = h_\vartheta x_\vartheta(t), \end{cases} \\ \begin{cases} \dot{x}_B(t) = \mathcal{A}_B(t)x_B(t), \quad x_B\left(t_0^+\right) = x_{B_0}, \\ B(t) = h_B x_B(t), \end{cases} \end{gathered} \tag{2.2}$$

where $x_\vartheta(t)\in\mathbb{R}^{n_\vartheta}$, $x_B(t)\in\mathbb{R}^{n_B}$ are exosystems state with unknown initial conditions $x_\vartheta\left(t_0^+\right)$, $x_B\left(t_0^+\right)$, $h_\vartheta\in\mathbb{R}^{n\times n_\vartheta}$, $h_B\in\mathbb{R}^{n\times n_B}$; $\mathcal{A}_\vartheta(t)\in\mathbb{R}^{n_\vartheta\times n_\vartheta}$, $\mathcal{A}_B(t)\in\mathbb{R}^{n_B\times n_B}$ denote known vectors and matrices.

Assumption 2. Lower $\beta_{\min} > 0$ and upper $\beta_{\max} > \beta_{\min}$ bounds are known for $|\beta(t)|$.

[2] The obtained results can be generalized to MIMO systems in case if the structures of matrices $A(t)\in\mathbb{R}^{n\times n}$ and $B(t)\in\mathbb{R}^{n\times m}$ are known.

[3] In general case, the matrices $\mathcal{A}_\vartheta(t)$, $\mathcal{A}_B(t)$ can depend on the system state $x(t)$ in a nonlinear fashion.

Assumption 3. The sign of the high-frequency gain $\beta(t)$ is constant but unknown ($\mathrm{sgn}(\beta(t)) = const$).

The required control quality for a closed-loop system with the control signal $u(t)$ and plant (2.1) is defined using the time-invariant reference model:

$$\forall t \geqslant t_0^+ \; \dot{x}_{ref}(t) = A_{ref}x_{ref}(t) + B_{ref}r(t), \quad x_{ref}\left(t_0^+\right) = x_{0ref}, \tag{2.3}$$

where $x_{ref}(t) \in \mathbb{R}^n$ is a reference model state with known initial conditions x_{0ref}, $r(t) \in \mathbb{R}$ stands for a reference signal, $A_{ref} \in \mathbb{R}^{n\times n}$ denotes a Hurwitz state matrix of the reference model, $B_{ref} \in \mathbb{R}^n$ is a reference model control input vector.

Having the plant (2.1), reference model (2.3) and a controllable pair $(A(t), B(t))$, if Assumption 3 is met, then it is assumed that the matching conditions are satisfied.

Assumption 4. There exists a matrix $K_x(t) = A_{ref} - A(t) \in \mathbb{R}^{n\times n}$ and a vector $K_r(t) = \left[B^{\mathrm{T}}(t)B(t)\right]^{-1}B^{\mathrm{T}}(t) \in \mathbb{R}^{1\times n}$ such that the following holds

$$A(t) + B(t)K_r(t)K_x(t) = A_{ref}, \quad B(t)K_r(t)B_{ref} = B_{ref}. \tag{2.4}$$

Considering Assumption 2, the error equation between the plant (2.1) and the reference model (2.3) is written as

$$\begin{aligned}\dot{e}_{ref}(t) &= A_{ref}e_{ref}(t) + B(t)u(t) - \left(A_{ref} - A(t)\right)x(t) - B_{ref}r(t)\\ &= A_{ref}e_{ref}(t) + B(t)\left(u(t) - u^*(t)\right),\end{aligned} \tag{2.5}$$

where $e_{ref}(t) = x(t) - x_{ref}(t), \quad u^*(t) = K_r(t)\left(K_x(t)x(t) + B_{ref}r(t)\right)$.

The aim is to derive a control law $u(t)$ that ensures achievement of the following goal:

$$\Phi(t) \in \mathrm{FE} \;\Rightarrow \lim_{t\to\infty}\|e_{ref}(t)\| = 0 \;(\exp), \tag{2.6}$$

where $\Phi(t)$ is some generalized vector of measurable signals.

Remark 1. Assumption 1 is to single out a group of systems, for which the problem of exponentially stable control (2.6) *is stated and solved in this paper, from the general class of linear systems with time-varying parameters. The proposed solution uses information about* $\beta_{\max} > \beta_{\min} > 0$, *which is required by Assumption 2. Mathematical modeling indicates that one can choose* $\beta_{\max} \to \infty$, $\beta_{\min} \to 0$, *which somewhat relaxes the strictness of this condition. Assumption 3 guarantees the continuity of the coefficients* $K_x(t)$, $K_r(t)$ *of the control law* $u^*(t)$. *Assumption 4 implies that the respective matrices of the reference model* (2.3) *and the plant* (2.1) *have the same structure.*

Remark 2. Systems (2.1) *with matched parametric uncertainty* (2.4) *are quite widespread, as far as practical scenarios are concerned. For example, the Euler angles dynamics equations of a solid body under the assumption of its symmetry are represented as a second-order system with matched uncertainty. Another good example of a control problem with matched uncertainty is the control of manipulator coordinates using the Euler–Lagrange formalism.*

3. MAIN RESULT

In Subsection 3.1 the stated problem of exponentially stable control (2.6) is reduced to the one of identification of the initial conditions x_{B_0}, x_{ϑ_0}. In subsection 3.2 a regression equation with respect to x_{B_0}, x_{ϑ_0} is derived on the basis of the measurable signals, and an adaptive law is introduced that allows one to achieve the goal (2.6).

3.1. Control Law Parametrization

The control law $u^*(t)$ is to be written via the plant measurable state (2.1), (2.2) and unknown parameters x_{B_0}, x_{ϑ_0}. For this equation (2.2) is solved:

$$\begin{gathered}\vartheta(t) = h_\vartheta \Phi_\vartheta(t) x_{\vartheta_0},\\ B(t) = h_B \Phi_B(t) x_{B_0},\end{gathered} \tag{3.1.1}$$

where, if Assumptions 2 and 3 are met, the following inequality holds

$$\begin{gathered}0 < \beta_{\min}^2 \leqslant x_{B_0}^{\mathrm{T}} G(t) x_{B_0} \leqslant \lambda_{\max}\left(G(t)\right) \left\|x_{B_0}\right\|^2,\\ G(t) = \Phi_B^{\mathrm{T}}(t) h_B^{\mathrm{T}} h_B \Phi_B(t),\end{gathered} \tag{3.1.2}$$

and the fundamental matrices $\Phi_\vartheta(t)$ and $\Phi_B(t)$ are measurable and defined as

$$\begin{gathered}\dot{\Phi}_\vartheta(t) = \mathcal{A}_\vartheta(t)\Phi_\vartheta(t), \quad \Phi_\vartheta\left(t_0^+\right) = I_{n_\vartheta},\\ \dot{\Phi}_B(t) = \mathcal{A}_B(t)\Phi_B(t), \quad \Phi_B\left(t_0^+\right) = I_{n_B}.\end{gathered}$$

Considering (2.4) and (3.1.1), the ideal control law $u^*(t)$ is rewritten in the required form

$$\begin{aligned} u^*(t) &= K_r(t)\left(K_x(t)x(t) + B_{ref}r(t)\right)\\ &= \frac{x_{B_0}^{\mathrm{T}}\Phi_B^{\mathrm{T}}(t)h_B^{\mathrm{T}}}{x_{B_0}^{\mathrm{T}}\Phi_B^{\mathrm{T}}(t)h_B^{\mathrm{T}}h_B\Phi_B(t)x_{B_0}}\Big((A_{ref} - A_0)x(t) - e_1\vartheta^{\mathrm{T}}(t)x(t) + B_{ref}r(t)\Big)\\ &= \frac{x_{B_0}^{\mathrm{T}}\Phi_B^{\mathrm{T}}(t)h_B^{\mathrm{T}}}{F(t)}\, e_1\Big(e_1^{\mathrm{T}}(A_{ref} - A_0)x(t) - x_{\vartheta_0}^{\mathrm{T}}\Phi_\vartheta^{\mathrm{T}}(t)h_\vartheta^{\mathrm{T}}x(t) + e_1^{\mathrm{T}}B_{ref}r(t)\Big),\end{aligned} \tag{3.1.3}$$

where $F(t) = x_{B_0}^{\mathrm{T}} G(t) x_{B_0} > 0$.

