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Adaptive Observer of State and Disturbances for Linear Overparameterized Systems

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Abstract—The problem of state reconstruction is considered for a class of linear systems with time-invariant unknown parameters and overparameterization that are affected by external perturbations generated by a known exosystem with unknown initial conditions. An extended adaptive observer is proposed, which, in contrast to existing approaches, solves state and perturbation adaptive estimation problems for systems that are not represented in the observer canonical form. The obtained theoretical results are validated via mathematical modeling.

Keywords: estimation, identification, adaptive observer, persistent excitation, convergence, overparameterization

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1. INTRODUCTION

One of the problems of automatic control theory is the reconstruction of unmeasured state of completely observable linear systems:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= C^{\mathrm{T}}x(t) \end{aligned} \tag{1.1}$$

with unknown matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n}$, $C \in \mathbb{R}^{n}$.

To solve it, various observers based on the invariant ellipsoid technique [1], high-gain methods [2, 3], sliding mode approach [3, 4], and parametric identification theory [5, 6] have been proposed.

In contrast to other approaches, observers based on the methods of identification theory [5, 6] use parameter adaptation algorithms and, therefore, usually require less *a priori* information about the system parameters. However, since the baseline solutions [7–10], the class of systems, for which the adaptive observers can be designed, is traditionally restricted to models in the observer canonical form:

$$\begin{aligned} \dot{\xi}(t) &= A_0\xi(t) + \psi_a y(t) + \psi_b u(t) = A_a\xi(t) + \psi_b u(t), \\ y(t) &= C_0^{\mathrm{T}}\xi(t), \end{aligned} \tag{1.2}$$

$$A_0 = \begin{bmatrix} 0_n & \begin{matrix} I_{n-1} \\ 0_{n-1}^{\mathrm{T}} \end{matrix} \end{bmatrix}, \quad A_a = \begin{bmatrix} \psi_a & \begin{matrix} I_{n-1} \\ 0_{n-1}^{\mathrm{T}} \end{matrix} \end{bmatrix},$$

$$\psi_a = \begin{bmatrix} -a_{n-1} \\ -a_{n-2} \\ \vdots \\ -a_0 \end{bmatrix}, \quad \psi_b = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_0 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where ψ_a and ψ_b are parameters of the characteristic polynomials of the following linear operator

$$W_{uy}(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0},$$

and they are related to the matrices of the model (1.1) via a transformation matrix T:

$$\begin{gathered}\psi_a = TAT^{-1}C_0, \;\; \psi_b = TB, \;\; C_0^{\mathrm{T}} = C^{\mathrm{T}}T^{-1}, \\ \mathcal{O}_n = \mathcal{O}\Big[0_{1\times(n-1)} \;\; 1\Big]^{\mathrm{T}}, \\ \mathcal{O}^{-1} = \Big[C \;\; A^{\mathrm{T}}C \;\; \cdots \;\; (A^{n-1})^{\mathrm{T}}C\Big]^{\mathrm{T}}, \;\; T^{-1} = \Big[A^{n-1}\mathcal{O}_n \;\; A^{n-2}\mathcal{O}_n \;\; \cdots \;\; \mathcal{O}_n\Big].\end{gathered} \tag{1.3}$$

The point is that the measurable control $u(t)$ and output $y(t)$ signals allow one to uniquely identify the parameters of such canonical state space form only [5, p. 269]. The states $\xi(t) \in \mathbb{R}^n$ of the model (1.2) are virtual and related to the plant physical states $x(t) \in \mathbb{R}^n$ via a non-singular transformation $\xi(t) = Tx(t)$.

Therefore, the estimates $\hat{\xi}(t)$ obtained by classical adaptive state observers [5, 6] of the following form ($\hat{\psi}_a(t), \hat{\psi}_b(t)$ are estimates of the parameters (1.3), L stands for the correction matrix, and the specific structures of functions $f_a(.), f_b(.), f_v(.)$ are defined in [5, 6]):

$$\begin{gathered}\dot{\hat{\xi}}(t) = A_0\hat{\xi}(t) + \hat{\psi}_a(t)y(t) + \hat{\psi}_b(t)u(t) + L\left(\hat{y}(t) - y(t)\right) + v(t), \\ \hat{y}(t) = {C_0}^{\mathrm{T}}\hat{\xi}(t), \\ \dot{\hat{\psi}}_a(t) = f_a\left(u, y, \hat{y}, \hat{\psi}_a\right), \\ \dot{\hat{\psi}}_b(t) = f_b\left(u, y, \hat{y}, \hat{\psi}_b\right), \\ \mathrm{Re}\left\{\lambda_i\left(A_0 + LC^{\mathrm{T}}\right)\right\} < 0, \\ v(t) = f_v\left(u, y, \dot{\hat{\psi}}_a, \dot{\hat{\psi}}_b\right) \text{ or } v(t) = 0_n,\end{gathered} \tag{1.4}$$

not only do not coincide with $x(t)$, but also turn out to be useless, for example, to solve the problems of failure diagnostics, monitoring and storage of unmeasured variables of technological processes, design and online adjustment of digital twins and other practical scenarios.

The solution to this problem is to identify the linear transformation matrix T together with the parameters ψ_a and ψ_b. For one specific class of linear systems, an algorithm is proposed in [7] that forms an estimate of $\hat{T}(t)$ on the basis of the ones of $\hat{\psi}_a(t)$ and $\hat{\psi}_b(t)$. In the general case, the mapping $\hat{T}(t) = f_{\mathrm{T}}\left(\hat{\psi}_a(t), \; \hat{\psi}_b(t)\right)$ can be singular for certain values of the estimates $\hat{\psi}_a(t)$, $\hat{\psi}_b(t)$ (see Section VIII of [7]). In more recent papers [10–12] devoted to the development of methods to design adaptive observers (and even in the fundamental books on adaptive observers for linear systems [5, 6]), to the best of the authors' knowledge, the problem of the physical state $x(t)$ reconstruction and the estimation of the linear transformation matrix T with the help of adaptive observers was no longer touched upon.

In a recent paper [13] a new approach of adaptive reconstruction of the linear system physical state is presented instead of identification of the linear transformation matrix. It is proposed to overparameterize the matrices of the system (1.1) with respect to some physical parameters $\theta \in \mathbb{R}^{n_\theta}$ (such overparameterization is always possible if the model is obtained directly on the basis of the laws of mathematical physics—Kirchhoff, Euler–Lagrange, etc.):

$$\begin{gathered}\dot{x}(t) = A(\theta)x(t) + B(\theta)u(t) = \Phi^{\mathrm{T}}(x, \; u)\,\Theta_{AB}(\theta), \\ y(t) = C^{\mathrm{T}}x(t),\end{gathered} \tag{1.5}$$

and, using the following change of notation $\psi_a := \psi_a(\theta)$, $\psi_b := \psi_b(\theta)$, to take into account the dependence of the model (1.1) parameters from θ.

The fact that the overparameterization is considered allows one to link the matrices of the models (1.1) and (1.2) not via the above-mentioned transformation, but by means of some new functional transformations of the following form ($\theta = \mathcal{F}(\psi_{ab})$ is an inverse function, $\mathcal{L}_{ab} \in \mathbb{R}^{n_\theta \times 2n}$ stands for a matrix that defines some linear transformation, which ensures $\dim\{\psi_{ab}\} = \dim\{\theta\}$)

$$\Theta_{AB}(\theta) = (\Theta_{AB} \circ \mathcal{F})(\psi_{ab}),$$
$$\psi_{ab}(\theta) = \mathcal{L}_{ab}\begin{bmatrix}\psi_a(\theta)\\ \psi_b(\theta)\end{bmatrix} : \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_\theta},$$

which provides much room to design the adaptive observers of physical states $x(t)$.

In [13] it is shown that, if the condition

$$\det^2\{\nabla_\theta \psi_{ab}(\theta)\} > 0, \quad \psi_{ab}(\theta) = \mathcal{L}_{ab}\begin{bmatrix}\psi_a(\theta)\\ \psi_b(\theta)\end{bmatrix} : \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_\theta} \tag{1.6}$$

of existence of the inverse function $\mathcal{F} : \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_\theta}$ holds and $\psi_{ab}(\theta)$ and $\Theta_{AB}(\theta)$ depend from θ in a polynomial fashion, then, using only the measurable signals $y(t), u(t)$ and known vector C, the following regression equations can be obtained without identification of the parameters $\psi_{ab}(\theta)$ and θ (where $\mathcal{Y}_{AB}(t)$, $\mathcal{Y}_L(t)$, $\mathcal{M}_{AB}(t)$, $\mathcal{M}_L(t)$ stand for the measurable signals):

$$\mathcal{Y}_{AB}(t) = \mathcal{M}_{AB}(t)\Theta_{AB}(\theta),$$
$$\mathcal{Y}_L(t) = \mathcal{M}_L(t)L(\theta)$$

and, as a consequence, an adaptive observer of the system (1.5) state can be implemented in the following form:

$$\begin{aligned}\dot{\hat{x}}(t) &= \Phi^{\mathrm{T}}(\hat{x}, u)\,\hat{\Theta}_{AB}(t) - \hat{L}(t)\,(\hat{y}(t) - y(t)),\\ \dot{\hat{\Theta}}_{AB}(t) &= f_{\Theta_{AB}}\left(\mathcal{Y}_{AB}, \mathcal{M}_{AB}, \hat{\Theta}_{AB}\right),\\ \dot{\hat{L}}(t) &= f_L\left(\mathcal{Y}_L, \mathcal{M}_L, \hat{L}\right),\end{aligned} \tag{1.7}$$

where $\hat{L}(t)$ is an estimate of the matrix $L(\theta)$ such that $A(\theta) - L(\theta)C^{\mathrm{T}}$ is a Hurwitz one.

In other words, owing to the fact that $\Theta_{AB}(\theta)$ and $\psi_{ab}(\theta)$ are related to each other via the physical parameters θ, if the condition (1.6) is met, then, in accordance with [13], $\Theta_{AB}(\theta)$ and $L(\theta)$ can be identified without direct estimation of θ or $\psi_{ab}(\theta)$. In contrast to (1.4), the (1.7) observer allows one to obtain estimates of the physical state $x(t)$, and, contrary to [7], it is applicable to a wider class of systems and does not require direct identification of parameters $\psi_{ab}(\theta)$.

The aim of this paper is to extend the results of [13] to the class of linear systems with overparameterization that are affected by the external perturbations generated by a known exosystem with unknown initial conditions.

Main definitions

The definition of a heterogeneous mapping, the regressor persistent excitation condition, and the property of the Kreisselmeyer filtering [10] given below will be used throughout this paper[1].

[1] For the sake of brevity, hereafter the arguments θ and t will be omitted except the cases in which it is necessary for understanding.

Definition 1. A mapping $\mathcal{F}: \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_\mathcal{F} \times m_\mathcal{F}}$ is heterogeneous of degree $\ell_\mathcal{F} \geqslant 1$, if there exists $\Pi_\mathcal{F}(\omega) \in \mathbb{R}^{n_\mathcal{F} \times n_\mathcal{F}}$, $\Xi_\mathcal{F}(\omega) = \overline{\Xi}_\mathcal{F}(\omega)\omega(t) \in \mathbb{R}^{\Delta_\mathcal{F} \times n_\theta}$, and mapping $\mathcal{T}_\mathcal{F}: \mathbb{R}^{\Delta_\mathcal{F}} \to \mathbb{R}^{n_\mathcal{F} \times m_\mathcal{F}}$ such that for all $\omega(t) \in \mathbb{R}$ and $\theta \in \mathbb{R}^{n_\theta}$ the following functional equation has a solution

$$\Pi_\mathcal{F}(\omega)\mathcal{F}(\theta) = \mathcal{T}_\mathcal{F}(\Xi_\mathcal{F}(\omega)\theta), \tag{1.8}$$

where

$$\begin{gathered} \det\{\Pi_\mathcal{F}(\omega)\} \geqslant \omega^{\ell_\mathcal{F}}(t), \\ \Xi_{\mathcal{F}ij}(\omega) = c_{ij}\omega^{\ell_{ij}}(t), \;\; \overline{\Xi}_{\mathcal{F}ij}(\omega) = c_{ij}\omega^{\ell_{ij}-1}(t), \\ c_{ij} \in \{0,1\}, \;\; \ell_{ij} \geqslant 1. \end{gathered}$$

For example, $\mathcal{F}(\theta) = \operatorname{col}\{\theta_1\theta_2, \theta_1\}$ with $\Pi_\mathcal{F}(\omega) = \operatorname{diag}\{\omega^2, \omega\}$, $\Xi_\mathcal{F}(\omega) = \operatorname{diag}\{\omega, \omega\}$ is heterogeneous of degree $\ell_\mathcal{F} = 3$.

Using a known function $\mathcal{Y}_\theta(t) = \omega(t)\theta$, the main property $\Xi_\mathcal{F}(\omega)\theta = \overline{\Xi}_\mathcal{F}(\omega)\omega(t)\theta$ from Definition 1 allows one to obtain a linear regression equation with respect to $\mathcal{F}(\theta)$ in the following way:

$$\begin{gathered} \Pi_\mathcal{F}(\omega)\mathcal{F}(\theta) = \mathcal{T}_\mathcal{F}\left(\overline{\Xi}_\mathcal{F}(\omega)\mathcal{Y}_\theta\right), \\ \begin{bmatrix} \omega^2(t) & 0 \\ 0 & \omega(t) \end{bmatrix} \mathcal{F}(\theta) = \begin{bmatrix} \mathcal{Y}_{1\theta}(t)\mathcal{Y}_{2\theta}(t) \\ \mathcal{Y}_{1\theta}(t) \end{bmatrix}. \end{gathered}$$

The elements of a mapping $\mathcal{F}(\theta)$ satisfy Definition 1 if they are polynomials or monomials of θ, as well as some of the irrational functions.

Definition 2. A regressor $\overline{\varphi}(t) \in \mathbb{R}^n$ is persistently exciting ($\overline{\varphi}(t) \in \mathrm{PE}$) if $\exists T > 0$ and $\alpha > 0$ such that $\forall t \geqslant t_0 \geqslant 0$ the following inequality holds

$$\int_t^{t+T} \overline{\varphi}(\tau)\overline{\varphi}^{\mathrm{T}}(\tau)\, d\tau \geqslant \alpha I_n, \tag{1.9}$$

where $\alpha > 0$ is an excitation level, I_n stands for an identity matrix.

For a determinant of state of a stable ($l > 0$) dynamical filter

$$\dot{\varphi}(t) = -l\varphi(t) + \overline{\varphi}(t)\overline{\varphi}^{\mathrm{T}}(t), \quad \varphi(t_0) = 0_{n\times n}$$

it holds that

Proposition 1. (a) *If $\overline{\varphi}(t) \in \mathrm{PE}$, then for all $t \geqslant t_0 + T$ the following inequality holds*

$$\Delta(t) = \det\{\varphi(t)\} \geqslant \alpha^n e^{-nlT} = \Delta_{\min} > 0. \tag{1.10}$$

(b) *If there exists $t_e \in [t_0, \infty)$ such that for all $t \geqslant t_e$ equation* (1.10) *holds, then $\overline{\varphi}(t) \in \mathrm{PE}$.*

Proof of Proposition 1 is given in [14].

2. PROBLEM STATEMENT

We consider the following class of SISO-systems with overparameterization affected by a bounded external perturbation:

$$\begin{gathered} \dot{x}(t) = A(\theta)x(t) + B(\theta)u(t) + D(\theta)\delta(t) = \Phi^{\mathrm{T}}(x, u, \delta)\Theta_{AB}(\theta), \\ y(t) = C^{\mathrm{T}}x(t), \;\; x(t_0) = x_0, \end{gathered} \tag{2.1}$$

where

$$\Phi^{\mathrm{T}}(x,u,\delta)=\left[I_n\otimes x^{\mathrm{T}}(t)\;\;I_n\otimes u^{\mathrm{T}}(t)\;\;I_n\otimes \delta^{\mathrm{T}}(t)\right]\mathcal{D}_{\Phi}\in\mathbb{R}^{n\times n_{\Theta}},$$
$$\Theta_{AB}(\theta)=\mathcal{L}_{\Phi}\left[vec^{\mathrm{T}}\left(A^{\mathrm{T}}(\theta)\right)\;\;B^{\mathrm{T}}(\theta)\;\;D^{\mathrm{T}}(\theta)\right]^{\mathrm{T}}\in\mathbb{R}^{n_{\Theta}},$$

$x(t)\in\mathbb{R}^n$ are physical states of the system with unknown initial conditions x_0, $\delta(t)$ stands for a bounded external perturbation, $\Theta_{AB}\in\mathbb{R}^{n_\Theta}$, $\theta\in\mathbb{R}^{n_\theta}$ denote unknown vectors such that $n_\Theta\geqslant n_\theta$, $\mathcal{D}_\Phi\in\mathbb{R}^{(n^2+2n)\times n_\Theta}$, $\mathcal{L}_\Phi\in\mathbb{R}^{n_\Theta\times(n^2+2n)}$ are unknown matrices, the vector $C\in\mathbb{R}^n$ and mapping $\Theta_{AB}:\mathbb{R}^{n_\theta}\to\mathbb{R}^{n_\Theta}$ are known. Only the control $u(t)\in\mathbb{R}$ and output $y(t)\in\mathbb{R}$ signals are measurable.