As according to (2.5) the parameters x_{B_0} and x_{ϑ_0} are unknown, then equation (3.1.3) motivates to introduce the control law with the adjustable parameters:

$$u(t) = \frac{\hat{x}_{B_0}^{\mathrm{T}}(t)\Phi_B^{\mathrm{T}}(t)h_B^{\mathrm{T}}}{\hat{F}(t)} e_1\Big(e_1^{\mathrm{T}}\left(A_{ref} - A_0\right)x(t) - \hat{x}_{\vartheta_0}^{\mathrm{T}}(t)\Phi_\vartheta^{\mathrm{T}}(t)h_\vartheta^{\mathrm{T}}x(t) + e_1^{\mathrm{T}}B_{ref}r(t)\Big), \tag{3.1.4}$$

where $\hat{F}(t) = \hat{x}_{B_0}^{\mathrm{T}}(t) G(t) \hat{x}_{B_0}(t)$.

Proposition 1. *The error $u(t) - u^*(t)$ between the actual* (3.1.4) *and ideal* (3.1.3) *control signals is written as*

$$u(t) - u^*(t) = \tilde{\theta}^{\mathrm{T}}(t)\omega(t), \tag{3.1.5}$$

where $\tilde{\theta}(t) = \begin{bmatrix}\tilde{x}_{\vartheta_0}^{\mathrm{T}}(t) & \tilde{x}_{B_0}^{\mathrm{T}}(t)\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{n_\vartheta + n_B}$ is a parametric error, $\omega(t) \in \mathbb{R}^{n_\vartheta + n_B}$ stands for an unmeasurable regressor.

Proof of Proposition 1 and functional definition of the regressor $\omega(t)$ are given in Appendix.

Having substituted (3.1.5) into (2.5), it is obtained that:

$$\dot{e}_{ref}(t) = A_{ref}e_{ref}(t) + B(t)\tilde{\theta}^{\mathrm{T}}(t)\omega(t). \tag{3.1.6}$$

Then, according to the first paper [19] in this series, the following transformations are to be defined

$$\begin{gathered}\Phi(t) = \mathcal{F}_1\left(t, x(t), u(t), \Phi_\vartheta(t), \Phi_B(t)\right), \quad z(t) = \mathcal{F}_2\left(x(t)\right),\\ \dot{\hat{\theta}}(t) = \mathcal{G}\left(\Phi(t), z(t)\right),\end{gathered}$$

which together ensure that the goal of exponentially stable control in the augmented tracking error space is achieved:

$$\Phi(t) \in \mathrm{FE} \;\Rightarrow \lim_{t\to\infty} \|\xi(t)\| = 0 \;\; (\exp), \tag{3.1.7}$$

where $\xi(t) = \left[e_{ref}^{\mathrm{T}}(t) \;\; \tilde{\theta}^{\mathrm{T}}(t)\right]^{\mathrm{T}}$ is the augmented tracking error.

3.2. Adaptive Law Design

The next aim is to derive the regression equation with respect to the unknown time-invariant parameters x_{B_0} and x_{ϑ_0} of the ideal control law (3.1.3). The result of such parametrization is represented as the following proposition.

Proposition 2. *Using (i) the states of the set of filters ($A_K \in \mathbb{R}^{n\times n}$ is a Hurwitz matrix)*

$$\begin{aligned} \dot{\bar{x}}(t) &= A_K\bar{x}(t) - A_K x(t),\; \bar{x}\left(t_0^+\right) = 0_n,\\ \dot{\varphi}(t) &= A_K\varphi(t) + e_1 x^{\mathrm{T}}(t) h_\vartheta \Phi_\vartheta(t),\; \varphi(t_0^+) = 0_{n\times n_\vartheta},\\ \dot{\psi}(t) &= A_K\psi(t) + h_B\Phi_B(t)u(t),\; \psi\left(t_0^+\right) = 0_{n\times n_B},\\ \dot{\upsilon}(t) &= A_K\upsilon(t) + A_0 x(t),\; \upsilon\left(t_0^+\right) = 0_n, \end{aligned} \qquad A_K = \begin{bmatrix} K \in \mathbb{R}^n & \begin{matrix} I_{(n-1)\times(n-1)} \\ 0_{1\times(n-1)} \end{matrix} \end{bmatrix}, \tag{3.2.1}$$

(ii) normalization procedures

$$\begin{gathered} z(t) := \frac{e_1^{\mathrm{T}}\left(x(t) - \bar{x}(t) - \upsilon(t)\right)}{1 + \Phi^{\mathrm{T}}(t)\Phi(t)} \quad \Psi^{\mathrm{T}}(t) := \frac{\Phi^{\mathrm{T}}(t)}{1 + \Phi^{\mathrm{T}}(t)\Phi(t)},\\ \Phi^{\mathrm{T}}(t) := \left[e_1^{\mathrm{T}} e^{A_K\left(t-t_0^+\right)} \;\; e_1^{\mathrm{T}}\varphi(t) \;\; e_1^{\mathrm{T}}\psi(t)\right], \end{gathered} \tag{3.2.2}$$

(iii) dynamic extension

$$\dot{\Delta}(t) = e^{-\sigma\left(t-t_0^+\right)}\Psi(t)\Psi^{\mathrm{T}}(t), \quad \Delta\left(t_0^+\right) = 0_{(n_\vartheta+n_B+n)\times(n_\vartheta+n_B+n)}, \tag{3.2.3a}$$

$$\dot{y}(t) = e^{-\sigma\left(t-t_0^+\right)}\Psi(t)z(t), \quad y\left(t_0^+\right) = 0_{(n_\vartheta+n_B+n)} \tag{3.2.3b}$$

(iv) and mixing

$$Y(t) := \mathrm{adj}\left\{\Delta(t)\right\} y(t), \quad \Omega(t) := \det\left\{\Delta(t)\right\}, \tag{3.2.3c}$$

the following regression equation with respect to the parameters x_{B_0} and x_{ϑ_0} is obtained:

$$\Upsilon(t) := \mathfrak{L}Y(t) = \Omega\left(\Phi(t)\right)\theta, \quad \mathfrak{L} = \left[0_{(n_\vartheta+n_B)\times n} \; I_{(n_\vartheta+n_B)\times(n_\vartheta+n_B)}\right], \tag{3.2.4}$$

where, if $\Phi(t) \in \mathrm{FE}$, then $\forall t \geqslant t_e$ it holds that $\Omega_{UB}(t) \geqslant \Omega(t) \geqslant \Omega_{LB} > 0$.

Proof of Proposition 2 is postponed to Appendix.

If $\omega(t)$ is measurable, then, using equation (3.2.4) and the results of Theorem 1 from [19], the adaptive law can be derived, which ensures that the goal (3.1.7) is achieved. The following theorem is to obtain the adaptive law, which ensures (3.1.7) in case $\omega(t)$ is unmeasurable.

Theorem 1. *Let Assumptions 1–4 be met and $\Phi(t) \in \mathrm{FE}$, then the adaptive law*

$$\begin{gathered} \dot{\hat{\theta}}(t) = -\gamma(t)\Omega(t)\left(\Omega(t)\hat{\theta}(t) - \Upsilon(t)\right) = -\gamma(t)\Omega^2(t)\tilde{\theta}(t), \quad \hat{\theta}\left(t_0^+\right) = \hat{\theta}_0,\\ \gamma(t) = \begin{cases} 0, \;\; \text{if} \;\; \Omega(t) < \rho \in (0; \Omega_{LB}], \\ \dfrac{\gamma_0\lambda_{\max}\left(\hat{\omega}(t)\hat{\omega}^{\mathrm{T}}(t)\right) + \gamma_1}{\Omega^2(t)} \;\; \text{otherwise} \end{cases} \end{gathered} \tag{3.2.5}$$

in case $\gamma_0 > 0,\ \gamma_1 \geqslant 0$ *ensures the following:*

1) $\left|\tilde{\theta}_i\left(t_a\right)\right| \leqslant \left|\tilde{\theta}_i\left(t_b\right)\right| \ \forall t_a \geqslant t_b$;

2) $\left\{\begin{array}{c} \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right) = \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)x_{B_0}\right) \\ \left|V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right| > \left|V_1^{\mathrm{T}}(t)x_{B_0}\right| \end{array}\right\} \Rightarrow \hat{F}(t) > 0$;

3) $\forall t \geqslant t_0^+$ *boundedness of* $\xi(t) \in L_\infty$;

4) *exponential convergence of* $\xi(t)$ *to zero for all* $t \geqslant t_e$.

Proof of theorem and definitions of $V_1(t)$, $\hat{\omega}(t)$ are presented in Appendix.

In case the conditions of the second statement of theorem are not met, then a division by zero may occur in the control law (3.1.4). That is why, considering a practical scenario, the law (3.1.4) should be augmented with the projection operator:

$$
\begin{gathered}
u(t) = \frac{\hat{x}_{B_0}^{\mathrm{T}}(t)\Phi_B^{\mathrm{T}}(t)h_B^{\mathrm{T}}}{\hat{F}_{\mathrm{prj}}(t)} e_1 \left(e_1^{\mathrm{T}}\left(A_{ref} - A_0\right)x(t) - \hat{x}_{\vartheta_0}^{\mathrm{T}}(t)\Phi_\vartheta^{\mathrm{T}}(t)h_\vartheta^{\mathrm{T}}x(t) + e_1^{\mathrm{T}}B_{ref}r(t)\right), \\
\hat{F}_{\mathrm{prj}}(t) := \begin{cases} \hat{x}_{B_0}^{\mathrm{T}}(t)G(t)\hat{x}_{B_0}(t), \text{ if } \hat{x}_{B_0}^{\mathrm{T}}(t)G(t)\hat{x}_{B_0}(t) > \beta_{\min}^2 > 0, \\ \beta_{\min}^2 \text{ otherwise.} \end{cases}
\end{gathered} \tag{3.2.6}
$$

The proposed transformations $\mathcal{F}_1\left(.\right)$ and $\mathcal{F}_2\left(.\right)$ are described by (3.2.1), (3.2.2), and $\mathcal{G}\left(.\right)$ — by (3.2.3)–(3.2.5), respectively. In general, the designed adaptive control system includes the control law (3.1.4), procedures of the measurable signals processing (3.2.1)–(3.2.4) and the adaptive law (3.2.5). The filtering (3.2.1) allows one, using the measurable signals $x(t)$, $u(t)$, $\Phi_\vartheta(t)$, $\Phi_B(t)$, to obtain the static regression equation with respect to the unknown parameters x_{B_0}, x_{ϑ_0}, x_0. The normalization (3.2.2) guarantees that the regressor $\Psi(t)$ is bounded, which, owing to Proposition 1 from [19], is sufficient to state that $\Omega(t)$ is upper bounded. The procedures of extension and mixing (3.2.3) are used to transform the vector regressor $\Psi(t)$, first, into a matrix one $\Delta(t)$, and then — into a scalar one $\Omega(t)$. The division by $\Omega^2(t)$ used in the adaptive law (3.2.5) is a safe operation as $\Omega(t) \geqslant \Omega_{\mathrm{LB}} > 0$, and in case of proper choice of ρ it allows one to ensure the convergence of $\tilde{\theta}(t)$ to zero with the rate defined as $\gamma_0\lambda_{\max}\left(\hat{\omega}(t)\hat{\omega}^{\mathrm{T}}(t)\right) + \gamma_1$.

According to the results of theorem, unlike most known approaches [3–15] to control linear systems with time-varying parameters, the proposed system ensures exponentially stable control (2.6).

Remark 3. The application of the projection operator (3.2.6) *is a classical and well-known tool to avoid singularity in adaptive control schemes* (*see, for example,* [1, p. 400]). *In case conditions from the second statement of theorem are met, the choice* $\beta_{\min} \to 0$ *guarantees that there are no switches in* (3.2.6). *Otherwise, the choice* $\beta_{\min} \to 0$ *ensures that the number of switches is finite.*

Remark 4. Over $\left[t_0^+; t_e\right]$ *or when* $\Phi(t) \notin \mathrm{FE}$, *the loop of adjustment* (3.2.5) *of the control law* (3.1.4) *parameters is open, and in case of arbitrary choice of the initial conditions* $\hat{\theta}\left(t_0^+\right)$ *the control quality can be arbitrarily poor up to loss of stability. Therefore, in practice, for the proposed adaptive system:*

i) *the choice of the initial conditions* $\hat{\theta}\left(t_0^+\right)$ *should be made using robust control techniques to ensure that the following system is asymptotically stable*

$$
\dot{x}(t) = \left(A(t) + B(t)\hat{K}_r(t)\hat{K}_x(t)\right)x(t) \ \textit{when} \ \dot{\hat{\theta}}(t) \equiv 0 \ \textit{for all} \ \ t \geqslant t_0^+,
$$

ii) *the control law* (3.1.4) *should be augmented with some robust term, which guarantees the boundedness of the error* $\xi(t)$ *and acceptable control quality.*

For example, a) *in case* $\mathrm{sgn}\,(\beta(t))$ *and bounds* $\beta_{\min}$, $\beta_{\max}$ *are known, the following control law can be applied*

$$u(t) = \{(3.1.4)\,,\ (3.2.6)\} - \gamma_3\,\mathrm{sgn}\,(\beta(t))\,e_{ref}(t)Pe_1\hat{\omega}(t)\hat{\omega}^{\mathrm{T}}(t), \quad \gamma_3 > 0$$

(*see Lemma 2.2 from* [5]), b) *if* $\mathrm{sgn}\,(\beta(t))$ *is unknown, but* $\beta_{\min}$, $\beta_{\max}$ *are known, then the law with the damping and Nussbaum function* [21] can be used:

$$u(t) = \{(3.1.4)\,,(3.2.6)\} - \gamma_3 N\left(w(t)\right)e_{ref}^{\mathrm{T}}(t)Pe_1\hat{\omega}(t)\hat{\omega}^{\mathrm{T}}(t), \quad \gamma_3 > 0,$$
$$N\left(w(t)\right) = w^2(t)\cos\left(w(t)\right),$$
$$\dot{w}(t) = \gamma_3\gamma_4 e_{ref}^{\mathrm{T}}(t)Pe_1e_1^{\mathrm{T}}Pe_{ref}(t)\hat{\omega}(t)\hat{\omega}^{\mathrm{T}}(t), \quad \gamma_4 > 0.$$

4. NUMERICAL EXPERIMENTS

In Matlab/Simulink the numerical experiments with the proposed adaptive control system have been conducted for cases when the conditions of the second statement of theorem are met and violated. The simulation was done using numerical integration by the explicit Euler method with a constant discretization step of $\tau_s = 10^{-4}$ s.

4.1. $\mathrm{sgn}\left(V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right) = \mathrm{sgn}\left(V_1^{\mathrm{T}}(t)x_{B_0}\right)$ *and* $\left|V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right| > \left|V_1^{\mathrm{T}}(t)x_{B_0}\right|$

The matrices of the plant (2.1) were defined as follows for all $t \geqslant 0$:

$$A(t) = \begin{bmatrix} 0 & 1 \\ a_1\sin\left(a_2 t\right) & a_4e^{a_3t} + a_5\left(1 - e^{a_3t}\right)\end{bmatrix},$$
$$B(t) = \begin{bmatrix} 0 \\ b_1\cos\left(b_2t\right) + b_4e^{b_3t} + b_5\end{bmatrix}, \quad x\left(t_0^+\right) = \begin{bmatrix} -1 \\ 1\end{bmatrix}, \tag{4.1.1}$$

where a_2 =1, $a_3 = b_3 = -0.25$, $b_2 = \sqrt{12}$ are known constants, $a_1 = -10$, $a_4 = 1$, $a_5 = 7$, $b_1 = 0.25$, $b_4 = -2$, $b_5 = -4$ denotes unknown constants.

Then the matrices and exosystem (2.2) initial conditions took the form:

$$\mathcal{A}_\vartheta(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a_2^2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}, \quad x_{\vartheta_0} = \begin{bmatrix} -a_1a_2 \\ 0 \\ a_4 - a_5 \\ a_5\end{bmatrix}, \quad h_\vartheta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1\end{bmatrix},$$
$$\mathcal{A}_B(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -b_2^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_3\end{bmatrix}, \quad x_{B_0} = \begin{bmatrix} 0 \\ b_1 \\ b_5 \\ b_4\end{bmatrix}, \quad h_B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1\end{bmatrix}. \tag{4.1.2}$$

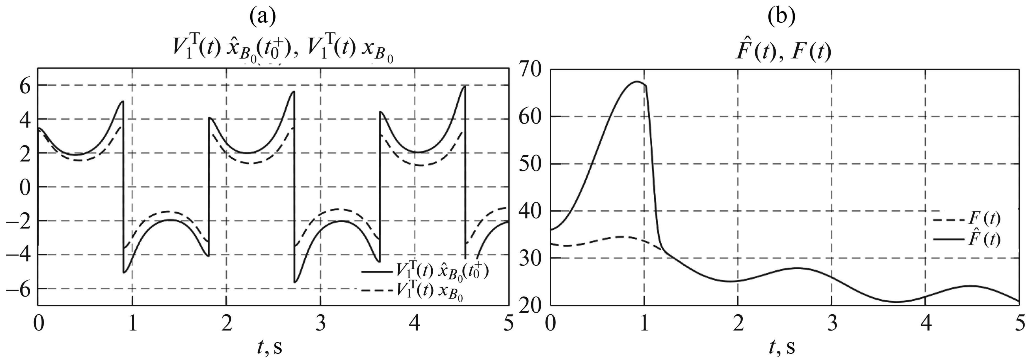

Fig. 1. Behavior of (a) $V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)$ and $V_1^{\mathrm{T}}(t)x_{B_0}$, (b) $\hat{F}(t)$ and $F(t)$.