The following assumptions are adopted for the control and disturbance signals.

Assumption 1. For all $t\geqslant t_0$ the control signal $u(t)$ ensures existence and boundedness of all trajectories of the system (2.1).

Assumption 2. The disturbance $\delta(t)$ is continuous and generated by a stable exosystem with time-invariant parameters:

$$\begin{gathered}\dot{x}_\delta(t)=\mathcal{A}_\delta x_\delta(t),\;\;x_\delta(t_0)=x_{\delta 0},\\ \delta(t)=h_\delta^{\mathrm{T}}x_\delta(t),\end{gathered}\tag{2.2}$$

where $x_\delta(t)\in\mathbb{R}^{n_\delta}$ are states of the exosystem with unknown initial conditions $x_{\delta0}$, $h_\delta\in\mathbb{R}^{n_\delta}$, $\mathcal{A}_\delta(t)\in\mathbb{R}^{n_\delta\times n_\delta}$ are known vector and matrix, which form an observable pair $\left(h_\delta^{\mathrm{T}},\mathcal{A}_\delta\right)$.

Taking into account the duality of the observation and control problems and following the results of the generalized pole placement theory [15, 16], we adopt an assumption that there exists a vector $L(\theta)\in\mathbb{R}^n$, which transforms an algebraic spectrum $\sigma\{.\}$ of the matrix $A^{\mathrm{T}}(\theta)-CL^{\mathrm{T}}(\theta)$ into a desired one.

Assumption 3. A pair $\left(A^{\mathrm{T}}(\theta),\,C\right)$ is controllable, there exists a known state matrix $\Gamma\in\mathbb{R}^{n\times n}$ of an exosystem

$$\begin{aligned}&\dot{\chi}(t)=\Gamma\chi(t),\\ &v(t)=B^{\mathrm{T}}(\theta)\,\chi(t)\end{aligned}\tag{2.3}$$

such that the pair $\left(B^{\mathrm{T}}(\theta),\,\Gamma\right)$ is observable and $\sigma\{A(\theta)\}\cap\sigma\{\Gamma\}=0$.

If Assumptions 1–3 are met, then the following observer of the state and perturbation can be introduced:

$$\begin{gathered}\dot{\hat{x}}(t)=\Phi^{\mathrm{T}}\left(\hat{x},u,\hat{\delta}\right)\hat{\Theta}_{AB}(t)-\hat{L}(t)\left(\hat{y}(t)-y(t)\right),\\ \hat{\delta}(t)=h_\delta^{\mathrm{T}}\Phi_\delta(t)\hat{x}_{\delta0}(t),\\ \dot{\Phi}_\delta(t)=\mathcal{A}_\delta\Phi_\delta(t),\quad\Phi_\delta(t_0)=I_{n_\delta}.\end{gathered}\tag{2.4}$$

The aim is to augment the observer (2.4) with the estimation laws, which ensure that the following equalities hold

$$\begin{gathered}\lim_{t\to\infty}\|\tilde{x}(t)\|=0\;\;(\exp),\;\lim_{t\to\infty}\left\|\tilde{\delta}(t)\right\|=0\;\;(\exp),\;\lim_{t\to\infty}\|\tilde{\kappa}(t)\|=0\;\;(\exp),\\ \tilde{\kappa}(t)=\left[\tilde{x}_{\delta0}^{\mathrm{T}}(t)\;\;\tilde{\Theta}_{AB}^{\mathrm{T}}(t)\;\;\tilde{L}^{\mathrm{T}}(t)\right]^{\mathrm{T}},\end{gathered}\tag{2.5}$$

where $\tilde{x}(t)=\hat{x}(t)-x(t)$ is the state (2.1) observation error, $\tilde{\delta}(t)=\hat{\delta}(t)-\delta(t)$ stands for the disturbance observation error, $\tilde{\Theta}_{AB}(t)=\hat{\Theta}_{AB}(t)-\Theta_{AB}(\theta)$ denotes the error of the system (2.1) parameters estimation, $\tilde{x}_{\delta0}(t)=\hat{x}_{\delta0}(t)-x_{\delta0}$ is the observation error of the exosystem (2.2) initial conditions, $\tilde{L}(t)=\hat{L}(t)-L(\theta)$ stands for the error of $L(\theta)$ estimation.

Remark 1. Assumptions 1 and 3 are conventional for the adaptive observation [10–12] *and pole placement design* [15, 16] *problems, respectively. Assumption 2 restricts the class of permissible external perturbations.*

3. PREREQUISITES AND PRELIMINARY TRANSFORMATIONS

Before presenting the solution of the problem (2.5), the identifiability of the unknown parameters κ from the measurements of $y(t)$ and $u(t)$ is investigated. For this purpose, using the transformations (1.3), the system (2.1) is represented in the form (1.2):

$$\dot{\xi}(t) = A_0\xi(t) + \psi_a(\theta)\, y(t) + \psi_b(\theta)\, u(t) + \psi_d(\theta)\,\delta(t), \tag{3.1}$$

$$y(t) = C^{\mathrm{T}}x(t) = C_0^{\mathrm{T}}\xi(t), \quad \xi(t_0) = Tx_0 = \xi_0, \tag{3.2}$$

where $\psi_d(\theta) = TD(\theta)$, $\xi(t) \in \mathbb{R}^n$ are unmeasurable virtual state of the observer canonical form, the vector $C_0 \in \mathbb{R}^n$ and mappings ψ_a, ψ_b, ψ_d: $\mathbb{R}^{n_\theta} \to \mathbb{R}^n$ are known.

The following parametrization can be obtained for the unknown parameters $\eta(\theta) = \mathrm{col}\{\psi_a(\theta), \psi_b(\theta)\}$ of equation (3.1) in case Assumptions 1 and 2 are met.

Lemma 1. *The unknown parameters $\eta(\theta)$ satisfy the following linear regression model* [2]

$$\begin{gathered}\mathcal{Y}(t) = \Delta(t)\eta(\theta) + \epsilon(t),\\ \mathcal{Y}(t) = k(t) \times \mathrm{adj}\{\varphi(t)\}\, q(t), \;\; \Delta(t) = k(t) \times \det\{\varphi(t)\},\end{gathered} \tag{3.3}$$

where

$$\begin{gathered}\dot{q}(t) = -k_2 q(t) + \overline{\varphi}_f(t)(\overline{q}(t) - k_1\overline{q}_f(t) - \beta^{\mathrm{T}}(F_f(t) + l y_f(t))),\; q(t_0) = 0_{2n},\\ \dot{\varphi}(t) = -k_2\varphi(t) + \overline{\varphi}_f(t)\overline{\varphi}_f^{\mathrm{T}}(t),\; \varphi(t_0) = 0_{2n\times 2n},\end{gathered} \tag{3.4}$$

$$\begin{gathered}\dot{\overline{q}}_f(t) = -k_1\overline{q}_f(t) + \overline{q}(t),\; \overline{q}_f(t_0) = 0,\\ \dot{\overline{\varphi}}_f(t) = -k_1\overline{\varphi}_f(t) + \overline{\varphi}(t),\; \overline{\varphi}_f(t_0) = 0_{2n},\\ \dot{F}_f(t) = -k_1 F_f(t) + F(t),\; F_f(t_0) = 0_{n_\delta},\\ \dot{y}_f(t) = -k_1 y_f(t) + y(t),\; y_f(t_0) = 0,\end{gathered} \tag{3.5}$$

$$\begin{gathered}\overline{q}(t) = y(t) - C_0^{\mathrm{T}}z(t),\; \overline{\varphi}(t) = \begin{bmatrix}\dot{\Omega}^{\mathrm{T}}(t)C_0 + N^{\mathrm{T}}(t)\beta\\ \dot{P}^{\mathrm{T}}(t)C_0 + H^{\mathrm{T}}(t)\beta\end{bmatrix},\\ \dot{z}(t) = A_K z(t) + Ky(t),\; z(t_0) = 0_n,\\ \dot{\Omega}(t) = A_K\Omega(t) + I_n y(t),\; \Omega(t_0) = 0_{n\times n},\\ \dot{P}(t) = A_K P(t) + I_n u(t),\; P(t_0) = 0_{n\times n},\\ \dot{F}(t) = GF(t) + Gly(t) - lC_0^{\mathrm{T}}\dot{z}(t),\; F(t_0) = 0_{n_\delta},\\ \dot{H}(t) = GH(t) - lC_0^{\mathrm{T}}\dot{P}(t),\; H(t_0) = 0_{n_\delta\times n},\\ \dot{N}(t) = GN(t) - lC_0^{\mathrm{T}}\dot{\Omega}(t),\; N(t_0) = 0_{n_\delta\times n},\end{gathered} \tag{3.6}$$

and, if $\overline{\varphi}(t) \in \mathrm{PE}$, then for all $t \geqslant t_0 + T$ it holds that $\Delta_{\max} \geqslant \Delta(t) \geqslant \Delta_{\min} > 0$.

Here $\epsilon(t)$ is an exponentially decaying term, $k(t) \geqslant k_{\min} > 0$ stands for an amplitude modulator (can be time-varying), $k_1 > 0$, $k_2 > 0$ denote filters constants, $A_K = A_0 - KC_0^{\mathrm{T}}$, G are stable matrices of respective dimension, the vector $l \in \mathbb{R}^{n_\delta}$ is such that the pair $(G,\, l)$ is controllable, and

[2] Without loss of generality, further the exponentially decaying term $\epsilon(t)$ is omitted.

G is chosen considering $\sigma\{\mathcal{A}_\delta\} \cap \sigma\{G\} = 0$, the parameter $\beta \in \mathbb{R}^{n_\delta}$ is a solution of the following set of equations

$$M_\delta \mathcal{A}_\delta - G M_\delta = l \overline{h}_\delta^{\mathrm{T}},\ \overline{h}_\delta^{\mathrm{T}} = h_\delta^{\mathrm{T}} \mathcal{A}_\delta,$$
$$\beta = \overline{h}_\delta^{\mathrm{T}} M_\delta^{-1}.$$

Proof of Lemma 1 is postponed to Appendix.

In the general case, the goal (2.5) cannot be achieved because only the parameters $\psi_a,\ \psi_b$ of the characteristic polynomials of the transfer function $W_{uy}(s) = C^{\mathrm{T}}(sI_n - A(\theta))^{-1} B(\theta)$ are identifiable on the basis of measurable signals $u(t),\ y(t)$ via parameterization (3.3) if $\overline{\varphi}(t) \in \mathrm{PE}$. However, in the case that is important for practical scenarios, according to the problem statement, the parameters $\Theta_{AB},\ \psi_d,\ L$ depend nonlinearly from the physical parameters θ in a known way. In their turn, the parameters $\psi_a,\ \psi_b$ of the characteristic polynomials of the transfer function $W_{uy}(s)$ also depend nonlinearly from θ. Therefore, if the following condition is met

$$\det{}^2\{\nabla_\theta \psi_{ab}(\theta)\} > 0,\ \psi_{ab}(\theta) = \mathcal{L}_{ab}\eta(\theta) \in \mathbb{R}^{n_\theta}, \tag{3.7}$$

then, owing to the inverse function theorem [17], there exists an inverse mapping $\theta = \mathcal{F}(\psi_{ab})$, and therefore, it becomes possible to: *i*) calculate the parameters of the system Θ_{AB} and observer L using ψ_{ab}, *ii*) obtain estimates $\hat{x}_{\delta 0}(t)$ of the initial conditions of the exosystem (2.2), *iii*) implement the adaptive observer (2.4), from which the estimates $\hat{x}(t)$ and $\hat{\delta}(t)$ are obtained.

In this paper, to solve the problem of reconstruction of unmeasurable state $x(t)$ and external perturbation $\delta(t)$ when the condition (3.7) is satisfied, the following hypotheses are additionally adopted with respect to $\psi_{ab}(\theta)$, $\Theta_{AB}(\theta)$, and $\psi_d(\theta)$.

Hypothesis 1. There exist the heterogeneous in the sense of (1.8) mappings $\mathcal{G}\colon \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_\theta \times n_\theta}$, $\mathcal{S}\colon \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_\theta}$ such that:

$$\begin{gathered}\mathcal{S}(\psi_{ab}) = \mathcal{G}(\psi_{ab})\mathcal{F}(\psi_{ab}) = \mathcal{G}(\psi_{ab})\theta,\\ \Pi_\theta(\omega)\mathcal{G}(\psi_{ab}) = \mathcal{T}_{\mathcal{G}}(\Xi_{\mathcal{G}}(\omega)\psi_{ab}),\\ \Pi_\theta(\omega)\mathcal{S}(\psi_{ab}) = \mathcal{T}_{\mathcal{S}}(\Xi_{\mathcal{S}}(\omega)\psi_{ab}),\end{gathered} \tag{3.8}$$

where $\Xi_{\mathcal{G}}(\omega) \in \mathbb{R}^{\Delta_{\mathcal{G}} \times n_\theta}$, $\Xi_{\mathcal{S}}(\omega) \in \mathbb{R}^{\Delta_{\mathcal{S}} \times n_\theta}$, $\det\{\Pi_\theta(\omega)\} \geqslant \omega^{\ell_\theta}(t)$, $\operatorname{rank}\{\mathcal{G}(\psi_{ab})\} = n_\theta$, $\ell_\theta \geqslant 1$, $\mathcal{T}_{\mathcal{G}} : \mathbb{R}^{\Delta_{\mathcal{G}}} \to \mathbb{R}^{n_\theta \times n_\theta}$, $\mathcal{T}_{\mathcal{S}} : \mathbb{R}^{\Delta_{\mathcal{S}}} \to \mathbb{R}^{n_\theta}$ and all mappings are known.

Hypothesis 2. There exist the heterogeneous in the sense of (1.8) mappings $\mathcal{X}\colon \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_\Theta \times n_\Theta}$, $\mathcal{Z}\colon \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_\Theta}$ such that:

$$\begin{gathered}\mathcal{Z}(\theta) = \mathcal{X}(\theta)\Theta_{AB}(\theta),\\ \Pi_\Theta(\omega)\mathcal{X}(\theta) = \mathcal{T}_{\mathcal{X}}(\Xi_{\mathcal{X}}(\omega)\theta),\\ \Pi_\Theta(\omega)\mathcal{Z}(\theta) = \mathcal{T}_{\mathcal{Z}}(\Xi_{\mathcal{Z}}(\omega)\theta),\end{gathered} \tag{3.9}$$

where $\Xi_{\mathcal{X}}(\omega) \in \mathbb{R}^{\Delta_{\mathcal{X}} \times n_\theta}$, $\Xi_{\mathcal{Z}}(\omega) \in \mathbb{R}^{\Delta_{\mathcal{Z}} \times n_\theta}$, $\det\{\Pi_\Theta(\omega)\} \geqslant \omega^{\ell_\Theta}(t)$, $\operatorname{rank}\{\mathcal{X}(\theta)\} = n_\Theta$, $\ell_\Theta \geqslant 1$, $\mathcal{T}_{\mathcal{X}}\colon \mathbb{R}^{\Delta_{\mathcal{X}}} \to \mathbb{R}^{n_\Theta \times n_\Theta}$, $\mathcal{T}_{\mathcal{Z}}\colon \mathbb{R}^{\Delta_{\mathcal{Z}}} \to \mathbb{R}^{n_\Theta}$ and all mappings are known.

Hypothesis 3. There exist the heterogeneous in the sense of (1.8) mappings $\mathcal{W} :\ \mathbb{R}^{n_\theta} \to \mathbb{R}^n$, $\mathcal{R} : \mathbb{R}^{n_\theta} \to \mathbb{R}^{n \times n}$ such that:

$$\begin{gathered}\mathcal{W}(\theta) = \mathcal{R}(\theta)\psi_d(\theta),\\ \Pi_{\psi_d}(\omega)\mathcal{R}(\theta) = \mathcal{T}_{\mathcal{R}}(\Xi_{\mathcal{R}}(\omega)\theta),\\ \Pi_{\psi_d}(\omega)\mathcal{W}(\theta) = \mathcal{T}_{\mathcal{W}}(\Xi_{\mathcal{W}}(\omega)\theta),\end{gathered} \tag{3.10}$$

where $\Xi_{\mathcal{W}}(\omega) \in \mathbb{R}^{\Delta_{\mathcal{W}} \times n_\theta}$, $\Xi_{\mathcal{R}}(\omega) \in \mathbb{R}^{\Delta_{\mathcal{R}} \times n_\theta}$, $\det\{\Pi_{\psi_d}(\omega)\} \geqslant \omega^{\ell_{\psi_d}}(t)$, $\operatorname{rank}\{\mathcal{R}(\theta)\} = n$, $\ell_{\psi_d} \geqslant 1$, $\mathcal{T}_{\mathcal{R}} :\ \mathbb{R}^{\Delta_{\mathcal{R}}} \to \mathbb{R}^{n \times n}$, $\mathcal{T}_{\mathcal{W}} :\ \mathbb{R}^{\Delta_{\mathcal{W}}} \to \mathbb{R}^n$ and all mappings are known.