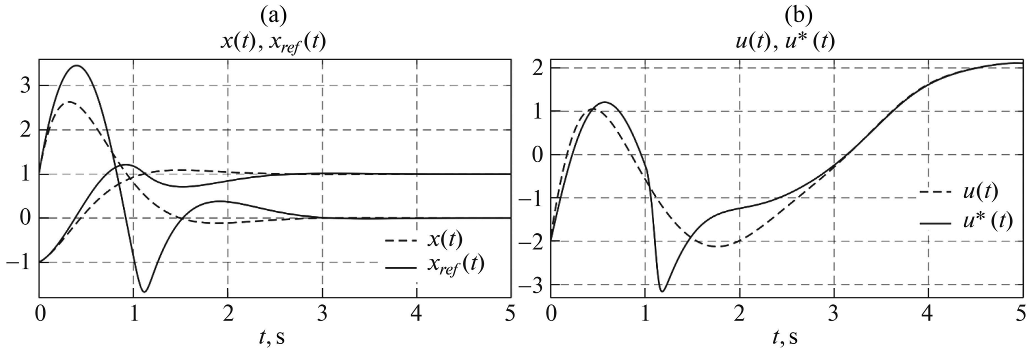

Fig. 2. Behavior of (a) $x(t)$ and $x_{ref}(t)$, (b) $u^*(t)$ and $u(t)$.

The matrices of the reference model (2.3), filters parameters (3.2.1), (3.2.3) and some parameters of the adaptive law (3.2.5) were picked as:

$$\begin{gathered} A_{ref} = \begin{bmatrix} 0 & 1 \\ -8 & -4 \end{bmatrix}, \quad B_{ref} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}, \quad A_K = \begin{bmatrix} -20 & 1 \\ -100 & 0 \end{bmatrix}, \quad \sigma = 5, \\ \rho = 10^{-62}, \quad \beta_{\min} = 0.1, \quad \beta_{\max} = 10, \quad \gamma_0 = 10^{-8}, \quad \gamma_1 = 0, \\ \hat{\theta}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & -8 & 1 \end{bmatrix}^{\mathrm{T}}. \end{gathered} \tag{4.1.3}$$

First of all, it was checked whether the requirements of the second statement of theorem were met for such an experiment. Figure 1 depicts comparison of the functions $V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)$ with $V_1^{\mathrm{T}}(t)x_{B_0}$ and $\hat{F}(t)$ with $F(t)$.

The discontinuities in Fig. 1 were caused by the change of the direction of the eigenvector $V_1(t)$ (the elements of the matrix $G(t)$ crossed zero). It follows from Fig. 1a that the chosen initial conditions (4.1.3) guaranteed that the conditions of the second statement of theorem were met. Together Figs. 1a and 1b confirm the implication

$$\left\{ \begin{gathered} \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right) = \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)x_{B_0}\right) \\ \left|V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right| > \left|V_1^{\mathrm{T}}(t)x_{B_0}\right| \end{gathered} \right\} \Rightarrow \hat{F}(t) > 0.$$

Having validated that $\hat{F}(t) > 0$, the modelling was continued. Figure 2a presents comparison of states of the reference model $x_{ref}(t)$ (when $x_{0ref} = x_0$) and the plant $x(t)$, and in Fig. 2b the ideal $u^*(t)$ and actual $u(t)$ control signals are compared.

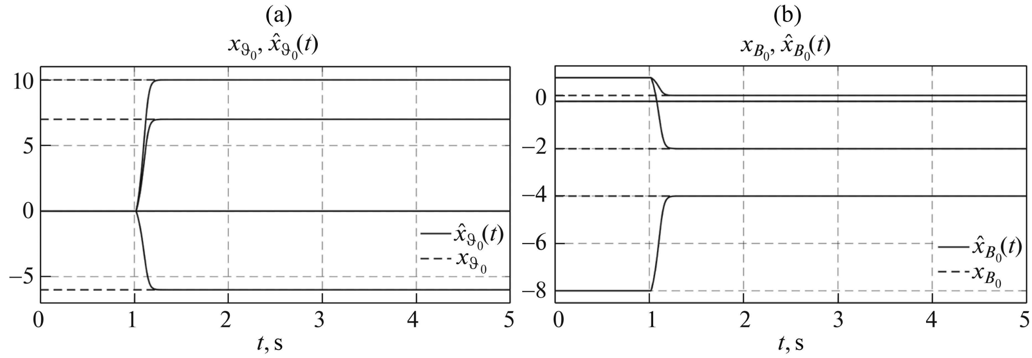

Fig. 3. Behaviour of (a) x_{ϑ_0} and $\hat{x}_{\vartheta_0}(t)$, (b) x_{B_0} and $\hat{x}_{B_0}(t)$.

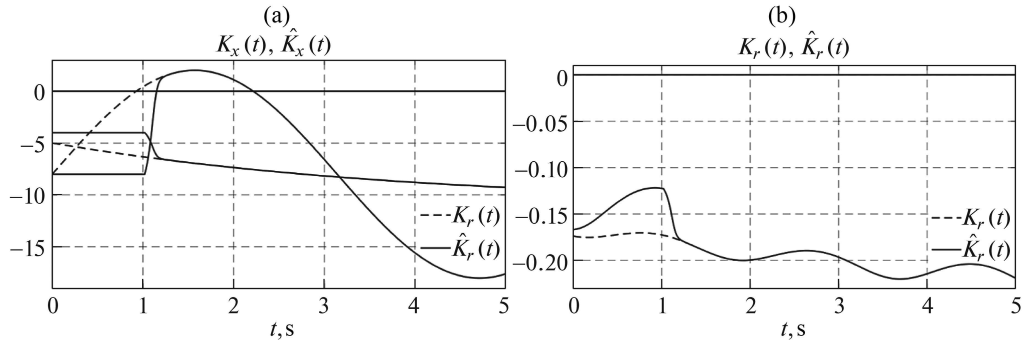

Fig. 4. Behavior of (a) $\hat{K}_x(t)$ and $K_x(t)$, (b) $\hat{K}_r(t)$ and $K_r(t)$.

In Fig. 3a the parameters x_{ϑ_0} and $\hat{x}_{\vartheta_0}(t)$ are compared, while in Fig. 3b — x_{B_0} and $\hat{x}_{B_0}(t)$.

Figure 4 demonstrates the comparison of the parameters $K_x(t)$, $K_r(t)$ and their estimates $\hat{K}_x(t)$, $\hat{K}_r(t)$ calculated with the help of $\hat{x}_{\vartheta_0}(t)$, $\hat{x}_{B_0}(t)$.

The simulation results confirm the theoretical conclusions of Theorem 1. Indeed, when $\gamma_0 > 0$, $\gamma_1 \geqslant 0$ the proposed adaptive system guaranteed that the goal (2.6) was achieved.

The transients shown in Figs. 2–4 confirm the shortcoming of the proposed system noted in Remark 3. Over the time interval [0; 1] the control system functioned with an open adaptive loop (3.2.5), which resulted in oscillations of $x(t)$.

4.2. $\operatorname{sgn}\left(V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right) \neq \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)x_{B_0}\right)$

The same plant was considered (4.1.1), (4.1.2) and the same parameters (4.1.3) of the reference model (2.3), filters (3.2.1), (3.2.3), adaptive law (3.2.5) were used, but under more realistic scenario for practice $\operatorname{sgn}\left(V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right) \neq \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)x_{B_0}\right)$. The modified control law (3.2.6) was chosen and, according to the first set of experiments, we picked $\gamma_0 = 10^{-10}$, $\gamma_1 = 10$ and $\beta_{\min} = 1$, $\rho = 10^{-81}$, $\hat{\theta}_0 = \begin{bmatrix} 0 & -8 & -2 & -2 & 0 & 1 & -8 & 1 \end{bmatrix}^{\mathrm{T}}$.

Figure 5 depicts the comparison of the function $V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)$ with $V_1^{\mathrm{T}}(t)x_{B_0}$, and $\hat{F}(t)$, $\hat{F}_{\mathrm{prj}}(t)$ with $F(t)$.

The discontinuities in Fig. 5 were caused by the change of the direction of the eigenvector $V_1(t)$ (the elements of the matrix $G(t)$ crossed zero). It follows from Fig. 5a that the chosen initial conditions guaranteed that the condition $\operatorname{sgn}\left(V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right) \neq \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)x_{B_0}\right)$ was met. Figure 5b demonstrates the effect of the projection operator (3.2.6) application. Together Figs. 5a and 5b

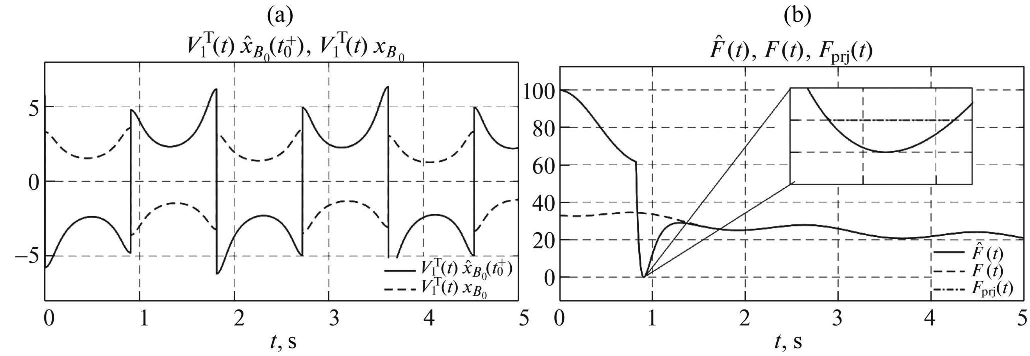

Fig. 5. Behavior of (a) $\hat{\Theta}_0^i(t)$ and $\Theta_0^i(t)$, (b) $\hat{F}(t)$, $\hat{F}_{\mathrm{prj}}(t)$ and $F(t)$.

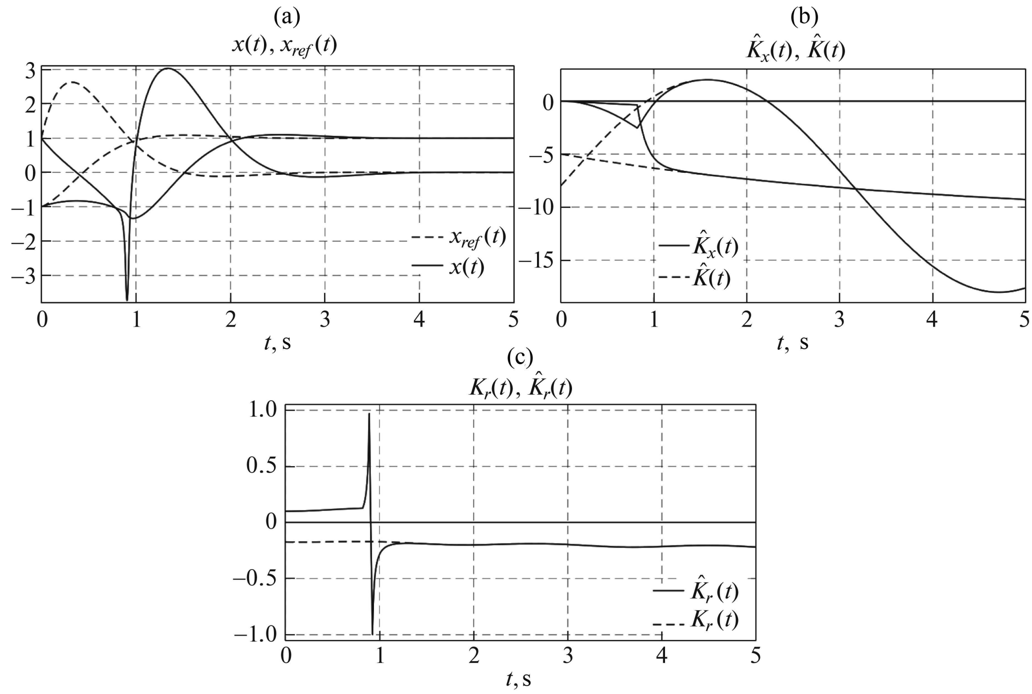

Fig. 6. Comparison of: (a) $x(t)$ and $x_{ref}(t)$, (b) $K_x(t)$ and $\hat{K}_x(t)$, (c) $K_r(t)$ and $\hat{K}_r(t)$.

confirm the implication

$$\operatorname{sgn}\left(V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right) \neq \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)x_{B_0}\right) \Rightarrow \hat{F}(t) \geqslant 0.$$

Figure 6 depicts the comparison of the states of the reference model $x_{ref}(t)$ (when $x_{0ref} = x_0$) and the plant $x(t)$, as well as the parameters $K_x(t)$, $K_r(t)$ and their estimates $\hat{K}_x(t)$, $\hat{K}_r(t)$.

The experimental results confirm the capability of the adjustable control law (3.2.6) to effectively avoid possible division by zero when $\operatorname{sgn}\left(V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right) \neq \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)x_{B_0}\right)$.

Conducted experiments fully confirmed all theoretical conclusions of Theorem 1, Remarks 3 and 4.

5. CONCLUSION

The results of the first paper of this series were extended to the class of linear systems with time-varying unknown parameters described by known nonstationary exosystems with unknown initial conditions.

For this class of systems, the control system was proposed that solved the problem of tracking the trajectories of a time-invariant reference model by a time-varying plant. The control signal was computed using measurable signals and unknown initial conditions of the exosystems that generated the system parameters. To identify such initial conditions, an adaptive law was proposed that ensured exponential stability of the tracking error $e_{ref}(t)$ if the regressor was finitely exciting. The solution did not require to know the sign of the high-frequency gain, but requires known bounds of its absolute value.

The result had a drawback in common with [19, 22], namely it required the condition of the regressor finite excitation to be met to ensure boundedness of the tracking error. In Remark 4, some ways of dealing with this problem for single-input systems were given. For systems with multiple inputs, the problem to ensure tracking error boundedness without knowing the sign of the control allocation matrix is an open one.

The scope of further research could be to extend the results to (a) output-feedback control problems for systems with time-varying parameters, (b) control problems when matching conditions are violated, and (c) systems with multiple inputs.

FUNDING

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APPENDIX

Proof of Proposition 1. The following estimates are defined:

$$\begin{aligned} \hat{\vartheta}(t) &= h_{\vartheta}\Phi_{\vartheta}(t)\hat{x}_{\vartheta_0}(t), \\ \hat{B}(t) &= h_B\Phi_B(t)\hat{x}_{B_0}(t), \\ v(t) &= e_1^{\mathrm{T}}\left(A_{ref} - A_0\right)x(t) + e_1^{\mathrm{T}}B_{ref}r(t). \end{aligned} \tag{A.1}$$

Considering (A.1), the subtraction $u(t) - u^*(t)$ is written as follows (for the sake of brevity, time dependencies are omitted for a while):

$$\begin{aligned} u - u^* &= \frac{\hat{B}^{\mathrm{T}}}{\hat{F}}e_1\left(v - \hat{\vartheta}^{\mathrm{T}}x\right) - \frac{B^{\mathrm{T}}}{F}e_1\left(v - \vartheta^{\mathrm{T}}x\right) \\ \pm\frac{B^{\mathrm{T}}}{F}e_1\left(v - \hat{\vartheta}^{\mathrm{T}}x\right) &= -\frac{B^{\mathrm{T}}e_1}{F}\tilde{\vartheta}^{\mathrm{T}}x + \left(\frac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}} - \frac{B^{\mathrm{T}}e_1}{F}\right)\left(v - \hat{\vartheta}^{\mathrm{T}}x\right). \end{aligned} \tag{A.2}$$

The subtraction $\frac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}} - \frac{B^{\mathrm{T}}e_1}{F}$ is transformed into a linear regression with respect to $\tilde{B}$ and $\tilde{F}$:

$$\begin{aligned} \frac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}} - \frac{B^{\mathrm{T}}e_1}{F} &= \frac{\hat{B}^{\mathrm{T}}e_1F \pm \hat{B}^{\mathrm{T}}e_1\hat{F} - B^{\mathrm{T}}e_1\hat{F}}{\hat{F}F} \\ &= \frac{-\hat{B}^{\mathrm{T}}e_1\left(\hat{F} - F\right) + \left(\hat{B}^{\mathrm{T}} - B^{\mathrm{T}}\right)e_1\hat{F}}{\hat{F}F} \\ &= \frac{-\hat{B}^{\mathrm{T}}e_1\tilde{F} + \tilde{B}^{\mathrm{T}}e_1\hat{F}}{\hat{F}F} = \frac{-\hat{B}^{\mathrm{T}}e_1}{\hat{F}F}\tilde{F} + \tilde{B}^{\mathrm{T}}\frac{e_1}{F}. \end{aligned} \tag{A.3}$$

The error $\tilde{F}$ is considered separately:

$$\begin{aligned}\tilde{F} &= \hat{B}^{\mathrm{T}}\hat{B} - B^{\mathrm{T}}B + B^{\mathrm{T}}\hat{B} - B^{\mathrm{T}}\hat{B}\\ &= \left(\hat{B}^{\mathrm{T}} - B^{\mathrm{T}}\right)\hat{B} + B^{\mathrm{T}}\left(\hat{B} - B\right) = \tilde{B}^{\mathrm{T}}\hat{B} + B^{\mathrm{T}}\tilde{B}.\end{aligned} \tag{A.4}$$