Hypotheses 1–3 are met if the corresponding mappings are defined using elementary algebraic functions in a polynomial form. For example, for vectors $\Theta_{AB}(\theta) = col\left\{\theta_2^2\theta_1^2 + (\theta_2+\theta_1)^3,\ \theta_2\right\}$ and $\psi_{ab}(\theta) = col\left\{\theta_1\theta_2 + \theta_1^2,\ \theta_2+\theta_1\right\}$ the mappings from (3.9) and (3.8) are written as follows

$$\mathcal{T}_{\mathcal{X}}\left(\Xi_{\mathcal{X}}\left(\mathcal{M}_\theta\right)\theta\right) = \begin{bmatrix}\mathcal{M}_\theta^4 & 0\\ 0 & \mathcal{M}_\theta\end{bmatrix},\ \mathcal{T}_{\mathcal{Z}}\left(\Xi_{\mathcal{Z}}\left(\mathcal{M}_\theta\right)\theta\right) = \begin{bmatrix}\mathcal{M}_\theta^4\theta_2^2\theta_1^2 + \mathcal{M}_\theta^4(\theta_2+\theta_1)^3\\ \mathcal{M}_\theta\theta_2\end{bmatrix},$$

$$\mathcal{S}\left(\psi_{ab}\right) = \begin{bmatrix}\psi_{1ab}\\ \psi_{2ab}^2 - \psi_{1ab}\end{bmatrix},\ \mathcal{G}\left(\psi_{ab}\right) = \begin{bmatrix}\psi_{2ab} & 0\\ 0 & \psi_{2ab}\end{bmatrix}, \tag{3.11}$$

$$\mathcal{T}_{\mathcal{G}}\left(\Xi_{\mathcal{G}}\left(\Delta\right)\psi_{ab}\right) = \begin{bmatrix}\Delta\psi_{2ab} & 0\\ 0 & \Delta^2\psi_{2ab}\end{bmatrix},\ \mathcal{T}_{\mathcal{S}}\left(\Xi_{\mathcal{S}}\left(\Delta\right)\psi_{ab}\right) = \begin{bmatrix}\psi_{1ab}\Delta\\ \Delta^2\psi_{2ab}^2 - \Delta^2\psi_{1ab}\end{bmatrix}.$$

The essence of Hypotheses 1–3 is that, owing to the property $\Xi_{(.)}(\omega) = \overline{\Xi}_{(.)}(\omega)\,\omega(t)$, the linear regression equations with respect to the unknown parameters $\theta,\ \Theta_{AB}(\theta),\ \psi_d(\theta)$ can be parametrized on the basis of the measurable signals $\mathcal{Y}_{ab}(t) = \mathcal{L}_{ab}\mathcal{Y}(t) = \Delta(t)\psi_{ab}(\theta)$ and $\mathcal{Y}_\theta(t) = \mathcal{M}_\theta(t)\theta$, respectively.

For instance, equation (3.11) can be rewritten as:

$$\mathcal{T}_{\mathcal{Z}}\left(\overline{\Xi}_{\mathcal{Z}}\left(\mathcal{M}_\theta\right)\mathcal{Y}_\theta\right) = \begin{bmatrix}\mathcal{Y}_{2\theta}^2\mathcal{Y}_{1\theta}^2 + \mathcal{M}_\theta(\mathcal{Y}_{2\theta}+\mathcal{Y}_{1\theta})^3\\ \mathcal{Y}_{2\theta}\end{bmatrix},$$

$$\mathcal{T}_{\mathcal{G}}\left(\overline{\Xi}_{\mathcal{G}}\left(\Delta\right)\mathcal{Y}_{ab}\right) = \begin{bmatrix}\mathcal{Y}_{2ab} & 0\\ 0 & \Delta\mathcal{Y}_{2ab}\end{bmatrix},\ \mathcal{T}_{\mathcal{S}}\left(\overline{\Xi}_{\mathcal{S}}\left(\Delta\right)\mathcal{Y}_{ab}\right) = \begin{bmatrix}\mathcal{Y}_{1ab}\\ \mathcal{Y}_{2ab}^2 - \Delta\mathcal{Y}_{1ab}\end{bmatrix},$$

and therefore, we directly have the following measurable linear regression equations[3]

$$\mathcal{T}_{\mathcal{Z}}\left(\overline{\Xi}_{\mathcal{Z}}\left(\mathcal{M}_\theta\right)\mathcal{Y}_\theta\right) = \mathcal{T}_{\mathcal{X}}\left(\overline{\Xi}_{\mathcal{X}}\left(\mathcal{M}_\theta\right)\mathcal{Y}_\theta\right)\Theta_{AB}(\theta),$$
$$\mathcal{T}_{\mathcal{S}}\left(\overline{\Xi}_{\mathcal{S}}\left(\Delta\right)\mathcal{Y}_{ab}\right) = \mathcal{T}_{\mathcal{G}}\left(\overline{\Xi}_{\mathcal{G}}\left(\Delta\right)\mathcal{Y}_{ab}\right)\theta,$$

where the signals $\mathcal{Y}_\theta(t)$ and $\mathcal{M}_\theta(t)$ are calculated in the following way using the second equation:

$$\mathcal{Y}_\theta(t) = \operatorname{adj}\left\{\mathcal{T}_{\mathcal{G}}\left(\overline{\Xi}_{\mathcal{G}}\left(\Delta\right)\mathcal{Y}_{ab}\right)\right\}\mathcal{T}_{\mathcal{S}}\left(\overline{\Xi}_{\mathcal{S}}\left(\Delta\right)\mathcal{Y}_{ab}\right),$$
$$\mathcal{M}_\theta(t) = \det\left\{\mathcal{T}_{\mathcal{G}}\left(\overline{\Xi}_{\mathcal{G}}\left(\Delta\right)\mathcal{Y}_{ab}\right)\right\}.$$

The requirement (3.7) and Hypotheses 1–3, despite being seemed as mathematically restrictive, are practice-oriented and met for a large number of models of real technical systems.

[3] A measurable regression equation means that its regressor and regressand are measurable or can be computed, while its parameters are unknown.

4. MAIN RESULT

The solvability conditions (3.7)–(3.10) are assumed to be met and the error equations for the differences between (2.4) and (2.1), $\hat{\delta}(t)$ and $\delta(t)$ are written:

$$\begin{gathered}\dot{\tilde{x}}(t) = \Phi^{\mathrm{T}}\left(\hat{x}, u, \hat{\delta}\right)\hat{\Theta}_{AB}(t) - \hat{L}(t)\tilde{y}(t) - \Phi^{\mathrm{T}}\left(x, u, \delta\right)\Theta_{AB} \\ = \Phi^{\mathrm{T}}\left(\hat{x}, u, \hat{\delta}\right)\hat{\Theta}_{AB}(t) - \hat{L}(t)\tilde{y}(t) - \Phi^{\mathrm{T}}(x, u, \delta)\Theta_{AB} \pm \Phi^{\mathrm{T}}\left(\hat{x}, u, \hat{\delta}\right)\Theta_{AB} \\ = \Phi^{\mathrm{T}}\left(\hat{x}, u, \hat{\delta}\right)\tilde{\Theta}_{AB}(t) - \hat{L}(t)\tilde{y}(t) - \Phi^{\mathrm{T}}(x, u, \delta)\Theta_{AB} + \Phi^{\mathrm{T}}\left(\hat{x}, u, \hat{\delta}\right)\Theta_{AB} \\ = A\left(\theta\right)\tilde{x}(t) + D\left(\theta\right)\tilde{\delta}(t) + \Phi^{\mathrm{T}}\left(\hat{x}, u, \hat{\delta}\right)\tilde{\Theta}_{AB}(t) - \hat{L}(t)\tilde{y}(t) \pm L\left(\theta\right)\tilde{y}(t) \\ = A_m\tilde{x}(t) + D\left(\theta\right)h_\delta^{\mathrm{T}}\Phi_\delta(t)\tilde{x}_{\delta 0}(t) + \Phi^{\mathrm{T}}\left(\hat{x}, u, \hat{\delta}\right)\tilde{\Theta}_{AB}(t) - \tilde{L}(t)\tilde{y}(t) \\ = A_m\tilde{x}(t) + \phi^{\mathrm{T}}(t)\tilde{\kappa}(t), \\ \tilde{\delta}(t) = h_\delta^{\mathrm{T}}\Phi_\delta(t)\hat{x}_{\delta 0} - h_\delta^{\mathrm{T}}\Phi_\delta(t)x_{\delta 0} = h_\delta^{\mathrm{T}}\Phi_\delta(t)\tilde{x}_{\delta 0}(t),\end{gathered} \tag{4.1}$$

where

$$\phi^{\mathrm{T}}(t) = \begin{bmatrix} D\left(\theta\right)h_\delta^{\mathrm{T}}\Phi_\delta(t) & \Phi^{\mathrm{T}}\left(\hat{x},\ u,\ \hat{\delta}\right) & -\tilde{y}(t)I_n \end{bmatrix}$$

and $A_m = A\left(\theta\right) - L\left(\theta\right)C^{\mathrm{T}}$ is a Hurwitz matrix in accordance with Assumption 3.

In order to achieve the goal (2.5), using equations (4.1), an estimation law is required to be designed that ensures exponential convergence to zero of the error $\tilde{\kappa}(t)$ and exponential stability of the equilibrium point of the state observation error $\tilde{x}(t)$. Thus, the problem of reconstruction of the perturbation $\delta(t)$ and unmeasurable state $x(t)$ of the system (2.1) is reduced to the problem of parametric identification. Such problem, in its turn, can be solved if the assumptions (3.7)–(3.10) are met. To design an estimation law that is based on Hypotheses 1–3 and the results of Lemma 1, and ensures achievement of (2.5), we first parameterize the static regression equation with respect to κ.

Lemma 2. *The vector of unknown parameters κ satisfies the linear regression equation*

$$\begin{gathered}\mathcal{Y}_\kappa(t) = \mathcal{M}_\kappa(t)\kappa, \\ \mathcal{Y}_\kappa(t) = \operatorname{adj}\left\{\operatorname{blkdiag}\left\{\mathcal{M}_{x_{\delta 0}}(t)I_{n_\delta},\ \mathcal{M}_{AB}(t)I_{n_\Theta},\ \mathcal{M}_L(t)I_n\right\}\right\}\begin{bmatrix}\mathcal{Y}_{x_{\delta 0}}(t) \\ \mathcal{Y}_{AB}(t) \\ \mathcal{Y}_L(t)\end{bmatrix}, \\ \mathcal{M}_\kappa(t) = \det\left\{\operatorname{blkdiag}\left\{\mathcal{M}_{x_{\delta 0}}(t)I_{n_\delta},\ \mathcal{M}_{AB}(t)I_{n_\Theta},\ \mathcal{M}_L(t)I_n\right\}\right\},\end{gathered} \tag{4.2}$$

where:

1) *the regressand and regressor of the regression $\mathcal{Y}_{AB}(t) = \mathcal{M}_{AB}(t)\Theta\left(\theta\right)$, using the auxilary calculations*

$$\begin{gathered}\mathcal{Y}_\theta(t) = \operatorname{adj}\left\{\mathcal{T}_\mathcal{G}\left(\Xi_\mathcal{G}\left(\Delta\right)\mathcal{Y}_{ab}\right)\right\}\mathcal{T}_\mathcal{S}\left(\Xi_\mathcal{S}\left(\Delta\right)\mathcal{Y}_{ab}\right), \\ \mathcal{M}_\theta(t) = \det\left\{\mathcal{T}_\mathcal{G}\left(\Xi_\mathcal{G}\left(\Delta\right)\mathcal{Y}_{ab}\right)\right\},\end{gathered}$$

are defined as:

$$\begin{gathered}\mathcal{Y}_{AB}(t) = \operatorname{adj}\left\{\mathcal{T}_\mathcal{X}\left(\Xi_\mathcal{X}\left(\mathcal{M}_\theta\right)\mathcal{Y}_\theta\right)\right\}\mathcal{T}_\mathcal{Z}\left(\Xi_\mathcal{Z}\left(\mathcal{M}_\theta\right)\mathcal{Y}_\theta\right), \\ \mathcal{M}_{AB}(t) = \det\left\{\mathcal{T}_\mathcal{X}\left(\Xi_\mathcal{X}\left(\mathcal{M}_\theta\right)\mathcal{Y}_\theta\right)\right\}.\end{gathered}$$

2) *the regressand and regressor of the regression* $\mathcal{Y}_L(t) = \mathcal{M}_L(t)L(\theta)$ *are calculated as:*

$$\mathcal{Y}_L(t) = \mathrm{adj}\left\{\mathcal{T}_{\mathcal{P}}\left(\overline{\Xi}_{\mathcal{P}}\left(\mathcal{M}_{AB}\right)\mathcal{Y}_{AB}\right)\right\}\mathcal{T}_{\mathcal{Q}}\left(\overline{\Xi}_{\mathcal{Q}}\left(\mathcal{M}_{AB}\right)\mathcal{Y}_{AB}\right),$$
$$\mathcal{M}_L(t) = \det\left\{\mathcal{T}_{\mathcal{P}}\left(\overline{\Xi}_{\mathcal{P}}\left(\mathcal{M}_{AB}\right)\mathcal{Y}_{AB}\right)\right\},$$
$$\mathcal{T}_{\mathcal{P}}\left(\Xi_{\mathcal{P}}\left(\mathcal{M}_{AB}\right)\mathcal{Y}_{AB}\right) = vec^{-1}\Big\{\mathcal{M}_{AB}\,\mathrm{adj}\Big\{I_n \otimes vec^{-1}\left(\mathcal{L}_{A^{\mathrm{T}}}\mathcal{D}_{\Phi}\mathcal{Y}_{AB}\right) - \mathcal{M}_{AB}\Gamma^{\mathrm{T}} \otimes I_n\Big\}vec\left(C(\mathcal{L}_B\mathcal{D}_{\Phi}\mathcal{Y}_{AB})^{\mathrm{T}}\right)\Big\}^{\mathrm{T}},$$
$$\mathcal{T}_{\mathcal{Q}}\left(\Xi_{\mathcal{Q}}\left(\mathcal{M}_{AB}\right)\mathcal{Y}_{AB}\right) = \det\Big\{I_n \otimes vec^{-1}\left(\mathcal{L}_{A^{\mathrm{T}}}\mathcal{D}_{\Phi}\mathcal{Y}_{AB}\right) - \mathcal{M}_{AB}\Gamma^{\mathrm{T}} \otimes I_n\Big\}\mathcal{L}_B\mathcal{D}_{\Phi}\mathcal{Y}_{AB}.$$

3) *the regression* $\mathcal{Y}_{x_{\delta 0}}(t) = \mathcal{M}_{x_{\delta 0}}(t)x_{\delta 0}$, *considering the equations*

$$p(t) = \Delta(t)\overline{q}(t) - C_0^{\mathrm{T}}\Omega(t)\mathcal{L}_a\mathcal{Y}(t) - C_0^{\mathrm{T}}P(t)\mathcal{L}_b\mathcal{Y}(t),$$
$$\mathcal{Y}_{\psi_d}(t) = \mathrm{adj}\left\{\mathcal{T}_{\mathcal{R}}\left(\overline{\Xi}_{\mathcal{R}}\left(\mathcal{M}_{\theta}\right)\mathcal{Y}_{\theta}\right)\right\}\mathcal{T}_{\mathcal{W}}\left(\overline{\Xi}_{\mathcal{W}}\left(\mathcal{M}_{\theta}\right)\mathcal{Y}_{\theta}\right),$$
$$\mathcal{M}_{\psi_d}(t) = \det\left\{\mathcal{T}_{\mathcal{R}}\left(\overline{\Xi}_{\mathcal{R}}\left(\mathcal{M}_{\theta}\right)\mathcal{Y}_{\theta}\right)\right\}$$

and filtering

$$\begin{aligned}
&\dot{V}(t) = A_K V(t) + \left(h_\delta^{\mathrm{T}}\Phi_\delta(t) \otimes I_n\right), \quad V(t_0) = 0_{n\times nn_\delta},\\
&\dot{p}_f(t) = -k_2\varphi(t) + \Delta(t)(I_{n_\delta} \otimes \mathcal{Y}_{\psi_d}(t))^{\mathrm{T}}V^{\mathrm{T}}(t)C_0\mathcal{M}_{\psi_d}(t)p(t),\ p_f(t_0) = 0_{n_\delta},\\
&\dot{V}_f(t) = -k_2V_f(t) + \Delta^2(t)(I_{n_\delta} \otimes \mathcal{Y}_{\psi_d}(t))^{\mathrm{T}}V^{\mathrm{T}}(t)C_0C_0^{\mathrm{T}}V(t)\left(I_{n_\delta} \otimes \mathcal{Y}_{\psi_d}(t)\right),\ V_f(t_0) = 0_{n_\delta\times n_\delta},
\end{aligned} \tag{4.3}$$

is defined as follows:

$$\mathcal{Y}_{x_{\delta 0}}(t) = \mathcal{M}_{x_{\delta 0}}(t)x_{\delta 0}, \quad \mathcal{Y}_{x_{\delta 0}}(t) = \mathrm{adj}\left\{V_f(t)\right\}p_f(t), \quad \mathcal{M}_{x_{\delta 0}}(t) = \det\left\{V_f(t)\right\},$$

and, if the conditions $\overline{\varphi}(t) \in \mathrm{PE}$, $\left(h_\delta^{\mathrm{T}}\Phi_\delta(t) \otimes I_n\right) \in \mathrm{PE}$ *are met, then for all* $t \geqslant t_0 + T$ *it holds that* $|\mathcal{M}_\kappa(t)| \geqslant \underline{\mathcal{M}_\kappa} > 0$.