The substitution of (A.4), (A.3) into (A.2) allows one to obtain:

$$\begin{aligned}u - u^{*} &= -\frac{B^{\mathrm{T}}e_1}{F}\tilde{\vartheta}^{\mathrm{T}}x - \left(\frac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}F}\tilde{B}^{\mathrm{T}}\hat{B} + \frac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}F}B^{\mathrm{T}}\tilde{B} - \tilde{B}^{\mathrm{T}}\frac{e_1}{F}\right)\left(v - \hat{\vartheta}^{\mathrm{T}}x\right)\\ &= -\frac{B^{\mathrm{T}}e_1}{F}x^{\mathrm{T}}\tilde{\vartheta} - \left(\frac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}F}\hat{B}^{\mathrm{T}} + \frac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}F}B^{\mathrm{T}} - \frac{e_1^{\mathrm{T}}}{F}\right)\left(v - \hat{\vartheta}^{\mathrm{T}}x\right)\tilde{B}\\ &= -\frac{B^{\mathrm{T}}e_1}{F}x^{\mathrm{T}}h_{\vartheta}\Phi_{\vartheta}\tilde{x}_{\vartheta_0} - \left(\frac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}F}\hat{B}^{\mathrm{T}} + \frac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}F}B^{\mathrm{T}} - \frac{e_1^{\mathrm{T}}}{F}\right)\left(v - \hat{\vartheta}^{\mathrm{T}}x\right)h_B\Phi_B\tilde{x}_{B_0} = \tilde{\theta}^{\mathrm{T}}\omega,\end{aligned} \tag{A.5}$$

where

$$\tilde{\theta} = \begin{bmatrix}\tilde{x}_{\vartheta_0}^{\mathrm{T}} & \tilde{x}_{B_0}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}, \quad \omega = \begin{bmatrix} -\left(\dfrac{B^{\mathrm{T}}e_1}{F}x^{\mathrm{T}}h_{\vartheta}\Phi_{\vartheta}\right)^{\mathrm{T}} \\ -\Phi_B^{\mathrm{T}}h_B^{\mathrm{T}}\left[\left(\dfrac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}F}\hat{B}^{\mathrm{T}} + \dfrac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}F}B^{\mathrm{T}} - \dfrac{e_1^{\mathrm{T}}}{F}\right)\left(v - \hat{\vartheta}^{\mathrm{T}}x\right)\right]^{\mathrm{T}}\end{bmatrix},$$

which completes proof of Proposition 1.

Proof of Proposition 2. The error $\chi(t) = x(t) - \bar{x}(t)$ is introduced. Differentiating $\chi(t)$ with respect to time, it is obtained that:

$$\begin{aligned}\dot{\chi}(t) &= \dot{x}(t) - \dot{\bar{x}}(t)\\ &= A(t)x(t) + B(t)u(t) - A_K\bar{x}(t) + A_Kx(t)\\ &= A_K\left(x(t) - \bar{x}(t)\right) + A(t)x(t) + B(t)u(t)\\ &= A_K\chi(t) + A_0x(t) + e_1x^{\mathrm{T}}(t)\vartheta(t) + B(t)u(t)\\ &= A_K\chi(t) + A_0x(t) + e_1x^{\mathrm{T}}(t)h_{\vartheta}\Phi_{\vartheta}(t)x_{\vartheta_0} + h_B\Phi_B(t)u(t)x_{B_0}.\end{aligned} \tag{A.6}$$

The solution of the differential equation (A.6), which is multiplied by e_1^{T}, takes the following form:

$$\begin{aligned}e_1^{\mathrm{T}}\left[\chi(t) - v(t)\right] &= e_1^{\mathrm{T}}\left[x(t) - \bar{x}(t) - v(t)\right]\\ &= e_1^{\mathrm{T}}e^{A_K\left(t - t_0^{+}\right)}x\left(t_0^{+}\right) + e_1^{\mathrm{T}}\varphi(t)x_{\vartheta_0} + e_1^{\mathrm{T}}\psi(t)x_{B_0}\\ &= e_1^{\mathrm{T}}e^{A_K\left(t - t_0^{+}\right)}x_0 + e_1^{\mathrm{T}}\varphi(t)x_{\vartheta_0} + e_1^{\mathrm{T}}\psi(t)x_{B_0}\\ &= \begin{bmatrix}e_1^{\mathrm{T}}e^{A_K\left(t - t_0^{+}\right)} & e_1^{\mathrm{T}}\varphi(t) & e_1^{\mathrm{T}}\psi(t)\end{bmatrix}\begin{bmatrix}x_0\\ x_{\vartheta_0}\\ x_{B_0}\end{bmatrix} := \Phi^{\mathrm{T}}(t)\eta.\end{aligned} \tag{A.7}$$

Having applied to the regression equation (A.7) the procedures of normalization (3.2.2), dynamic extension (3.2.3a), (3.2.3b) and mixing (3.2.3c), and using the property $\operatorname{adj}\{\Delta(t)\}\Delta(t) = \det\{\Delta(t)\}I_{(n_{\vartheta}+n_B+n)\times(n_{\vartheta}+n_B+n)}$, the measurable regression equation (3.2.4) is obtained.

Proof of the fact that for all $t \geqslant t_e$ the inequality $\Omega_{\rm UB}(t) \geqslant \Omega(t) \geqslant \Omega_{LB} > 0$ holds if $\Phi(t) \in {\rm FE}$ has been obtain in Proposition 4 of [23].

Proof of Theorem 1. Proof of the first part of theorem coincides with the one of the first part of the theorem from [19].

To prove the second part of theorem, the eigenvalue decomposition is applied to the matrix $G(t)$:

$$\begin{gathered}\forall t \geqslant t_0^+ \ F(t) = x_{B_0}^{\rm T} V(t)\Lambda(t)V^{\rm T}(t)x_{B_0} = x_{B_0}^{\rm T} V_1(t)\Lambda_1(t)V_1^{\rm T}(t)x_{B_0} = \Theta^{\rm T}(t)\Lambda_1(t)\Theta(t),\\ V(t) = \begin{bmatrix} V_1(t) & V_2(t)\end{bmatrix}, \quad \Lambda(t) = \begin{bmatrix} \Lambda_1(t) & 0_{r_G(t)\times \bar r_G(t)} \\ 0_{\bar r_G(t)\times r_G(t)} & 0_{\bar r_G(t)}\end{bmatrix},\\ \Lambda_1(t) = {\rm diag}\left\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_{r_G(t)}(t)\right\}, \quad \lambda_{\min}\left(\Lambda_1(t)\right) > 0,\end{gathered}$$

where $V_1(t) \in \mathbb{R}^{n_B \times r_G(t)}$, $V_2(t) \in \mathbb{R}^{n_B \times \bar r_G(t)}$, $\Lambda(t) \in \mathbb{R}^{n_B\times n_B}$, $r_G(t) = {\rm rank}\left\{G(t)\right\}$, $\bar r_G(t) = n_B - r_G(t)$.

Using the above-introduced decomposition, the lower bound of $\hat F(t)$ is written:

$$\begin{gathered}\forall t \geqslant t_0^+ \ \hat F(t) = \hat B^{\rm T}(t)\hat B(t) = \hat x_{B_0}^{\rm T}(t)V_1(t)\Lambda_1(t)V_1^{\rm T}(t)\hat x_{B_0}(t)\\ = \hat\Theta^{\rm T}(t)\Lambda_1(t)\hat\Theta(t) \geqslant \lambda_{\min}\left(\Lambda_1(t)\right)\left\|\hat\Theta(t)\right\|^2 > 0.\end{gathered} \tag{A.8}$$

Based on (A.8), it is necessary and sufficient to satisfy the following inequality to ensure that $\hat F(t) > 0$

$$\begin{gathered}\forall t \geqslant t_0^+ \ \left\|\hat\Theta(t)\right\|^2 = \sum_{i=1}^{r_G(t)} \left(\hat\Theta_i(t)\right)^2 \neq 0,\\ \Updownarrow\\ \forall i \in \overline{1,\ r_G(t)}\ \left|\hat\Theta_i(t)\right| \neq 0,\end{gathered} \tag{A.9}$$

where $\hat\Theta^i(t)$ denotes the $i^{\rm th}$ element of the vector $\hat\Theta(t) \in \mathbb{R}^{r_G(t)}$.