Proof of Lemma 2 and definitions of the matrices $\mathcal{L}_{A^{\mathrm{T}}}, \mathcal{L}_B, \mathcal{L}_a, \mathcal{L}_b$ *are given in Appendix.*

Having at hand the regression equation (4.2) with a scalar regressor $\mathcal{M}_\kappa(t)$, which is bounded away from zero for all $t \geqslant t_0 + T$, and using the results from [13, 18], the estimation law is derived, which ensures that the goal (2.5) is achieved.

Theorem 1. *Let the vector* $D_{\max} \in \mathbb{R}^n$ *be known such that* $\|D(\theta)\| \leqslant \|D_{\max}\|$, *then, if* $\overline{\varphi}(t) \in \mathrm{PE}$, $\left(h_\delta^{\mathrm{T}}\Phi_\delta(t) \otimes I_n\right) \in \mathrm{PE}$ *and* $\gamma_0 > 0$, $\gamma_1 > 0$, *then the estimation law*

$$\begin{aligned}
&\dot{\hat{\kappa}}(t) = \dot{\tilde{\kappa}}(t) = -\gamma(t)\mathcal{M}_\kappa(t)\left(\mathcal{M}_\kappa(t)\hat{\kappa}(t) - \mathcal{Y}_\kappa(t)\right) = -\gamma(t)\mathcal{M}_\kappa^2(t)\tilde{\kappa}(t),\\
&\gamma(t) := \begin{cases} 0, \text{ if } \Delta(t) < \rho \in [\Delta_{\min}; \Delta_{\max}),\\ \dfrac{\gamma_0\lambda_{\max}\left(\phi_{\max}(t)\phi_{\max}^{\mathrm{T}}(t)\right) + \gamma_1}{\mathcal{M}_\kappa^2(t)} \text{ otherwise},\end{cases}\\
&\phi_{\max}^{\mathrm{T}}(t) = \left[D_{\max}h_\delta^{\mathrm{T}}\Phi_\delta(t) \quad \Phi^{\mathrm{T}}\left(\hat{x},\ u,\ \hat{\delta}\right) \quad -\tilde{y}(t)I_n\right]
\end{aligned} \tag{4.4}$$

ensures the following properties:

1) $\forall t \geqslant t_0$ $\left[\tilde{x}^{\mathrm{T}}(t) \quad \tilde{\kappa}^{\mathrm{T}}(t)\right]^{\mathrm{T}} \in L_\infty$;
2) $\forall t \geqslant t_0 + T$ *the error* $\left[\tilde{x}^{\mathrm{T}}(t) \quad \tilde{\kappa}^{\mathrm{T}}(t)\right]^{\mathrm{T}}$ *converges exponentially to zero with the rate, which minimum value is directly proportional to* $\gamma_1 > 0$.

Proof of the first part of theorem is similar to the proof of the second part of Theorem 1 from [18], *proof of the second part of theorem coincides up to the notation with the proof of Theorem 1 from* [13].

Owing to the boundedness of $h_\delta^{\mathrm{T}}\Phi_\delta(t)$, the exponential convergence of the error $\tilde{\delta}(t)$ follows from the above-given theorem, which together with the exponential convergence of $\left[\tilde{x}^{\mathrm{T}}(t)\ \ \tilde{\kappa}^{\mathrm{T}}(t)\right]^{\mathrm{T}}$ means that the goal (2.5) is achieved.

Remark 2. The results of Lemma 2 describe the procedure to transform the regression equation (3.3) *with a scalar regressor with respect to the numerator and denominator parameters of the transfer function $W_{uy}(s)$ into a new equation* (4.2) *with respect to the observer parameters* (2.4). *Considering such recalculation, the division operations by time-dependent signals are not used, the parameters $\eta(\theta)$, $\psi_{ab}(\theta)$ or θ are not identified, and $\mathcal{Y}_\kappa(t)$ and $\mathcal{M}_\kappa(t)$ are calculated solely using the signals $\mathcal{Y}(t)$ and $\Delta(t)$ that are measurable according to the results of Lemma 1.*

Remark 3. The exponential stability conditions from the theorem are conservative. In practice, the knowledge of $D_{\max} \in \mathbb{R}^n$ and ρ, as well as the implementation of the procedure to compute the eigenvalue $\lambda_{\max}\left(\phi_{\max}(t)\phi_{\max}^{\mathrm{T}}(t)\right)$ are not required, and the goal (2.5) *can be achieved using any sufficiently large constant coefficient $\gamma \geqslant \left(\gamma_{\min} \sim \frac{1}{\mathcal{M}_\kappa^2(t)}\right) > 0$, which is a majorant for $\lambda_{\max}\left(\phi_{\max}(t)\phi_{\max}^{\mathrm{T}}(t)\right)$.*

5. DISCUSSION

In this section, four additional technical comments on the obtained results are given.

Comment 1. In accordance with the lower bound from (A.48), the regressor $\mathcal{M}_\kappa(t)$ is proportional to a power function $\Delta^{\ell_\theta\ell_\Theta n_\Theta + \ell_\theta\ell_\Theta n(n^3+n) + n_\delta^2\left(2\ell_\theta\ell_{\psi_d}+2\right)}(t)$. Therefore, if $\Delta(t) \ll 1$ or $\Delta(t) \gg 1$, then the computational elimination of the regressor excitation may occur inside a software implementation of the proposed approach:

$$\Delta(t) \ll 1 \Rightarrow \mathcal{M}_\kappa(t) \to 0 \text{ or } \Delta(t) \gg 1 \Rightarrow \mathcal{M}_\kappa(t) \to \infty,$$

i.e. $\mathcal{M}_\kappa(t)$ can become so small or so large that it can not be processed by a computer as its CPU has a limited registers length (for example, in Matlab/Simulink the numbers that are smaller than 10^{-309} or larger than 10^{309} are considered equal to zero and infinity, respectively).

This problem does not concern the theoretical results of the paper, but is related solely to the shortcomings of the existing computational devices. To prevent computational elimination of the regressor excitation, a time-varying amplitude modulator $k(t)$ should be used in accordance with the method of regressor excitation normalization:

$$\begin{gathered} k(t) \sim \frac{1}{\Delta(t)} \ \text{ or } \ k(t) := \begin{cases} 1, \text{ if } \Delta(t) < \rho \in [\Delta_{\min}; \Delta_{\max}), \\ \dfrac{1}{\Delta(t)} \text{ otherwise,} \end{cases} \\ \text{or } \ k(t) := \begin{cases} 1, \text{ if } t < t_e \in [t_0; \infty), \\ \dfrac{1}{\Delta(t)} \text{ otherwise.} \end{cases} \end{gathered} \tag{5.1}$$

Moreover, implementing the parameterization (4.2) in practice, it is advisable to apply a multiplication by an amplitude modulator similar to (5.1) after each multiplication by the adjoint matrix adj{.}. The problem of computational elimination of the regressor excitation was discussed in more detail in Section 3.3 of [18].

Comment 2. The existing identification methods with the relaxed regressor excitation requirements do not allow one to ensure parametric convergence if the parameterized regression equation is affected even by an exponentially decaying perturbation [19].

To solve this problem, in [20] it is proposed to use integral filtering with periodic resetting after a given time interval. The method from [20] allows one to reduce the upper bound of the steady-state parametric error iteratively.

An alternative approach is to extend the identification problem via parameterization of the exponentially decaying perturbation as a linear regression with measurable regressor and unknown parameters—unmeasurable initial conditions [11–13]. This approach allows one to ensure the exponential convergence of the parametric error to zero when the relaxed regressor excitation requirements are met, but it is applicable only to perturbations that can be reduced to a linear regression model. The exponentially decaying perturbation $\epsilon(t)$ of (3.3) cannot be represented in such a way.

Therefore, in contrast to the results of [11–13], in this paper, to achieve the goal (2.5), instead of the relaxed conditions, a stricter one of the regressor persistent excitation (1.9) is required. If this condition is met, then the filters with memory from [11–13] are not required, and the exponential convergence of the parametric error is guaranteed even in case of existence of an exponentially decaying perturbation in the parameterization in use.

It is possible to relax the requirement of the regressor persistent excitation by application of the following filter instead of (3.4):

$$\begin{aligned}\dot{q}(t) &= \int\limits_{t_\epsilon}^{t} e^{-k_2\tau}\overline{\varphi}_f(\tau)(\overline{q}(\tau) - k_1\overline{q}_f(\tau) - \beta^{\mathrm{T}}(F_f(\tau) + l y_f(\tau)))d\tau,\ \ q\left(t_\epsilon\right) = 0_{2n},\\ \dot{\varphi}(t) &= \int\limits_{t_\epsilon}^{t} e^{-k_2\tau}\overline{\varphi}_f\left(\tau\right)\overline{\varphi}_f^{\mathrm{T}}\left(\tau\right)d\tau, \quad \varphi\left(t_\epsilon\right) = 0_{2n\times 2n},\end{aligned} \tag{5.2}$$

where $t_\epsilon \gg t_0$ is a known time instant when the filtering is started.

If the time instant t_ϵ is chosen so that to satisfy the condition $\varepsilon(t) = o\left(\varphi(t)\eta(\theta)\right)$ from (A.26) for all $t \geqslant t_\epsilon$, and the regressor is finitely exciting over $[t_\epsilon,\, t_e]$, then the goal (2.5) is achieved under the relaxed regressor excitation requirement. More detailed properties of the extended observer on the basis of the parameterization with filtering (5.2) are studied in [21].

Comment 3. According to theorem 1, the proposed observer (2.4) + (4.4) ensures convergence of the state observation error to zero only if the persistent excitation requirements $\overline{\varphi}(t) \in \mathrm{PE}$ and $\left(h_\delta^{\mathrm{T}}\Phi_\delta(t)\otimes I_n\right) \in \mathrm{PE}$ are met. As the signal $h_\delta^{\mathrm{T}}\Phi_\delta(t)\otimes I_n$ is known for all $t \in [t_0,\, \in fty)$, then the condition $\left(h_\delta^{\mathrm{T}}\Phi_\delta(t)\otimes I_n\right) \in \mathrm{PE}$ can be validated offline—before the observer implementation. The condition $\overline{\varphi}(t) \in \mathrm{PE}$ is, strictly speaking, unverifiable both offline and online, since it depends on all previous and future values of the regressor $\overline{\varphi}(t)$. Usually, to meet the regressor persistent excitation condition in linear systems parametrizations of the form (A.25), the control signal is formed so as to belong to a class of functions that are sufficiently rich of some order [5, 6], i.e., the functions that include a sufficient number of spectral lines (harmonics), as far as their Fourier expansion is considered. Considering the parametrization (A.25), (3.4) from this paper, unfortunately, at this stage, it is difficult to define the exact number of spectral lines that the control signal is to include in order to meet the condition $\overline{\varphi}(t) \in \mathrm{PE}$. This is one of the main disadvantages of the proposed solution, which significantly reduces its practical value.

However, owing to the implication from Proposition 1

$$\overline{\varphi}(t) \in \mathrm{PE} \Leftrightarrow \exists t \geqslant t_e \in [t_0, \infty)\ \ \Delta(t) \geqslant \Delta_{\mathrm{LB}} > 0\ ,$$

the proposed observer can be augmented with the following heuristic procedure to obtain the control signal that ensures $\overline{\varphi}(t) \in \mathrm{PE}$.

Initialization. *Set* $k = 1$ *and* $m = 1$.

Step 1. *Choose*

$$u(t) = u_b(t) + \sum_{i=1}^{m} a_i \sin(\omega_i t), \tag{5.3}$$

where $u_b(t)$ *is a stabilizing component of the control signal, for example, a P-controller,* a_i *stands for an arbitrary amplitude of the* i^{th} *harmonic, and the frequencies* $\omega_i(t)$ *are such that* $\omega_i \neq \omega_j$ *for all* $i \neq j$.

Step 2. *Apply the control signal* $u(t)$ *to the system and calculate the value of* $\Delta(t)$ *over* $[t_{k-1}, t_k]$, *where* $t_k - t_{k-1}$ *is a sufficiently large value.*

Step 3. *If there exists* $\Delta_{LB} > 0$ *such that* $\Delta(t) \geqslant \Delta_{LB}$ *for all* $t \in [t_{k-1}, t_k]$, *then, according to Proposition 1, it holds that*

$$\int_{t}^{t+T} \overline{\varphi}(\tau)\, \overline{\varphi}^{\mathrm{T}}(\tau)\, d\tau \geqslant \alpha I_n \tag{5.4}$$

for all $t \in [t_{k-1}, t_k]$.

Assume that the result obtained at $[t_{k-1}, t_k]$ *can be interpolated to the entire time axis* $[t_0, \infty)$, *then, based on Proposition 1, a control signal is found that satisfies the condition* $\overline{\varphi}(t) \in \mathrm{PE}$.

If there is no $\Delta_{LB} > 0$ *such that* $\Delta(t) \geqslant \Delta_{LB}$ *for all* $t \in [t_{k-1}, t_k]$, *then set* $m = m + 1$ *and* $k = k + 1$, *and go to Step 1.*

The essence of the above-given algorithm is to increase iteratively the number of harmonics in the control signal until the scalar regressor $\Delta(t)$ becomes to be bounded away from zero over a sufficiently large time interval. Assuming that there exists $m_{\max}$ such that, when $m = m_{\max}$, then the control signal (5.3) ensures that the condition $\overline{\varphi}(t) \in \mathrm{PE}$ is met, then it can be claimed that this algorithm, in a finite number of iterations $m_{\max}$, allows one to generate a control signal that ensures that the condition $\overline{\varphi}(t) \in \mathrm{PE}$ is met. It should be noted that the above procedure is not mathematically rigorous because it is designed under the strict assumption that the fact that the condition (5.4) is met over a sufficiently large time interval means that the condition (5.4) is satisfied over the entire time axis. Generally speaking, such a conclusion cannot be made. However, as far as practical scenarios are concerned, the above-mentioned simplification is acceptable, and the described procedure can be efficient.

Comment 4. If, in addition to Hypotheses 1–3, in the extended system

$$\begin{gathered}\dot{x}_e(t) = \left(A_e(\theta) + A_\delta\right) x_e(t) + B_e u(t) = \Phi^{\mathrm{T}}(x_e,\ u)\, \Theta_{AB}(\theta) + A_\delta x_e(t),\\ y(t) = C_e^{\mathrm{T}} x_e(t), \quad x_e(t_0) = \begin{bmatrix} x_0 & x_{\delta 0}\end{bmatrix}^{\mathrm{T}},\end{gathered} \tag{5.5}$$

$$A_e(\theta) = \begin{bmatrix} A(\theta) & D(\theta)\, h_\delta^{\mathrm{T}} \\ 0_{n_\delta \times n} & 0_{n_\delta \times n_\delta}\end{bmatrix}, \quad A_\delta = \begin{bmatrix} 0_{n\times n} & 0_{n \times n_\delta} \\ 0_{n_\delta \times n} & \mathcal{A}_\delta \end{bmatrix},$$

$$B_e(\theta) = \begin{bmatrix} B(\theta) \\ 0_{n_\delta}\end{bmatrix}, \quad x_e(t) = \begin{bmatrix} x(t) \\ x_\delta(t)\end{bmatrix}, \quad C_e = \begin{bmatrix} C \\ 0_{n_\delta}\end{bmatrix},$$

$$\Phi^{\mathrm{T}}(x_e,\ u) = \begin{bmatrix} I_{n+n_\delta} \otimes x_e^{\mathrm{T}}(t) & I_{n+n_\delta} \otimes u^{\mathrm{T}}(t)\end{bmatrix} \mathcal{D}_\Phi \in \mathbb{R}^{(n+n_\delta)\times n_\Theta},$$

$$\Theta_{AB}(\theta) = \mathcal{L}_\Phi \begin{bmatrix} vec^{\mathrm{T}}\left(A_e^{\mathrm{T}}(\theta)\right) & B_e^{\mathrm{T}}(\theta)\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{n_\Theta}$$

the pair $\left(C_e^{\mathrm{T}},\ A_e\right)$ is observable, then the extended observer

$$\dot{\hat{x}}_e(t) = \Phi^{\mathrm{T}}\left(\hat{x}_e,\ u\right)\hat{\Theta}_{AB}(t) + A_\delta \hat{x}_\delta(t) - \hat{L}_e(t)\left(\hat{y}(t) - y(t)\right), \tag{5.6}$$

which is augmented only with the estimation laws for $\Theta_{AB}(\theta)$ and $L_e(\theta)$, (i) does not require to parametrize (4.3) the regression equation $\mathcal{Y}_{x_{\delta 0}}(t) = \mathcal{M}_{x_{\delta 0}}(t)x_{\delta 0}$, (ii) does not require to identify the initial conditions $x_{\delta 0}$ of the exosystem (2.2), and, (iii) if the condition (1.9) is met in case (3.4) is used or the regressor finite excitation condition is met in case (5.2) is used, ensures the exponential convergence to zero of the errors $\tilde{x}(t)$, $\tilde{\delta}(t)$. In (5.5) $\hat{L}_e(t)$ is an estimate of the vector $L_e(\theta) \in \mathbb{R}^{n+n_\delta}$, which ensures that the matrix $A_e(\theta) + A_\delta$ has the desired algebraic spectrum. The linear regression equation with respect to $L_e(\theta)$ is parameterised in the same way as $\mathcal{Y}_L(t) = \mathcal{M}_L(t)L(\theta)$, but in the space $n + n_\delta$. More detailed properties of the alternative version of the extended observer are given in [21].