The next aim is to obtain the functional definition of the estimate $\hat\Theta_i(t)$ for all $t \geqslant t_0^+$. For this the differential equation (3.2.5) is solved

$$\forall t \geqslant t_0^+ \ \tilde x_{B_0}(t) = \phi\left(t,\ t_0^+\right)\tilde x_{B_0}\left(t_0^+\right), \tag{A.10}$$

then equation (A.10) is multiplied by $V_1^{\rm T}(t)$ and $\Theta(t)$ is added to both left- and right-hand sides of the obtained multiplication:

$$\begin{gathered}V_1^{\rm T}\tilde x_{B_0}(t) + \Theta(t) = \hat\Theta(t) = \phi\left(t,\ t_0^+\right)V_1^{\rm T}(t)\tilde x_{B_0}\left(t_0^+\right) + \Theta(t),\\ \Updownarrow\\ \hat\Theta_i(t) = \phi\left(t,\ t_0^+\right)\tilde\Theta_i^0(t) + \Theta_i(t),\end{gathered} \tag{A.11}$$

where $\phi\left(t,\ t_0^+\right) = e^{-\beta_{\max}\int\limits_{t_0^+}^{t}\begin{cases}0, \text{ if } t<t_e,\\ \gamma_0\lambda_{\max}\left(\hat\omega(\tau)\hat\omega^{\rm T}(\tau)\right)+\gamma_1 \text{ otherwise}\end{cases}d\tau}$, $\tilde\Theta_i^0(t) = \hat\Theta_i^0(t) - \Theta_i(t)$, $\hat\Theta_i^0(t)$ is the $i^{\rm th}$ element of the vector $V_1^{\rm T}\hat x_{B_0}\left(t_0^+\right)$.

Then (A.9) is met, if it holds that

$$\begin{gathered}\hat{\Theta}_i(t)=\phi\left(t,\ t_0^+\right)\tilde{\Theta}_i^0(t)+\Theta_i(t)\neq 0\Rightarrow\phi\left(t,\ t_0^+\right)\tilde{\Theta}_i^0(t)\neq-\Theta_i(t)\\ \Rightarrow\underbrace{\operatorname{sgn}\left(\phi\left(t,\ t_0^+\right)\right)}_{=1}\neq\operatorname{sgn}\left(\frac{-\Theta_i(t)}{\tilde{\Theta}_i^0(t)}\right)=\operatorname{sgn}\left(\frac{-\Theta_i(t)}{\hat{\Theta}_i^0(t)-\Theta_i(t)}\right)\\ \Rightarrow\operatorname{sgn}\left(\hat{\Theta}_i^0(t)-\Theta_i(t)\right)\neq-\operatorname{sgn}\left(\Theta_i(t)\right)\\ \Rightarrow\left\{\begin{array}{l}\operatorname{sgn}\left(\hat{\Theta}_i^0(t)\right)=\operatorname{sgn}\left(\Theta_i(t)\right)\\ \left|\hat{\Theta}_i^0(t)\right|>\left|\Theta_i(t)\right|\end{array}\right\}\\ \Rightarrow\left\{\begin{array}{l}\operatorname{sgn}\left(V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right)=\operatorname{sgn}\left(V_1^{\mathrm{T}}(t)x_{B_0}\right)\\ \left|V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right|>\left|V_1^{\mathrm{T}}(t)x_{B_0}\right|\end{array}\right\},\end{gathered}\tag{A.12}$$

which completes proof of the second part of theorem.

The next step is to prove the third part of theorem. According to the Kalman–Yakubovich–Popov lemma, for the pair $(A_{ref}, I_{n\times n})$ and any constant matrix $D>0$ one can find matrices $Q\in\mathbb{R}^{n\times n}$, $K\in\mathbb{R}^{n\times n}$ and a constant $\mu>0$ such that there exists a solution of the following set of equations

$$\begin{gathered}A_{ref}^{\mathrm{T}}P+PA_{ref}=-QQ^{\mathrm{T}}-\mu P,\quad PI_{n\times n}=QK,\\ K^{\mathrm{T}}K=D+D^{\mathrm{T}},\end{gathered}\tag{A.13}$$

or a solution of the following Riccati equation, which is an equivalent of the above-given set of equations in a particular case $D=0.5k^2I_{n\times n}$, $K=k^2I_{n\times n}$, $k=1$:

$$A_{ref}^{\mathrm{T}}P+PA_{ref}+PP^{\mathrm{T}}+\mu P=0.\tag{A.14}$$

The following quadratic form is introduced to analyze the stability:

$$\begin{gathered}V=\xi^{\mathrm{T}}H\xi=\gamma_0e_{ref}^{\mathrm{T}}Pe_{ref}+\frac{\beta_{\max}^2}{2}\tilde{\theta}^{\mathrm{T}}\tilde{\theta},\\ H=\operatorname{blockdiag}\left\{\gamma_0P,\quad\frac{\beta_{\max}^2}{2}I_{(n_\vartheta+n_B)\times(n_\vartheta+n_B)}\right\},\\ \underbrace{\lambda_{\min}(H)}_{\lambda_m}\|\xi\|^2\leqslant V(\|\xi\|)\leqslant\underbrace{\lambda_{\max}(H)}_{\lambda_M}\|\xi\|^2,\end{gathered}\tag{A.15}$$

where the matrix P is a solution of the set (A.13) when $K=k^2I_{n\times n}$, $D=0.5k^2I_{n\times n}$, $k=1$ or equivalent Riccati equation (A.14).

Owing to equations (3.1.6) and (3.2.5), the derivative of the quadratic form (A.15) is written as

$$\begin{aligned}\dot{V}&=\gamma_0\left[e_{ref}^{\mathrm{T}}\left(A_{ref}^{\mathrm{T}}P+PA_{ref}\right)e_{ref}+2\tilde{\theta}^{\mathrm{T}}\omega e_{ref}^{\mathrm{T}}PB\right]-\beta_{\max}^2\tilde{\theta}^{\mathrm{T}}\gamma\Omega^2\tilde{\theta}\\ &=\gamma_0\left[-\mu e_{ref}^{\mathrm{T}}Pe_{ref}-e_{ref}^{\mathrm{T}}QQ^{\mathrm{T}}e_{ref}+2\tilde{\theta}^{\mathrm{T}}\omega e_{ref}^{\mathrm{T}}PI_{n\times n}B\right]-\beta_{\max}^2\tilde{\theta}^{\mathrm{T}}\gamma\Omega^2\tilde{\theta}\\ &=\gamma_0\left[-\mu e_{ref}^{\mathrm{T}}Pe_{ref}-e_{ref}^{\mathrm{T}}QQ^{\mathrm{T}}e_{ref}+2\tilde{\theta}^{\mathrm{T}}\omega B^{\mathrm{T}}Q^{\mathrm{T}}e_{ref}\right]-\beta_{\max}^2\tilde{\theta}^{\mathrm{T}}\gamma\Omega^2\tilde{\theta}.\end{aligned}\tag{A.16}$$

Completing the square in (A.16), it is obtained:

$$\begin{aligned}\dot{V} &= \gamma_0\Big[-\mu e_{ref}^{\mathrm{T}}Pe_{ref} - e_{ref}^{\mathrm{T}}QQ^{\mathrm{T}}e_{ref} + 2e_{ref}^{\mathrm{T}}QB\omega^{\mathrm{T}}\tilde{\theta} \pm 2\tilde{\theta}^{\mathrm{T}}\omega B^{\mathrm{T}}B\omega^{\mathrm{T}}\tilde{\theta}\Big] - \beta_{\max}^2\tilde{\theta}^{\mathrm{T}}\gamma\Omega^2\tilde{\theta} \\ &= \gamma_0\left[-\mu e_{ref}^{\mathrm{T}}Pe_{ref} - \left(e_{ref}^{\mathrm{T}}Q - B\omega^{\mathrm{T}}\tilde{\theta}\right)^2 + \tilde{\theta}^{\mathrm{T}}\omega B^{\mathrm{T}}B\omega^{\mathrm{T}}\tilde{\theta}\right] - \beta_{\max}^2\tilde{\theta}^{\mathrm{T}}\gamma\Omega^2\tilde{\theta} \\ &\leqslant \gamma_0\left[-\mu e_{ref}^{\mathrm{T}}Pe_{ref} + \tilde{\theta}^{\mathrm{T}}\omega F\omega^{\mathrm{T}}\tilde{\theta}\right] - \beta_{\max}^2\tilde{\theta}^{\mathrm{T}}\gamma\Omega^2\tilde{\theta}.\end{aligned} \tag{A.17}$$

Two situations need to be considered: $t < t_e$ and $t \geqslant t_e$. As for the first one, according to proposition 1, in the most conservative case it holds that $\Omega(t) = 0$ and $\left\|\tilde{\theta}(t)\right\| = \left\|\tilde{\theta}\left(t_0^+\right)\right\|$.