6. MATHEMATICAL MODELLING

In Matlab/Simulink the numerical experiments with the proposed adaptive observer have been conducted. The simulation was done using numerical integration by the explicit Euler method with a constant discretization step of $\tau_s = 10^{-4}$ s.

A two-mass elastic mechanical system shown in Fig. 1 was chosen as a plant.

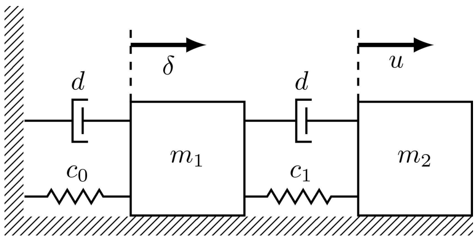

Fig. 1. Two-mass elastic mechanical system.

Here $c_0 > 0,\ c_1 > 0$ denote spring stiffness, $d > 0$ is a damping coefficient, $m_1 > 0,\ m_2 > 0$ are reduced masses of the bodies.

The mathematical model of the system under consideration was written as the following system of differential equations:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\theta_1\left(\theta_2 + \theta_3\right) & -2\theta_1\theta_4 & \theta_1\theta_3 & \theta_1\theta_4 \\ 0 & 0 & 0 & 1 \\ \theta_5\theta_3 & \theta_5\theta_4 & -\theta_3\theta_5 & -2\theta_4\theta_5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \theta_5 \end{bmatrix} u + \begin{bmatrix} 0 \\ \theta_1 \\ 0 \\ 0 \end{bmatrix} \delta, \\ y &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} x, \end{aligned} \tag{6.1}$$

where

$$\theta = col\left\{m_1^{-1},\ c_0,\ c_1,\ d,\ m_2^{-1}\right\},$$
$$\Theta(\theta) = col\left\{1,\ \theta_1\left(\theta_2 + \theta_3\right),\ \theta_1\theta_4,\ \theta_1\theta_3,\ \theta_3\theta_5,\ \theta_4\theta_5,\ \theta_1,\ \theta_5\right\}.$$

In the observer canonical form (3.1) the parameters of the system (6.1) were defined as follows:

$$\psi_a(\theta) = \begin{bmatrix} -2\theta_4(\theta_1+\theta_5) \\ -\theta_1\left(3\theta_5\theta_4^2+\theta_2+\theta_3\right)-\theta_3\theta_5 \\ -2\theta_1\theta_4\theta_5(\theta_2+\theta_3) \\ -\theta_1\theta_2\theta_3\theta_5 \end{bmatrix},\quad \psi_b(\theta) = \begin{bmatrix} 0 \\ \theta_5 \\ 2\theta_1\theta_4\theta_5 \\ \theta_1\theta_5(\theta_2+\theta_3) \end{bmatrix},$$
$$\psi_d(\theta) = \begin{bmatrix} 0 & 0 & \theta_1\theta_4\theta_5 & \theta_1\theta_3\theta_5 \end{bmatrix}^{\mathrm{T}}, \tag{6.2}$$

from which it followed that the condition (3.7) was met for

$$\psi_{ab}(\theta) = \begin{bmatrix} -2\theta_4(\theta_1+\theta_5) \\ -\theta_1\left(3\theta_5\theta_4^2+\theta_2+\theta_3\right)-\theta_3\theta_5 \\ \theta_1\theta_5(\theta_2+\theta_3) \\ 2\theta_1\theta_4\theta_5 \\ \theta_5 \end{bmatrix}.$$

The regression equation (4.2) was parameterised using the transformations introduced in Hypotheses 1–3 and Lemma 2.

Step 1. The derivation of the parametrization $\mathcal{Y}_\theta(t) = \mathcal{M}_\theta(t)\theta$. The following set of the nonlinear algebraic equations were solved with respect to θ

$$\psi_{ab}(\theta) = \begin{bmatrix} -2\theta_4(\theta_1+\theta_5) \\ -\theta_1\left(3\theta_5\theta_4^2+\theta_2+\theta_3\right)-\theta_3\theta_5 \\ \theta_1\theta_5(\theta_2+\theta_3) \\ 2\theta_1\theta_4\theta_5 \\ \theta_5 \end{bmatrix} = \begin{bmatrix} \psi_{1ab} \\ \psi_{2ab} \\ \psi_{3ab} \\ \psi_{4ab} \\ \psi_{5ab} \end{bmatrix},$$

and, using such solution, the mappings $\mathcal{S}(\psi_{ab})$ and $\mathcal{G}(\psi_{ab})$ from (3.8) were obtained:

$$\mathcal{S}(\psi_{ab}) = \begin{bmatrix} \psi_{4ab}\psi_{5ab} \\ \psi_{3ab}\psi_{5ab}(-\psi_{4ab}-\psi_{1ab}\psi_{5ab}) \\ +\,\psi_{4ab}\left((\psi_{2ab}\psi_{5ab}+\psi_{3ab})\psi_{5ab}+\frac{3}{4}\psi_{4ab}(-\psi_{4ab}-\psi_{1ab}\psi_{5ab})\right) \\ (\psi_{2ab}\psi_{5ab}+\psi_{3ab})\psi_{5ab}+\frac{3}{4}\psi_{4ab}(-\psi_{4ab}-\psi_{1ab}\psi_{5ab}) \\ -\psi_{4ab}-\psi_{1ab}\psi_{5ab} \\ \psi_{5ab}^2 \end{bmatrix},$$

$$\mathcal{G}(\psi_{ab}) = \mathrm{diag}\begin{bmatrix} -\psi_{4ab}-\psi_{1ab}\psi_{5ab} \\ \psi_{4ab}\psi_{5ab}^3 \\ -\psi_{5ab}^3 \\ 2\psi_{5ab}^2 \\ \psi_{5ab} \end{bmatrix}.$$

Then the mappings $\mathcal{T}_{\mathcal{S}}\left(\Xi_{\mathcal{S}}(\Delta)\mathcal{Y}_{ab}\right)$, $\mathcal{T}_{\mathcal{G}}\left(\Xi_{\mathcal{G}}(\Delta)\mathcal{Y}_{ab}\right)$ were defined as follows:

$$\mathcal{T}_{\mathcal{S}}\left(\Xi_{\mathcal{S}}(\Delta)\mathcal{Y}_{ab}\right)=\begin{bmatrix}\mathcal{Y}_{4ab}\mathcal{Y}_{5ab}\\ \mathcal{Y}_{3ab}\mathcal{Y}_{5ab}\left(-\Delta\mathcal{Y}_{4ab}-\mathcal{Y}_{1ab}\mathcal{Y}_{5ab}\right)\\ +\mathcal{Y}_{4ab}\left(\left(\mathcal{Y}_{2ab}\mathcal{Y}_{5ab}+\Delta\mathcal{Y}_{3ab}\right)\mathcal{Y}_{5ab}+\frac{3}{4}\Delta\mathcal{Y}_{4ab}\left(-\Delta\mathcal{Y}_{4ab}-\mathcal{Y}_{1ab}\mathcal{Y}_{5ab}\right)\right)\\ \left(\mathcal{Y}_{2ab}\mathcal{Y}_{5ab}+\Delta\mathcal{Y}_{3ab}\right)\mathcal{Y}_{5ab}+\frac{3}{4}\mathcal{Y}_{4ab}\left(-\Delta\mathcal{Y}_{4ab}-\mathcal{Y}_{1ab}\mathcal{Y}_{5ab}\right)\\ -\Delta\mathcal{Y}_{4ab}-\mathcal{Y}_{1ab}\mathcal{Y}_{5ab}\\ \mathcal{Y}_{5ab}^2\end{bmatrix},$$

$$\mathcal{T}_{\mathcal{G}}\left(\Xi_{\mathcal{G}}(\Delta)\mathcal{Y}_{ab}\right)=\mathrm{diag}\begin{bmatrix}-\Delta\mathcal{Y}_{4ab}-\mathcal{Y}_{1ab}\mathcal{Y}_{5ab}\\ \mathcal{Y}_{4ab}\mathcal{Y}_{5ab}^3\\ -\mathcal{Y}_{5ab}^3\\ 2\mathcal{Y}_{5ab}^2\\ \Delta\mathcal{Y}_{5ab}\end{bmatrix},$$

which allowed one to compute $\mathcal{Y}_\theta(t)$ and $\mathcal{M}_\theta(t)$.

Step 2. Using the above-obtained equation $\mathcal{Y}_\theta(t)=\mathcal{M}_\theta(t)\theta$, the following mappings were obtained

$$\mathcal{T}_{\mathcal{Z}}\left(\Xi_{\mathcal{Z}}(\mathcal{M}_\theta)\mathcal{Y}_\theta\right)=\mathrm{col}\left\{\mathcal{M}_\theta,\ \mathcal{Y}_{1\theta}\left(\mathcal{Y}_{2\theta}+\mathcal{Y}_{3\theta}\right),\ \mathcal{Y}_{1\theta}\mathcal{Y}_{4\theta},\ \mathcal{Y}_{1\theta}\mathcal{Y}_{3\theta},\ \mathcal{Y}_{3\theta}\mathcal{Y}_{5\theta},\ \mathcal{Y}_{4\theta}\mathcal{Y}_{5\theta},\ \mathcal{Y}_{1\theta},\ \mathcal{Y}_{5\theta}\right\},$$
$$\mathcal{T}_{\mathcal{X}}\left(\Xi_{\mathcal{X}}(\mathcal{M}_\theta)\mathcal{Y}_\theta\right)=\mathrm{blkdiag}\left\{\mathcal{M}_\theta,\ \mathcal{M}_\theta^2 I_5,\ \mathcal{M}_\theta,\ \mathcal{M}_\theta\right\},$$

therefore, we could calculate $\mathcal{Y}_{AB}(t)$ and $\mathcal{M}_{AB}(t)$.

Step 3. Having equation $\mathcal{Y}_{AB}(t)=\mathcal{M}_{AB}(t)\Theta_{AB}(\theta)$ at hand and using equations from the second statement of lemma 2, the values of $\mathcal{Y}_L(t)$ and $\mathcal{M}_L(t)$ were computed.

Step 4. Applying equation $\mathcal{Y}_\theta(t)=\mathcal{M}_\theta(t)\theta$ from the first step, the following mappings were obtained:

$$\mathcal{T}_{\mathcal{W}}\left(\Xi_{\mathcal{W}}(\mathcal{M}_\theta)\mathcal{Y}_\theta\right)=\mathrm{col}\left\{0,\ 0,\ \mathcal{Y}_{1\theta}\mathcal{Y}_{4\theta}\mathcal{Y}_{5\theta},\ \mathcal{Y}_{1\theta}\mathcal{Y}_{3\theta}\mathcal{Y}_{5\theta}\right\},$$
$$\mathcal{T}_{\mathcal{R}}\left(\Xi_{\mathcal{R}}(\mathcal{M}_\theta)\mathcal{Y}_\theta\right)=\mathrm{diag}\left\{\mathcal{M}_\theta,\ \mathcal{M}_\theta,\ \mathcal{M}_\theta^3,\ \mathcal{M}_\theta^3\right\},$$

which allowed one to calculate $\mathcal{Y}_{\psi_d}(t)$, $\mathcal{M}_{\psi_d}(t)$ and $\mathcal{Y}_{x_{\delta 0}}(t)$, $\mathcal{M}_{x_{\delta 0}}(t)$ on the basis of equations from the third statement of Lemma 2.

At this point, having $\mathcal{Y}_{AB}(t), \mathcal{Y}_L(t), \mathcal{Y}_{x_{\delta 0}}(t)$ at hand, equation (4.2) with measurable regressand $\mathcal{Y}_\kappa(t)$ and regressor $\mathcal{M}_\kappa(t)$ could be obtained, and the observer (2.4) with estimation law (4.4) was going to be implemented.

The unknown parameters of the system (6.1), the disturbance exosystem (2.2) and exosystem (2.3) parameters were picked as:

$$\begin{gathered}\theta=[1\ \ 0.5\ \ 0.75\ \ 0.25\ \ 0.5]^{\mathrm{T}},\ x_0=\begin{bmatrix}0 & 0 & -1 & 0\end{bmatrix}^{\mathrm{T}},\ x_{\delta 0}=\begin{bmatrix}-4 & 1\end{bmatrix}^{\mathrm{T}},\\ \mathcal{A}_\delta=\begin{bmatrix}0 & 1\\ -5 & 0\end{bmatrix},\ h_\delta^{\mathrm{T}}=\begin{bmatrix}1 & 0\end{bmatrix},\ \sigma\{\Gamma\}=\begin{bmatrix}-1 & -1 & -1 & -1\end{bmatrix}^{\mathrm{T}}.\end{gathered}\tag{6.3}$$

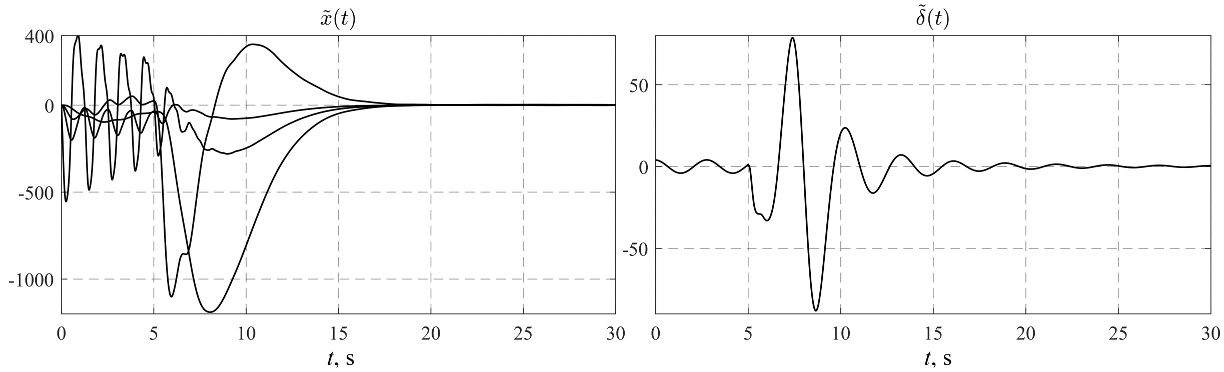

Fig. 2. Behavior of $\tilde{x}(t)$ and $\tilde{\delta}(t)$.

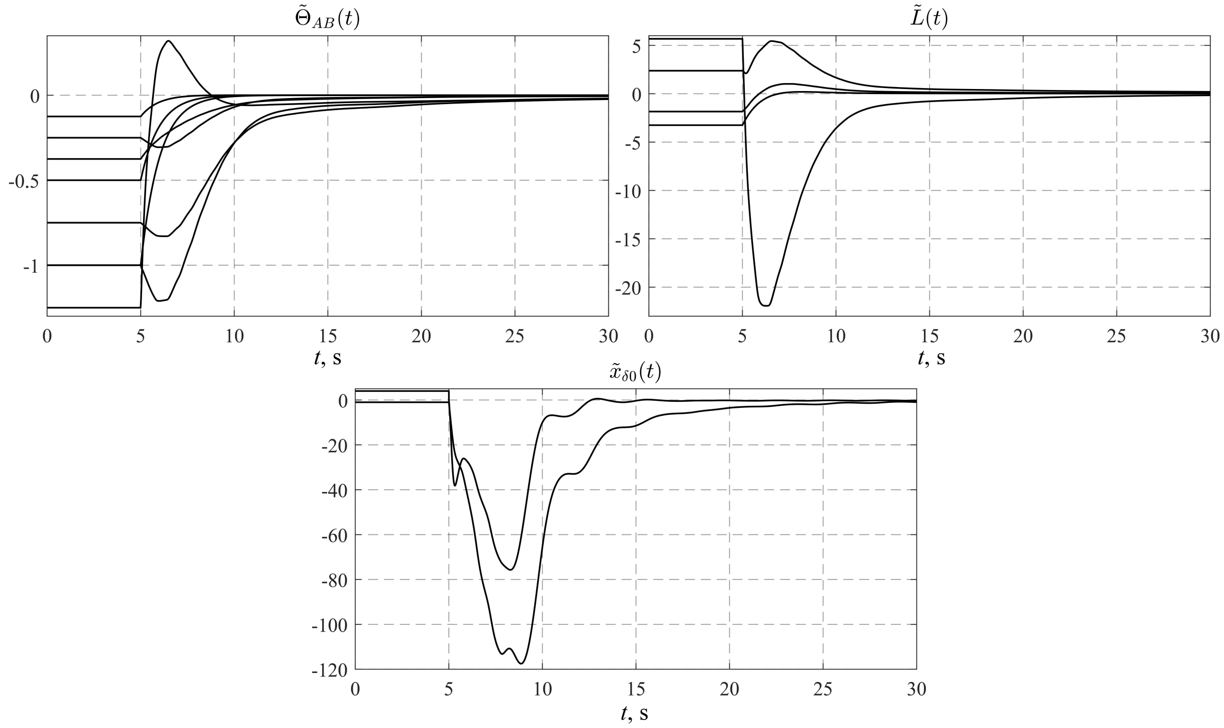

Fig. 3. Behaviour of $\tilde{\Theta}_{AB}(t),\ \tilde{L}(t),\ \tilde{x}_{\delta 0}(t)$.

The control signal $u(t)$ was obtained from a P-controller with the reference signal that was chosen by trial and error so as to ensure $\overline{\varphi}(t) \in \mathrm{PE}$:

$$\begin{aligned} & u(t) = 50\,(r(t) - y(t))\,, \\ & r(t) = 25\sin(10t) + 25\sin(20t) + 100\cos(0.1t)\,. \end{aligned} \tag{6.4}$$

The parameters of the filters (3.4)–(3.6), (4.3) and estimation law (4.4) were set as follows

$$\begin{gathered} \det\{sI_4 - A_K\} = (s+1)^4, \quad G = \begin{bmatrix} -4 & 1 \\ -2 & 0 \end{bmatrix}, \quad l = \begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathrm{T}}, \\ \beta = \begin{bmatrix} 15 & -5.5 \end{bmatrix}^{\mathrm{T}}, \quad k(t) = \begin{cases} 1, & \text{if } t < 5 \\ \Delta^{-1}(t), & \text{if } t \geqslant 5, \end{cases} \quad k_1 = 25, \quad k_2 = 0.1, \\ \rho = 10^{-4}, \quad \gamma_0 = 10^{-9}, \quad \gamma_1 = 1. \end{gathered} \tag{6.5}$$

Figure 2 shows the behavior of the state $\tilde{x}(t)$ and the external perturbation $\tilde{\delta}(t)$ observation errors.

The peaking in $\tilde{x}(t)$ over [5, 15] was caused primarily the fact that the error equation (4.1) with non-zero initial conditions [22] was included in the closed loop with $\hat{L}(t)\left(\hat{y}(t)-y(t)\right)$. The peaking in $\tilde{\delta}(t)$ could be explained by the fact that the behavior of $\tilde{x}_{\delta 0}(t)$ was affected by the perturbation $\epsilon(t)$.

Figure 3 depicts the behavior of the parametric errors $\tilde{\Theta}_{AB}(t)$, $\tilde{L}(t)$, and $\tilde{x}_{\delta 0}(t)$.

The oscillations of the obtained transients of the parametric errors $\tilde{\Theta}_{AB}(t)$, $\tilde{L}(t)$, $\tilde{x}_{\delta 0}(t)$ were caused by the influence of the exponentially decaying perturbation $\epsilon(t)$ from the parameterization (3.3). In general, the simulation results validated that the goal (2.5) was achieved.

7. CONCLUSION

An adaptive observer of state and perturbations for linear systems with overparameterization is developed. If the condition of the regressor persistent excitation (sufficient richness of the control/reference signal) is met, the solution provides exponential convergence to zero of the observation errors of the system state and the external perturbation generated by a known exosystem with unknown initial conditions. Unlike the closest analogues [10–12], the proposed observer allows one to reconstruct not virtual but physical state of the system represented in an arbitrary state space form.

The scopes of the further research can be:

— an application of the proposed observer to solve control problems with dynamic feedback;

— the relaxation of (1.9) by substituting (3.3) with a parametrization, which does not include $\epsilon(t)$ (a preliminary result for this problem has been obtained in comment 2 and [21]);

— the extension of the obtained results to systems with new, possibly nonlinear, models of the external perturbations;

— taking into consideration the additive disturbances, which affect the measurable output signal $y(t)$ directly;

— following [12], to reduce the transients peaking amplitude by estimation of the state $x(t)$ with the help of an algebraic equation instead of the differential one (a preliminary result for this problem has been obtained in [23]).

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APPENDIX

Proof of Lemma 1. The parameterization (3.3) is obtained as a combination of the results from [12, 24] with the dynamic regressor extension and mixing procedure from [14, 19]. The proof of lemma 1 is derived on the basis of Lemma 1 and Theorem 2 from [24]. To make it easier to understand the adopted notation and ensure that the results of the paper are self-contained, we next present the proof of this lemma in accordance with the one in [24]. In contrast to the results [24], in this paper, owing to Assumption 2, β is known, which allows one not to avoid overparameterization in (3.3) (see (A.23)).

Step 1. The following error is considered:

$$\tilde{\xi}(t) = \xi(t) - z(t) - \Omega(t)\psi_a(\theta) - P(t)\psi_b(\theta)\,. \tag{A.1}$$

The time derivative of (A.1) is written:

$$\begin{gathered}\dot{\tilde{\xi}}(t) = A_0\xi(t) + \psi_a(\theta)y(t) + \psi_b(\theta)u(t) + \psi_d(\theta)\delta(t) - A_K z(t) \\ -Ky(t) - (A_K\Omega(t) + I_n y(t))\psi_a(\theta) - (A_K P(t) + I_n u(t))\psi_b(\theta) \\ = A_0\xi(t) - A_K z(t) - Ky(t) - A_K\Omega(t)\psi_a(\theta) - A_K P(t)\psi_b(\theta) + \psi_d(\theta)\delta(t) \\ = A_K\tilde{\xi}(t) + \psi_d(\theta)\delta(t).\end{gathered} \tag{A.2}$$

The solution of equation (A.2) is obtained as

$$\tilde{\xi}(t) = e^{A_K(t-t_0)}\tilde{\xi}\left(t_0\right) + \overline{\delta}(t), \tag{A.3}$$

where the external perturbation $\overline{\delta}(t)$ is described as a set of equations

$$\begin{cases}\dot{\overline{\delta}}(t) = A_K\overline{\delta}(t) + \psi_d\left(\theta\right)\delta(t), \\ v_f(t) = C_0^{\mathrm{T}}\overline{\delta}(t).\end{cases} \tag{A.4}$$

Having substituted (A.3) into (A.1), it is written:

$$\begin{gathered}e^{A_K(t-t_0)}\tilde{\xi}\left(t_0\right) + \overline{\delta}(t) = \xi(t) - z(t) - \Omega(t)\psi_a\left(\theta\right) - P(t)\psi_b\left(\theta\right), \\ \Updownarrow \\ \xi(t) = e^{A_K(t-t_0)}\tilde{\xi}\left(t_0\right) + \overline{\delta}(t) + z(t) + \Omega(t)\psi_a\left(\theta\right) + P(t)\psi_b\left(\theta\right).\end{gathered} \tag{A.5}$$

Equation (A.5) is multiplied by C_0^{T} to obtain:

$$y(t) = C_0^{\mathrm{T}}\xi(t) = C_0^{\mathrm{T}}z(t) + C_0^{\mathrm{T}}\Omega(t)\psi_a\left(\theta\right) + C_0^{\mathrm{T}}P(t)\psi_b\left(\theta\right) + v_f(t) + C_0^{\mathrm{T}}e^{A_K(t-t_0)}\tilde{\xi}\left(t_0\right). \tag{A.6}$$

Considering (A.6), the function $\overline{q} = y(t) - C_0^{\mathrm{T}}z(t)$ is differentiated:

$$\dot{\overline{q}}(t) = C_0^{\mathrm{T}}\dot{\Omega}(t)\psi_a\left(\theta\right) + C_0^{\mathrm{T}}\dot{P}(t)\psi_b\left(\theta\right) + \dot{v}_f(t) + C_0^{\mathrm{T}}A_K e^{A_K(t-t_0)}\tilde{\xi}\left(t_0\right). \tag{A.7}$$

Step 2. The next aim is to parametrize the term $\dot{v}_f(t)$ of equation (A.7) as a linear regression equation with a measurable regressor. For this purpose, the system (A.4) is rewritten as a transfer function:

$$v_f(t) = C_0^{\mathrm{T}}(sI_n - A_K)^{-1}\psi_d\left(\theta\right)\delta(t) = W_f\left[\delta(t)\right]. \tag{A.8}$$

The derivative of the perturbation $\delta(t)$ is represented as:

$$\dot{\delta}(t) = h_\delta^{\mathrm{T}}\mathcal{A}_\delta x_\delta(t) + \delta\left(t_0\right)D_\delta(t), \tag{A.9}$$

where $D_\delta(t)$ is a Dirac delta function.

A virtual signal $\delta_d(t) = h_\delta^{\mathrm{T}}\mathcal{A}_\delta x_\delta(t)$ is introduced into consideration. Then the following equalities hold

$$\begin{gathered}\dot{x}_\delta(t) = \mathcal{A}_\delta x_\delta(t), \\ \delta_d(t) = \overline{h}_\delta^{\mathrm{T}}x_\delta(t),\ \overline{h}_\delta^{\mathrm{T}} = h_\delta^{\mathrm{T}}\mathcal{A}_\delta.\end{gathered} \tag{A.10}$$

Equation (A.8) is differentiated, and then (A.9), (A.10) are substituted into the obtained result to write:

$$\begin{aligned}\dot{v}_f &= sW_f\left[\delta(t)\right] = W_f\left[\dot{\delta}(t)\right] \\ &= W_f\left[h_\delta^{\mathrm{T}}A_\delta x_\delta(t) + \delta\left(t_0\right)D_\delta(t)\right] = \underbrace{W_f\left[\delta_d(t)\right]}_{v_f(t)} + W_f\left[\delta\left(t_0\right)D_\delta(t)\right].\end{aligned} \tag{A.11}$$

Thus, owing to the fact that the matrix A_K is a Hurwitz one, it is sufficient to parametrize $\upsilon_f(t)$ to parameterize $\dot{\upsilon}_f(t)$. For this purpose, an auxiliary signal $\zeta(t) = M_\delta x_\delta(t)$ is considered, where the transformation matrix M_δ is a solution of the Sylvester equation

$$M_\delta \mathcal{A}_\delta - G M_\delta = l\overline{h}_\delta^{\mathrm{T}}, \tag{A.12}$$

which has a unique solution [15, 16, 24] as, owing to Assumption 2, the pair $\left(h_\delta^{\mathrm{T}},\ \mathcal{A}_\delta\right)$ is observable and, following the premises of this lemma, $(G,\ l)$ is controllable and $\sigma\left\{\mathcal{A}_\delta\right\} \cap \sigma\left\{G\right\} = 0$.

We differentiate $\zeta(t)$ to obtain:

$$\dot{\zeta}(t) = M_\delta \mathcal{A}_\delta x_\delta(t) = G M_\delta x_\delta(t) + l\overline{h}_\delta^{\mathrm{T}} x_\delta(t) = G\zeta(t) + l\delta_d(t), \tag{A.13}$$

form which, considering $x_\delta(t) = M_\delta^{-1}\zeta(t)$, it follows that

$$\delta_d(t) = \overline{h}_\delta^{\mathrm{T}} M_\delta^{-1}\zeta = \beta^{\mathrm{T}}\zeta,\ \beta = \overline{h}_\delta^{\mathrm{T}} M_\delta^{-1}. \tag{A.14}$$

Taking into account (A.14), equation (A.11) is rewritten as:

$$\begin{aligned} \dot{\upsilon}_f(t) &= W_f\left[\beta^{\mathrm{T}}\zeta(t)\right] + W_f\left[\delta\left(t_0\right) D_\delta(t)\right] \\ &= \beta^{\mathrm{T}} W_f\left[\zeta(t)\right] + W_f\left[\delta\left(t_0\right) D_\delta(t)\right] = \beta^{\mathrm{T}}\zeta_w(t) + W_f\left[\delta\left(t_0\right) D_\delta(t)\right]. \end{aligned} \tag{A.15}$$

The signal $\upsilon_f(t)$ is filtered via (A.13) instead of $\delta_d(t)$:

$$\zeta_f(t) = (sI - G)^{-1} l\left[\upsilon_f(t)\right] + e^{G(t-t_0)}\zeta_f\left(t_0\right), \tag{A.16}$$

then, owing to $\zeta(t) = (sI - G)^{-1} l\left[\delta_d(t)\right] + e^{G(t-t_0)}\zeta\left(t_0\right)$, the following equality holds:

$$\begin{aligned} \zeta_w(t) = W_f\left[\zeta(t)\right] &= W_f\left[(sI - G)^{-1} l\left[\delta_d(t)\right] + e^{G(t-t_0)}\zeta\left(t_0\right)\right] \\ &= (sI - G)^{-1} l W_f\left[\delta_d(t)\right] + W_f\left[e^{G(t-t_0)}\zeta\left(t_0\right)\right] \\ &= (sI - G)^{-1} l\upsilon_f + W_f\left[e^{G(t-t_0)}\zeta\left(t_0\right)\right] \\ &= \zeta_f(t) - e^{G(t-t_0)}\zeta_f\left(t_0\right) + W_f\left[e^{G(t-t_0)}\zeta\left(t_0\right)\right]. \end{aligned} \tag{A.17}$$

Having substituted (A.17) into (A.15), it is written:

$$\dot{\upsilon}_f(t) = \beta^{\mathrm{T}}\zeta_f(t) - \beta^{\mathrm{T}} e^{G(t-t_0)}\zeta_f\left(t_0\right) + \beta^{\mathrm{T}} W_f\left[e^{G(t-t_0)}\xi\left(t_0\right)\right] + W_f\left[\delta\left(t_0\right) D_\delta(t)\right]. \tag{A.18}$$

The following observer of state $\zeta_f(t)$ is introduced:

$$\hat{\zeta}_f(t) = F(t) + H(t)\psi_b\left(\theta\right) + N(t)\psi_a\left(\theta\right) + ly(t). \tag{A.19}$$

Considering equations (A.7), (A.11), (A.16), (A.19), the error is differentiated $\tilde{\zeta}_f(t) = \zeta_f(t) - \hat{\zeta}_f(t)$ to obtain:

$$\begin{gathered}
\dot{\tilde{\zeta}}_f = G\zeta_f(t) + l\upsilon_f(t) - GF(t) - Gly(t) + lC_0^{\mathrm{T}}\dot{z}(t) \\
-\left(GH(t) - lC_0^{\mathrm{T}}\dot{P}(t)\right)\psi_b(\theta) - \left(GN(t) - lC_0^{\mathrm{T}}\dot{\Omega}(t)\right)\psi_a(\theta) \\
- lC_0^{\mathrm{T}}\dot{z}(t) - lC_0^{\mathrm{T}}\dot{\Omega}(t)\psi_a(\theta) - lC_0^{\mathrm{T}}\dot{P}(t)\psi_b(\theta) \\
- l\left(\upsilon_f(t) + W_f\left[\delta(t_0)D_\delta(t)\right]\right) - lC_0^{\mathrm{T}}A_K e^{A_K(t-t_0)}\tilde{\xi}(t_0) \\
= G\zeta_f(t) - \underbrace{GF(t) - Gly(t) - GH(t)\psi_b(\theta) - GN(t)\psi_a(\theta)}_{G\hat{\zeta}_f(t)} \\
- lW_f\left[\delta(t_0)D_\delta(t)\right] - lC_0^{\mathrm{T}}A_K e^{A_K(t-t_0)}\tilde{\xi}(t_0) \\
= G\tilde{\zeta}_f - lC_0^{\mathrm{T}}A_K e^{A_K(t-t_0)}\tilde{\xi}(t_0) - lW_f\left[\delta(t_0)D_\delta(t)\right].
\end{gathered} \tag{A.20}$$

The set of equations (A.20) is solved:

$$\tilde{\zeta}_f(t) = \zeta_f(t) - \hat{\zeta}_f(t) = e^{G(t-t_0)}\tilde{\zeta}_f(t_0) - \mathfrak{H}\left[C_0^{\mathrm{T}}A_K e^{A_K(t-t_0)}\tilde{\xi}(t_0) + W_f\left[\delta(t_0)D_\delta(t)\right]\right], \tag{A.21}$$

which allows one to rewrite (A.18) as follows:

$$\begin{gathered}
\dot{\upsilon}_f(t) = \beta^{\mathrm{T}}\hat{\zeta}_f(t) + \beta^{\mathrm{T}}e^{G(t-t_0)}\tilde{\zeta}_f(t_0) \\
-\beta^{\mathrm{T}}\mathfrak{H}\left[C_0^{\mathrm{T}}A_K e^{A_K(t-t_0)}\tilde{\xi}(t_0) + W_f\left[\delta(t_0)D_\delta(t)\right]\right] \\
-\beta^{\mathrm{T}}e^{G(t-t_0)}\zeta_f(t_0) + \beta^{\mathrm{T}}W_f\left[e^{G(t-t_0)}\xi(t_0)\right] + W_f\left[\delta(t_0)D_\delta(t)\right] \\
= \beta^{\mathrm{T}}\left(F(t) + ly(t)\right) + \beta^{\mathrm{T}}H(t)\psi_b(\theta) + \beta^{\mathrm{T}}N(t)\psi_a(\theta) \\
+\beta^{\mathrm{T}}e^{G(t-t_0)}\tilde{\zeta}_f(t_0) - \beta^{\mathrm{T}}\mathfrak{H}\left[C_0^{\mathrm{T}}A_K e^{A_K(t-t_0)}\tilde{\xi}(t_0) + W_f\left[\delta(t_0)D_\delta(t)\right]\right] \\
-\beta^{\mathrm{T}}e^{G(t-t_0)}\zeta_f(t_0) + \beta^{\mathrm{T}}W_f\left[e^{G(t-t_0)}\xi(t_0)\right] + W_f\left[\delta(t_0)D_\delta(t)\right],
\end{gathered} \tag{A.22}$$

where $\mathfrak{H}[.] = (sI_{n_\delta} - G)^{-1}l[.]$.