Then for all $t < t_e$ equation (A.17) is rewritten as

$$\begin{aligned}\dot{V} &\leqslant -\mu\gamma_0 e_{ref}^{\mathrm{T}}Pe_{ref} + \gamma_0\tilde{\theta}^{\mathrm{T}}\left(t_0^+\right)\omega F\omega^{\mathrm{T}}\tilde{\theta}\left(t_0^+\right) \pm \beta_{\max}^2\tilde{\theta}^{\mathrm{T}}\tilde{\theta} \\ &\leqslant -\mu\gamma_0 e_{ref}^{\mathrm{T}}Pe_{ref} - \beta_{\max}^2\tilde{\theta}^{\mathrm{T}}\tilde{\theta} + \gamma_0\tilde{\theta}^{\mathrm{T}}\left(t_0^+\right)\omega F\omega^{\mathrm{T}}\tilde{\theta}\left(t_0^+\right) + \beta_{\max}^2\tilde{\theta}^{\mathrm{T}}\left(t_0^+\right)\tilde{\theta}\left(t_0^+\right).\end{aligned} \tag{A.18}$$

The notion of the maximum eigenvalue of the matrix $\omega(t)B^{\mathrm{T}}(t)B(t)\omega^{\mathrm{T}}(t)$ over the time range $[0;\, t_e)$ is introduced:

$$\delta = \sup\max_{\forall t < t_e}\lambda_{\max}\left(\omega(t)F(t)\omega^{\mathrm{T}}(t)\right). \tag{A.19}$$

The function $F(t)$ is bounded according to Assumption 2, the rate of the regressor $\omega(t)$ change is no grater than exponential when Assumption 1 is met, therefore, it holds that $\delta \in L_\infty$.

Considering (A.19), equation (A.18) is rewritten as follows for $t < t_e$

$$\dot{V} \leqslant -\mu\gamma_0\lambda_{\min}(P)\left\|e_{ref}\right\|^2 - \beta_{\max}^2\left\|\tilde{\theta}\right\|^2 + \left(\gamma_0\delta + \beta_{\max}^2\right)\left\|\tilde{\theta}\left(t_0^+\right)\right\|^2 \leqslant -\eta_1 V + r_B, \tag{A.20}$$

where $\eta_1 = \min\left\{\frac{\mu\lambda_{\min}(P)}{\lambda_{\max}(P)};\, 2\right\}$, $r_B = \left(\gamma_0\delta + \beta_{\max}^2\right)\left\|\tilde{\theta}\left(t_0^+\right)\right\|^2$.

Having solved the differential equation (A.20), we have:

$$\forall t < t_e\colon V(t) \leqslant e^{-\eta_1\left(t - t_0^+\right)}V\left(t_0^+\right) + \frac{r_B}{\eta_1}. \tag{A.21}$$

Taking into consideration $\lambda_m\|\xi(t)\|^2 \leqslant V(t)$ and $V\left(t_0^+\right) \leqslant \lambda_M\left\|\xi\left(t_0^+\right)\right\|^2$, the following upper bound of the augmented tracking error is obtained for all $t < t_e$ from (A.21):

$$\|\xi(t)\| \leqslant \sqrt{\frac{\lambda_M}{\lambda_m}e^{-\eta_1\left(t - t_0^+\right)}\left\|\xi\left(t_0^+\right)\right\|^2 + \frac{r_B}{\lambda_m\eta_1}} \leqslant \sqrt{\frac{\lambda_M}{\lambda_m}\left\|\xi\left(t_0^+\right)\right\|^2 + \frac{r_B}{\lambda_m\eta_1}}, \tag{A.22}$$

from which it follows that $\xi(t)$ is bounded for all $t < t_e$.

As for the second situation, considering the definition of the adaptive gain γ and the fact that, owing to Proposition 1, for all $t \geqslant t_e$ the inequality $0 < \Omega_{\mathrm{LB}} \leqslant \Omega(t) \leqslant \Omega_{\mathrm{UB}}$ holds, it is obtained from (A.18) for $t \geqslant t_e$ that:

$$\begin{aligned}\dot{V} &\leqslant -\mu\gamma_0 e_{ref}^{\mathrm{T}}Pe_{ref} + \gamma_0\tilde{\theta}^{\mathrm{T}}\omega F\omega^{\mathrm{T}}\tilde{\theta} - \beta_{\max}^2\tilde{\theta}^{\mathrm{T}}\frac{\left(\gamma_0\lambda_{\max}\left(\hat{\omega}\hat{\omega}^{\mathrm{T}}\right) + \gamma_1\right)\Omega^2}{\Omega^2}\tilde{\theta} \\ &= -\mu\gamma_0 e_{ref}^{\mathrm{T}}Pe_{ref} + \gamma_0\tilde{\theta}^{\mathrm{T}}\omega F\omega^{\mathrm{T}}\tilde{\theta} - \beta_{\max}^2\tilde{\theta}^{\mathrm{T}}\left[\gamma_0\lambda_{\max}\left(\hat{\omega}\hat{\omega}^{\mathrm{T}}\right) + \gamma_1\right]\tilde{\theta}.\end{aligned} \tag{A.23}$$

The regressor $\hat{\omega}(t)$ is defined as follows:

$$\hat{\omega}(t)=\begin{bmatrix} -\left(\frac{\beta_{\max}}{\beta_{\min}^2}x^{\mathrm{T}}h_{\vartheta}\Phi_{\vartheta}\right)^{\mathrm{T}} \\ -\Phi_B^{\mathrm{T}}h_B^{\mathrm{T}}\left[\left(\frac{\hat{B}^{\mathrm{T}}e_1}{\beta_{\min}^2\hat{F}e_1^{\mathrm{T}}e_1}\hat{B}^{\mathrm{T}}+\frac{\hat{B}^{\mathrm{T}}e_1}{\beta_{\min}^2\hat{F}e_1^{\mathrm{T}}e_1}\beta_{\max}e_1^{\mathrm{T}}-\frac{e_1^{\mathrm{T}}}{\beta_{\min}^2e_1^{\mathrm{T}}e_1}\right)\left(v-\hat{\vartheta}^{\mathrm{T}}x\right)\right]^{\mathrm{T}}\end{bmatrix}.$$

It also holds for any $\omega(t)$ that

$$\begin{gathered}\gamma_0\tilde{\theta}^{\mathrm{T}}\omega F\omega^{\mathrm{T}}\tilde{\theta}-\tilde{\theta}^{\mathrm{T}}\gamma_0\beta_{\max}^2\lambda_{\max}\left(\hat{\omega}\hat{\omega}^{\mathrm{T}}\right)\tilde{\theta} \\ =\tilde{\theta}^{\mathrm{T}}\underbrace{\left(\gamma_0\omega F\omega^{\mathrm{T}}-\gamma_0\beta_{\max}^2\lambda_{\max}\left(\hat{\omega}\hat{\omega}^{\mathrm{T}}\right)I_{(n_\vartheta+n_B)\times(n_\vartheta+n_B)}\right)}_{\leqslant-\kappa I_{(n_\vartheta+n_B)\times(n_\vartheta+n_B)}}\tilde{\theta}\leqslant 0,\end{gathered}\tag{A.24}$$

so equation(A.23) is rewritten as

$$\begin{gathered}\dot{V}\leqslant-\gamma_0\mu e_{ref}^{\mathrm{T}}Pe_{ref}-\tilde{\theta}^{\mathrm{T}}\left(\kappa+\beta_{\max}^2\gamma_1\right)\tilde{\theta} \\ \leqslant-\mu\gamma_0\lambda_{\min}(P)\left\|e_{ref}\right\|^2-\left(\kappa+\beta_{\max}^2\gamma_1\right)\left\|\tilde{\theta}\right\|^2\leqslant-\eta_2V,\end{gathered}\tag{A.25}$$

where $\eta_2=\min\left\{\frac{\mu\lambda_{\min}(P)}{\lambda_{\max}(P)};\,2\left(\frac{\kappa}{\beta_{\max}^2}+\gamma_1\right)\right\}$.

Having solved the inequality (A.25), it is obtained that $V(t)\leqslant e^{-\eta_2(t-t_e)}V(t_e)$ for $t\geqslant t_e$.

Taking into account $\lambda_m\|\xi(t)\|^2\leqslant V(t)$, $V(t_e)\leqslant\lambda_M\|\xi(t_e)\|^2$ and equation (A.22), the upper bound of the augmented tracking error is obtained for $t\geqslant t_e$:

$$\|\xi(t)\|\leqslant\sqrt{\frac{\lambda_M}{\lambda_m}e^{-\eta_2(t-t_e)}\|\xi(t_e)\|^2}\leqslant\sqrt{\frac{\lambda_M}{\lambda_m}\left(\frac{\lambda_M}{\lambda_m}\left\|\xi\left(t_0^+\right)\right\|^2+\frac{r_B}{\lambda_m\eta_1}\right)},\tag{A.26}$$

from which together with (A.22) it follows that $\xi(t)\in L_\infty$, as well as exponential convergence of the error $\xi(t)$ to zero for all $t\geqslant t_e$ with the rate, which is directly proportional to the parameters γ_0, γ_1, Q.E.D.

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