Equation (A.22) is substituted into (A.7) to obtain:

$$\begin{gathered}
\dot{\overline{q}}(t) = C_0^{\mathrm{T}}\dot{\Omega}(t)\psi_a(\theta) + C_0^{\mathrm{T}}\dot{P}(t)\psi_b(\theta) \\
+ \beta^{\mathrm{T}}\left(F(t) + ly(t)\right) + \beta^{\mathrm{T}}H(t)\psi_b(\theta) + \beta^{\mathrm{T}}N(t)\psi_a(\theta) \\
+ \beta^{\mathrm{T}}e^{G(t-t_0)}\tilde{\zeta}_f(t_0) - \beta^{\mathrm{T}}\mathfrak{H}\left[C_0^{\mathrm{T}}A_K e^{A_K(t-t_0)}\tilde{\xi}(t_0) + W_f\left[\delta(t_0)D_\delta(t)\right]\right] \\
- \beta^{\mathrm{T}}e^{G(t-t_0)}\zeta_f(t_0) + \beta^{\mathrm{T}}W_f\left[e^{G(t-t_0)}\xi(t_0)\right] \\
+ W_f\left[\delta(t_0)D_\delta(t)\right] + C_0^{\mathrm{T}}A_K e^{A_K(t-t_0)}\tilde{\xi}(t_0) \\
= \overline{\varphi}^{\mathrm{T}}(t)\eta(\theta) + \beta^{\mathrm{T}}\left(F(t) + ly(t)\right) + \overline{\varepsilon}(t),
\end{gathered} \tag{A.23}$$

where $\overline{\varepsilon}(t)$ are aggregated exponentially vanishing functions.

Step 3. The next aim is to transform the regression equation (A.23) into the form of (3.3) via application of the dynamic regressor extension and mixing procedure. For this purpose, considering (A.23), (3.5), the signal $\chi(t) = \overline{q}(t) - k_1\overline{q}_f(t)$ is differentiated to obtain:

$$\begin{gathered}
\dot{\chi}(t) = \overline{\varphi}^{\mathrm{T}}(t)\eta(\theta) + \beta^{\mathrm{T}}\left(F(t) + ly(t)\right) + \overline{\varepsilon}(t) - k_1\left(-k_1\overline{q}_f(t) + \overline{q}(t)\right) \\
= -k_1\chi(t) + \overline{\varphi}^{\mathrm{T}}(t)\eta(\theta) + \beta^{\mathrm{T}}\left(F(t) + ly(t)\right) + \overline{\varepsilon}(t).
\end{gathered} \tag{A.24}$$

The solution of the differential equation (A.24) allows one to write:

$$\overline{q}(t) - k_1\overline{q}_f(t) - \beta^{\mathrm{T}}\left(F_f(t) + ly_f(t)\right) = e^{-k_1(t-t_0)}\overline{q}\left(t_0\right) + \overline{\varphi}_f^{\mathrm{T}}(t)\eta\left(\theta\right) + \overline{\varepsilon}_f(t), \tag{A.25}$$

where $\dot{\overline{\varepsilon}}_f(t) = -k_1\overline{\varepsilon}_f(t) + k_1\overline{\varepsilon}(t),\ \overline{\varepsilon}_f\left(t_0\right) = 0$.

Owing to (A.25), the solution of the first differential equation from (3.4) satisfies the following equation

$$q(t) = \varphi(t)\eta\left(\theta\right) + \varepsilon(t), \tag{A.26}$$

where $\dot{\varepsilon}(t) = -k_2\varepsilon(t) + \overline{\varphi}_f(t)\left(\overline{\varepsilon}_f(t) + e^{-k_1(t-t_0)}\overline{q}\left(t_0\right)\right),\ \varepsilon\left(t_0\right) = 0_{2n}$.

Having multiplied equation (A.26) by $k(t)\mathrm{adj}\left\{\varphi(t)\right\}$ and applied the property

$$\mathrm{adj}\left\{\varphi(t)\right\}\varphi(t) = \det\left\{\varphi(t)\right\}I_{2n},$$

equation (3.3) is obtained with $\epsilon(t) = k(t)\mathrm{adj}\left\{\varphi(t)\right\}\varepsilon(t)$.

In accordance with Lemma 6.8 from [6], when $\overline{\varphi}(t)\in\mathrm{PE}$, it also holds that $\overline{\varphi}_f(t)\in\mathrm{PE}$. Following Proposition 1, when $\overline{\varphi}_f(t)\in\mathrm{PE}$, then it holds that $\Delta(t)\geqslant\Delta_{\min}>0$. Since the signals $y(t),\ u(t)$ are bounded by Assumption 1, owing to the stability of the filters (3.4)–(3.6), the inequality $\Delta_{\max}\geqslant\Delta(t)$ holds for all $t\geqslant t_0$. Then for all $t\geqslant t_0+T$ it holds that $\Delta_{\max}\geqslant\Delta(t)\geqslant\Delta_{\min}>0$, which completes the proof of lemma.

Proof of Lemma 2. According to definition 1 and hypothesis 1 and owing to

$$\begin{gathered}\Xi_{\mathcal{S}}\left(\Delta\right) = \overline{\Xi}_{\mathcal{S}}\left(\Delta\right)\Delta(t),\ \Xi_{\mathcal{G}}\left(\Delta\right) = \overline{\Xi}_{\mathcal{G}}\left(\Delta\right)\Delta(t),\\ \mathcal{Y}_{ab}(t) = \mathcal{L}_{ab}\mathcal{Y}(t) = \Delta(t)\mathcal{L}_{ab}\eta\left(\theta\right) = \Delta(t)\psi_{ab}\left(\theta\right),\\ \overline{\Xi}_{\mathcal{S}}\left(\Delta\right)\Delta(t)\psi_{ab}\left(\theta\right) = \overline{\Xi}_{\mathcal{S}}\left(\Delta\right)\mathcal{Y}_{ab}(t),\\ \overline{\Xi}_{\mathcal{G}}\left(\Delta\right)\Delta(t)\psi_{ab}\left(\theta\right) = \overline{\Xi}_{\mathcal{G}}\left(\Delta\right)\mathcal{Y}_{ab}(t)\end{gathered}$$

it follows from (3.9) that

$$\mathcal{T}_{\mathcal{S}}\left(\overline{\Xi}_{\mathcal{S}}\left(\Delta\right)\mathcal{Y}_{ab}\right) = \mathcal{T}_{\mathcal{G}}\left(\overline{\Xi}_{\mathcal{G}}\left(\Delta\right)\mathcal{Y}_{ab}\right)\theta. \tag{A.27}$$

Then, having multiplied (A.27) by $\mathrm{adj}\left\{\mathcal{T}_{\mathcal{G}}\left(\overline{\Xi}_{\mathcal{G}}\left(\Delta\right)\mathcal{Y}_{ab}\right)\right\}$, the following regression equation is obtained

$$\mathcal{Y}_{\theta}(t) = \mathcal{M}_{\theta}(t)\theta, \tag{A.28}$$

which is used together with (3.8) to write:

$$\mathcal{T}_{\mathcal{Z}}\left(\overline{\Xi}_{\mathcal{Z}}\left(\mathcal{M}_{\theta}\right)\mathcal{Y}_{\theta}\right) = \mathcal{T}_{\mathcal{X}}\left(\overline{\Xi}_{\mathcal{X}}\left(\mathcal{M}_{\theta}\right)\mathcal{Y}_{\theta}\right)\Theta_{AB}\left(\theta\right), \tag{A.29a}$$

$$\mathcal{T}_{\mathcal{W}}\left(\overline{\Xi}_{\mathcal{W}}\left(\mathcal{M}_{\theta}\right)\mathcal{Y}_{\theta}\right) = \mathcal{T}_{\mathcal{R}}\left(\overline{\Xi}_{\mathcal{R}}\left(\mathcal{M}_{\theta}\right)\mathcal{Y}_{\theta}\right)\psi_d\left(\theta\right). \tag{A.29b}$$

Having multiplied (A.29a) by $\mathrm{adj}\left\{\mathcal{T}_{\mathcal{X}}\left(\overline{\Xi}_{\mathcal{X}}\left(\mathcal{M}_{\theta}\right)\mathcal{Y}_{\theta}\right)\right\}$, the regression equation $\mathcal{Y}_{AB}(t) = \mathcal{M}_{AB}(t)\Theta_{AB}\left(\theta\right)$ is obtained.

The next aim it to parametrize equation with respect to $L\left(\theta\right)$. If Assumption 2 is met, then, following the generalized pole placement theory [15, 16], the vector $L\left(\theta\right)$ can be obtained as a solution of the following set of equations

$$\begin{cases} A^{\mathrm{T}}(\theta)M - M\Gamma = CB^{\mathrm{T}}(\theta),\\ B^{\mathrm{T}}(\theta) = L^{\mathrm{T}}(\theta)M,\end{cases} \tag{A.30}$$

which has a unique solution [15, 16], as, following Assumption 3, the pair $\left(A^{\mathrm{T}}(\theta),\ C\right)$ is controllable, the pair $\left(B^{\mathrm{T}}(\theta),\ \Gamma\right)$ is observable and $\sigma\left\{A^{\mathrm{T}}(\theta)\right\}\cap\sigma\left\{\Gamma\right\}=0$.

Having vectorized the first equation from (A.30) and considered the property $vec\,(AB) = (I\otimes A)\,vec\,(B) = \left(B^{\mathrm{T}}\otimes I\right)vec\,(A)$, it is obtained that:

$$\left(I_n\otimes A^{\mathrm{T}}(\theta)-\Gamma^{\mathrm{T}}\otimes I_n\right)vec\,(M) = vec\left(CB^{\mathrm{T}}(\theta)\right). \tag{A.31}$$

As equations (A.30), (A.31) have unique solutions, then

$$\det\left\{I_n\otimes A^{\mathrm{T}}(\theta)-\Gamma^{\mathrm{T}}\otimes I_n\right\}\neq 0,$$

and therefore, having multiplied (A.31) by an adjoint matrix $\mathrm{adj}\left\{I_n\otimes A^{\mathrm{T}}(\theta)-\Gamma^{\mathrm{T}}\otimes I_n\right\}$, it is written:

$$\begin{gathered}\det\left\{I_n\otimes A^{\mathrm{T}}(\theta)-\Gamma^{\mathrm{T}}\otimes I_n\right\}vec\,(M)\\ = \mathrm{adj}\left\{I_n\otimes A^{\mathrm{T}}(\theta)-\Gamma^{\mathrm{T}}\otimes I_n\right\}vec\left(CB^{\mathrm{T}}(\theta)\right).\end{gathered} \tag{A.32}$$

The obtained result is devectorized $\left(vec^{-1}\left\{.\right\}\right)$ and substituted into the second equation of (A.30):

$$\begin{gathered}\underbrace{\det\left\{I_n\otimes A^{\mathrm{T}}(\theta)-\Gamma^{\mathrm{T}}\otimes I_n\right\}B(\theta)}_{\mathcal{Q}(\Theta_{AB})}\\ = \underbrace{vec^{-1}\Big\{\mathrm{adj}\left\{I_n\otimes A^{\mathrm{T}}(\theta)-\Gamma^{\mathrm{T}}\otimes I_n\right\}vec\left(CB^{\mathrm{T}}(\theta)\right)\Big\}^{\mathrm{T}}}_{\mathcal{P}(\Theta_{AB})}L(\theta),\end{gathered} \tag{A.33}$$

where $\det\left\{\mathcal{P}\left(\Theta_{AB}\right)\right\}\neq 0$.

The following equalities are introduced:

$$\begin{gathered}\mathcal{M}_{AB}(t)A^{\mathrm{T}}(\theta) = vec^{-1}\left(\mathcal{L}_{A^{\mathrm{T}}}\mathcal{D}_{\Phi}\mathcal{Y}_{AB}(t)\right),\\ \mathcal{M}_{AB}(t)B^{\mathrm{T}}(\theta) = \left[\mathcal{L}_{B}\mathcal{D}_{\Phi}\mathcal{Y}_{AB}(t)\right]^{\mathrm{T}},\\ \mathcal{M}_{AB}(t)B(\theta) = \mathcal{L}_{B}\mathcal{D}_{\Phi}\mathcal{Y}_{AB}(t).\end{gathered} \tag{A.34}$$

Having multiplied (A.33) by $\Pi_L\left(\mathcal{M}_{AB}\right) = \mathcal{M}_{AB}^{n^2+1}I_n$, used the properties $c^n\det\left\{A\right\} = \det\left\{cA\right\}$, $c^{n-1}\mathrm{adj}\left\{A\right\} = \mathrm{adj}\left\{cA\right\}$, $A\in\mathbb{R}^{n\times n}$ and substituted (A.34), it is obtained:

$$\begin{gathered}\mathcal{T}_{\mathcal{P}}\left(\Xi_{\mathcal{P}}\left(\mathcal{M}_{AB}\right)\Theta_{AB}\right) = \Pi_L\left(\mathcal{M}_{AB}\right)\mathcal{P}\left(\Theta_{AB}\right) = \mathcal{M}_{AB}^{n^2+1}\mathcal{P}\left(\Theta_{AB}\right)\\ = \mathcal{M}_{AB}^{n^2+1}vec^{-1}\Big\{\mathrm{adj}\left\{I_n\otimes A^{\mathrm{T}}(\theta)-\Gamma^{\mathrm{T}}\otimes I_n\right\}vec\left(CB^{\mathrm{T}}(\theta)\right)\Big\}^{\mathrm{T}}\\ = vec^{-1}\Big\{\mathcal{M}_{AB}\mathrm{adj}\{I_n\otimes vec^{-1}\left(\mathcal{L}_{A^{\mathrm{T}}}\mathcal{D}_{\Phi}\mathcal{Y}_{AB}\right)-\mathcal{M}_{AB}\Gamma^{\mathrm{T}}\otimes I_n\}\,vec\left(C(\mathcal{L}_{B}\mathcal{D}_{\Phi}\mathcal{Y}_{AB})^{\mathrm{T}}\right)\Big\}^{\mathrm{T}},\\ \mathcal{T}_{\mathcal{Q}}\left(\Xi_{\mathcal{Q}}\left(\mathcal{M}_{AB}\right)\Theta_{AB}\right) = \Pi_L\left(\mathcal{M}_{AB}\right)\mathcal{Q}\left(\Theta_{AB}\right) = \mathcal{M}_{AB}^{n^2+1}\mathcal{Q}\left(\Theta_{AB}\right)\\ = \mathcal{M}_{AB}^{n^2+1}\det\left\{I_n\otimes A^{\mathrm{T}}(\theta)-\Gamma^{\mathrm{T}}\otimes I_n\right\}B(\theta)\\ = \det\left\{I_n\otimes vec^{-1}\left(\mathcal{L}_{A^{\mathrm{T}}}\mathcal{D}_{\Phi}\mathcal{Y}_{AB}(t)\right)-\mathcal{M}_{AB}(t)\Gamma^{\mathrm{T}}\otimes I_n\right\}\left(\mathcal{L}_{B}\mathcal{D}_{\Phi}\mathcal{Y}_{AB}(t)\right),\end{gathered} \tag{A.35}$$

where $\Xi_{\mathcal{P}}\left(\mathcal{M}_{AB}\right) = \Xi_{\mathcal{Q}}\left(\mathcal{M}_{AB}\right) = \mathcal{M}_{AB}(t)$.

The following regression equation is written on the basis of equations (A.33) and (A.35):

$$\mathcal{T}_{\mathcal{Q}}\left(\overline{\Xi}_{\mathcal{Q}}\left(\mathcal{M}_{AB}\right)\mathcal{Y}_{AB}\right)=\mathcal{T}_{\mathcal{P}}\left(\overline{\Xi}_{\mathcal{P}}\left(\mathcal{M}_{AB}\right)\mathcal{Y}_{AB}\right)L(\theta), \tag{A.36}$$

where $\overline{\Xi}_{\mathcal{P}}\left(\mathcal{M}_{AB}\right)=\overline{\Xi}_{\mathcal{Q}}\left(\mathcal{M}_{AB}\right)=1$.

Having multiplied (A.36) by $\operatorname{adj}\left\{\mathcal{T}_{\mathcal{P}}\left(\overline{\Xi}_{\mathcal{P}}\left(\mathcal{M}_{AB}\right)\mathcal{Y}_{AB}\right)\right\}$, the regression equation $\mathcal{Y}_L(t)=\mathcal{M}_L(t)L(\theta)$ is obtained.

The next aim is to derive the regression equation with respect to $x_{\delta 0}$. Using the properties of the vectorization operation

$$vec\left(\psi_d(\theta)h_\delta^{\mathrm{T}}\Phi_\delta(t)x_{\delta 0}\right)=\underbrace{\left(x_{\delta 0}^{\mathrm{T}}\otimes\psi_d(\theta)\right)}_{n\times n_\delta}\underbrace{vec\left(h_\delta^{\mathrm{T}}\Phi_\delta\right)}_{n_\delta},$$
$$vec\left(\left(x_{\delta 0}^{\mathrm{T}}\otimes\psi_d(\theta)\right)vec\left(h_\delta^{\mathrm{T}}\Phi_\delta(t)\right)\right)=\underbrace{\left(h_\delta^{\mathrm{T}}\Phi_\delta(t)\otimes I_n\right)}_{n\times nn_\delta}\underbrace{vec\left(x_{\delta 0}^{\mathrm{T}}\otimes\psi_d(\theta)\right)}_{nn_\delta},$$

equation (3.1) is rewritten as follows:

$$\begin{aligned}\dot{\xi}(t)&=A_0\xi(t)+\psi_a(\theta)y(t)+\psi_b(\theta)u(t)\\&+\left(h_\delta^{\mathrm{T}}\Phi_\delta(t)\otimes I_n\right)vec\left(x_{\delta 0}^{\mathrm{T}}\otimes\psi_d(\theta)\right).\end{aligned} \tag{A.37}$$

The following error is introduced:

$$e(t)=\xi(t)-z(t)-\Omega(t)\psi_a(\theta)-P(t)\psi_b(\theta)-V(t)vec\left(x_{\delta 0}^{\mathrm{T}}\otimes\psi_d(\theta)\right). \tag{A.38}$$

Having differentiated (A.38), equation $\dot{e}(t)=A_Ke(t)$ is obtained in a similar way as (A.2). Then, having multiplied (A.38) by C_0^{T}, it is written:

$$\begin{aligned}\overline{q}(t)&=C_0^{\mathrm{T}}e^{A_K(t-t_0)}e\left(t_0\right)+C_0^{\mathrm{T}}\Omega(t)\psi_a(\theta)\\&+C_0^{\mathrm{T}}P(t)\psi_b(\theta)+C_0^{\mathrm{T}}V(t)vec\left(x_{\delta 0}^{\mathrm{T}}\otimes\psi_d(\theta)\right).\end{aligned} \tag{A.39}$$

Using the properties

$$x_{\delta 0}^{\mathrm{T}}\otimes\psi_d(\theta)=\psi_d(\theta)x_{\delta 0},$$
$$vec\left(\psi_d(\theta)x_{\delta 0}\right)=\underbrace{\left(I_{n_\delta}\otimes\psi_d(\theta)\right)}_{nn_\delta\times n_\delta}x_{\delta 0},$$

equation (A.39) is transformed into

$$\begin{aligned}\overline{q}(t)&=C_0^{\mathrm{T}}e^{A_K(t-t_0)}e\left(t_0\right)+C_0^{\mathrm{T}}\Omega(t)\psi_a(\theta)\\&+C_0^{\mathrm{T}}P(t)\psi_b(\theta)+C_0^{\mathrm{T}}V(t)vec\left(x_{\delta 0}^{\mathrm{T}}\otimes\psi_d(\theta)\right).\end{aligned} \tag{A.40}$$

To compensate for the unknown terms $C_0^{\mathrm{T}}\Omega(t)\psi_a(\theta)+C_0^{\mathrm{T}}P(t)\psi_b(\theta)$, the following auxiliary signal is introduced

$$\begin{aligned}\overline{p}_e(t)&=\Delta(t)C_0^{\mathrm{T}}\Omega(t)\psi_a(\theta)+\Delta(t)C_0^{\mathrm{T}}P(t)\psi_b(\theta)\\&=C_0^{\mathrm{T}}\Omega(t)\mathcal{L}_a\mathcal{Y}(t)+C_0^{\mathrm{T}}P(t)\mathcal{L}_b\mathcal{Y}(t),\end{aligned} \tag{A.41}$$

where

$$\mathcal{L}_a\mathcal{Y}(t) = \mathcal{L}_a\Delta(t)\eta(\theta) = \Delta(t)\mathcal{L}_a\eta(\theta) = \Delta(t)\psi_a(\theta),$$
$$\mathcal{L}_b\mathcal{Y}(t) = \Delta(t)\mathcal{L}_b\eta(\theta) = \Delta(t)\psi_b(\theta).$$

Having multiplied (A.40) by $\Delta(t)$ and subtracted (A.41) from the obtained result, it is written:

$$\begin{aligned} p(t) &= \Delta(t)\overline{q}(t) - \overline{p}_e(t) \\ &= \Delta(t)C_0^{\mathrm{T}}V(t)\left(I_{n_\delta}\otimes\psi_d(\theta)\right)x_{\delta 0} + \Delta(t)C_0^{\mathrm{T}}e^{A_K(t-t_0)}e\left(t_0\right) \\ &= \Delta(t)C_0^{\mathrm{T}}V(t)\left(I_{n_\delta}\otimes\psi_d(\theta)\right)x_{\delta 0} + \Delta(t)C_0^{\mathrm{T}}e^{A_K(t-t_0)}e\left(t_0\right). \end{aligned} \tag{A.42}$$

To implement the multiplier $\psi_d(\theta)$ indirectly, equation (A.29b) is multiplied by $\operatorname{adj}\left\{\mathcal{T}_{\mathcal{R}}\left(\overline{\Xi}_{\mathcal{R}}\left(\mathcal{M}_\theta\right)\mathcal{Y}_\theta\right)\right\}$:

$$\begin{gathered} \mathcal{Y}_{\psi_d}(t) = \mathcal{M}_{\psi_d}(t)\psi_d(\theta), \\ \mathcal{Y}_{\psi_d}(t) = \operatorname{adj}\left\{\mathcal{T}_{\mathcal{R}}\left(\overline{\Xi}_{\mathcal{R}}\left(\mathcal{M}_\theta\right)\mathcal{Y}_\theta\right)\right\}\mathcal{T}_{\mathcal{W}}\left(\overline{\Xi}_{\mathcal{W}}\left(\mathcal{M}_\theta\right)\mathcal{Y}_\theta\right), \\ \mathcal{M}_{\psi_d}(t) = \det\left\{\mathcal{T}_{\mathcal{R}}\left(\overline{\Xi}_{\mathcal{R}}\left(\mathcal{M}_\theta\right)\mathcal{Y}_\theta\right)\right\}. \end{gathered} \tag{A.43}$$

The multiplication of (A.42) by $\mathcal{M}_{\psi_d}(t)$ and substitution of (A.43) into the obtained result allow one to write:

$$\begin{aligned} \mathcal{M}_{\psi_d}(t)p(t) &= \mathcal{M}_{\psi_d}(t)\Delta(t)C_0^{\mathrm{T}}V(t)\left(I_{n_\delta}\otimes\psi_d(\theta)\right)x_{\delta 0} \\ &= \Delta(t)C_0^{\mathrm{T}}V(t)\left(I_{n_\delta}\otimes\mathcal{Y}_{\psi_d}(t)\right)x_{\delta 0}. \end{aligned} \tag{A.44}$$

Having filtered (A.44) via (4.3) and multiplied the obtained result by $\operatorname{adj}\{V_f(t)\}$, the regression equation $\mathcal{Y}_{x_{\delta 0}}(t) = \mathcal{M}_{x_{\delta 0}}(t)x_{\delta 0}$ is obtained, which completes proof of statement that equations (4.2) can be formed on the basis of the measurable signals.

Following Lemma 1, if $\overline{\varphi}(t)\in\mathrm{PE}$, then for all $t \geqslant t_0 + T$ it holds that $\Delta(t) \geqslant \Delta_{\min} > 0$, and, owing to Hypotheses 1–3 and proved inequalities:

$$\begin{gathered} \det{}^2\{\mathcal{X}(\theta)\} > 0, \quad \det{}^2\{\mathcal{R}(\theta)\} > 0, \\ \det{}^2\{\mathcal{G}(\psi_{ab})\} > 0, \quad \det{}^2\{\mathcal{P}(\Theta_{AB})\} > 0, \\ \det\{\Pi_\theta(\Delta)\} \geqslant \Delta^{\ell_\theta}(t), \quad \det\{\Pi_\Theta(\mathcal{M}_\theta)\} \geqslant \mathcal{M}_\theta^{\ell_\Theta}(t), \\ \det\{\Pi_{\psi_d}(\mathcal{M}_\theta)\} \geqslant \mathcal{M}_\theta^{\ell_{\psi_d}}(t), \quad \det\{\Pi_L(\mathcal{M}_{AB})\} \geqslant \mathcal{M}_{AB}^{n^3+n}(t), \end{gathered}$$

we have that, if $\overline{\varphi}(t)\in\mathrm{PE}$, then for all $t \geqslant t_0 + T$ the following holds:

$$\begin{gathered} |\mathcal{M}_\theta(t)| = \left|\det\left\{\mathcal{T}_{\mathcal{G}}\left(\overline{\Xi}_{\mathcal{G}}(\Delta)\mathcal{Y}_{ab}\right)\right\}\right| = \left|\det\{\Pi_\theta(\Delta)\}\det\{\mathcal{G}(\psi_{ab})\}\right| \\ \geqslant \left|\det\{\mathcal{G}(\psi_{ab})\}\right|\Delta_{\min}^{\ell_\theta} = \underline{\mathcal{M}_\theta} > 0, \\ |\mathcal{M}_{AB}(t)| = \left|\det\left\{\mathcal{T}_{\mathcal{X}}\left(\overline{\Xi}_{\mathcal{X}}(\mathcal{M}_\theta)\mathcal{Y}_\theta\right)\right\}\right| = \left|\det\{\Pi_\Theta(\mathcal{M}_\theta)\}\det\{\mathcal{X}(\theta)\}\right| \\ \geqslant \left|\det{}^{\ell_\Theta}\{\mathcal{G}(\psi_{ab})\}\right|\left|\det\{\mathcal{X}(\theta)\}\right|\Delta_{\min}^{\ell_\theta\ell_\Theta} = \underline{\mathcal{M}_{AB}} > 0, \\ |\mathcal{M}_{\psi_d}(t)| = \det\left\{\mathcal{T}_{\mathcal{R}}\left(\overline{\Xi}_{\mathcal{R}}(\mathcal{M}_\theta)\mathcal{Y}_\theta\right)\right\} = \left|\det\{\Pi_{\psi_d}(\mathcal{M}_\theta)\}\det\{\mathcal{R}(\theta)\}\right| \\ \geqslant \left|\det{}^{\ell_{\psi_d}}\{\mathcal{G}(\psi_{ab})\}\right|\left|\det\{\mathcal{R}(\theta)\}\right|\Delta_{\min}^{\ell_\theta\ell_{\psi_d}} = \underline{\mathcal{M}_{\psi_d}} > 0, \\ |\mathcal{M}_L(t)| = \left|\det\left\{\mathcal{T}_{\mathcal{P}}\left(\overline{\Xi}_{\mathcal{P}}(\mathcal{M}_{AB})\mathcal{Y}_{AB}\right)\right\}\right| = \left|\det\{\Pi_L(\mathcal{M}_{AB})\}\det\{\mathcal{P}(\Theta_{AB})\}\right| \\ \geqslant \left|\det\{\mathcal{P}(\Theta_{AB})\}\right|\mathcal{M}_{AB}^{n^3+n} \geqslant \left|\det\{\mathcal{P}(\Theta_{AB})\}\right|\underline{\mathcal{M}_{AB}^{n^3+n}} = \underline{\mathcal{M}_L} > 0. \end{gathered}$$

To obtain the lower bound for the regressor $\mathcal{M}_{x\delta 0}(t)$, first of all, such bound needs to be derived for the solution of the differential equation for $V_f(t)$ in case $\overline{\varphi}(t)\in\mathrm{PE}$ and $\left(h_\delta^{\mathrm{T}}\Phi_\delta(t)\otimes I_n\right)\in\mathrm{PE}$:

$$\begin{aligned}
V_f(t) &= \int\limits_{t_0}^{t} e^{-k_2(t-\tau)}\Delta^2(\tau)(I_{n_\delta}\otimes\mathcal{Y}_{\psi_d}(\tau))^{\mathrm{T}}V^{\mathrm{T}}(\tau)C_0C_0^{\mathrm{T}}V(\tau)\left(I_{n_\delta}\otimes\mathcal{Y}_{\psi_d}(\tau)\right)d\tau\\
&= (I_{n_\delta}\otimes\psi_d(\theta))^{\mathrm{T}}\int\limits_{t_0}^{t} e^{-k_2(t-\tau)}\mathcal{M}_{\psi_d}^2(\tau)\Delta^2(\tau)V^{\mathrm{T}}(\tau)C_0C_0^{\mathrm{T}}V(\tau)d\tau\left(I_{n_\delta}\otimes\psi_d(\theta)\right)\\
&\geqslant \underline{\mathcal{M}_{\psi_d}^2}\Delta_{\min}^2(I_{n_\delta}\otimes\psi_d(\theta))^{\mathrm{T}}\int\limits_{t_0}^{t} e^{-k_2(t-\tau)}V^{\mathrm{T}}(\tau)C_0C_0^{\mathrm{T}}V(\tau)d\tau\left(I_{n_\delta}\otimes\psi_d(\theta)\right)\\
&\geqslant \underline{\mathcal{M}_{\psi_d}^2}\Delta_{\min}^2(I_{n_\delta}\otimes\psi_d(\theta))^{\mathrm{T}}\left[\int\limits_{t_0}^{t-\overline{k}T} e^{-k_2(t-\tau)}V^{\mathrm{T}}(\tau)C_0C_0^{\mathrm{T}}V(\tau)d\tau\right.\\
&\qquad\left.+\sum_{k=1}^{\overline{k}}\int\limits_{t-kT}^{t-kT+T} e^{-k_2(t-\tau)}V^{\mathrm{T}}(\tau)C_0C_0^{\mathrm{T}}V(\tau)d\tau\right]\left(I_{n_\delta}\otimes\psi_d(\theta)\right)\\
&\geqslant \underline{\mathcal{M}_{\psi_d}^2}\Delta_{\min}^2e^{-k_2t}(I_{n_\delta}\otimes\psi_d(\theta))^{\mathrm{T}}\sum_{k=1}^{\overline{k}}\int\limits_{t-kT}^{t-kT+T} e^{k_2\tau}V^{\mathrm{T}}(\tau)C_0C_0^{\mathrm{T}}V(\tau)d\tau(I_{n_\delta}\otimes\psi_d(\theta))\\
&\geqslant \underline{\mathcal{M}_{\psi_d}^2}\Delta_{\min}^2(I_{n_\delta}\otimes\psi_d(\theta))^{\mathrm{T}}\sum_{k=1}^{\overline{k}}e^{-k_2kT}\int\limits_{t-kT}^{t-kT+T} V^{\mathrm{T}}(\tau)C_0C_0^{\mathrm{T}}V(\tau)d\tau\left(I_{n_\delta}\otimes\psi_d(\theta)\right),
\end{aligned}$$

where $\overline{k}\geqslant k\geqslant 1$ are integers.

In accordance with Lemma 6.8 from [6], if $\left(h_\delta^{\mathrm{T}}\Phi_\delta(t)\otimes I_n\right)\in\mathrm{PE}$, then the following inequality holds

$$\int\limits_{t}^{t+T} V^{\mathrm{T}}(\tau)C_0C_0^{\mathrm{T}}V(\tau)d\tau\geqslant\alpha I_{nn_\delta}, \tag{A.45}$$

and, using the properties of the Kronecker product, it is obtained that:

$$\begin{aligned}
(I_{n_\delta}\otimes\psi_d(\theta))^{\mathrm{T}}\underbrace{(I_{n_\delta}\otimes\psi_d(\theta))}_{nn_\delta\times n_\delta} &= \left(I_{n_\delta}^{\mathrm{T}}\otimes\psi_d^{\mathrm{T}}(\theta)\right)(I_{n_\delta}\otimes\psi_d(\theta))\\
&= I_{n_\delta}\otimes\psi_d^{\mathrm{T}}(\theta)\psi_d(\theta)=\underbrace{\psi_d^{\mathrm{T}}(\theta)\psi_d(\theta)}_{>0}I_{n_\delta}.
\end{aligned}\tag{A.46}$$

Then for all $t\geqslant t_0+T$ it holds that:

$$V_f(t)\geqslant\underbrace{\underline{\mathcal{M}_{\psi_d}^2}\Delta_{\min}^2\alpha\sum_{k=1}^{\overline{k}}e^{-k_2kT}\psi_d^{\mathrm{T}}(\theta)\psi_d(\theta)}_{>0}I_{n_\delta}\geqslant\sqrt[n_\delta]{\underline{\mathcal{M}_{x\delta 0}}}I_{n_\delta}, \tag{A.47}$$

from which for all $t\geqslant t_0+T$ we have $\mathcal{M}_{x\delta 0}\geqslant\underline{\mathcal{M}_{x\delta 0}}>0$, which allows one to obtain:

$$\forall t\geqslant t_0+T\ \ |\mathcal{M}_\kappa(t)|=\left|\mathcal{M}_{AB}^{n_\Theta}(t)\mathcal{M}_L^n(t)\mathcal{M}_{x\delta 0}^{n_\delta}(t)\right|\geqslant\underline{\mathcal{M}_{AB}^{n_\Theta}}\underline{\mathcal{M}_L^n}\underline{\mathcal{M}_{x\delta 0}^{n_\delta}}=\underline{\mathcal{M}_\kappa}>0. \tag{A.48}$$

This completes proof of Lemma 2.

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