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Density Systems: Analysis and Control

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Abstract—This paper considers a class of systems called density systems. For such systems, the derivative of a quadratic function depends on some function termed the density function. The latter function is used to define the properties of the space affecting the behavior of the systems under consideration. The role of density systems in control law design is shown. Control systems are constructed for plants with known and unknown parameters. The theoretical results are illustrated by numerical simulation.

Keywords: dynamic system, quadratic function, stability, control

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1. INTRODUCTION

We study a class of dynamic normal-from systems whose right-hand side depends on some function defining the properties of the space and affecting the system behavior. This function will be called the density function. All relevant definitions will be provided in the main part of the paper.

A particular class of such systems was considered in [1–8]. The (in)stability of a system $\dot{x} = f(x)$ in the plane was first analyzed by introducing a new system $\dot{x} = \rho(x)f(x)$ with an auxiliary function $\rho(x) > 0$ for all x in the pioneering book [1]. Then, the (in)stability of such systems was studied using the properties of the divergence and phase velocity vector flow; see [2–8]. The function $\rho(x)$ was called the density function [4]. In [5–8], a connection was established between the obtained results and the continuity equation [9], which arises in electromagnetism, wave theory, fluid dynamics, mechanics of deformable solids, and quantum mechanics.

Several control methods proposed in [10–14] ensure that controlled signals are in given sets. This goal is achieved by introducing an auxiliary function through an appropriate control law; the form of this function determines the corresponding properties in the closed loop system. For example, funnel control and control with prescribed performance were presented in [10, 12] and in [11], respectively; under these control laws, transients belong to a pipe converging to the neighborhood of zero. The method proposed in [13, 14] generalizes the results of [10–12], ensuring that the output variables will be in a given pipe (possibly asymmetric with respect to the equilibrium and without convergence to a given constant).

This paper is devoted to a class of systems depending, explicitly or implicitly, on a density function. This function will be used to define the density of the space in the sense of distinguishing (in)stability domains and forbidden domains (where the system has no solutions). The behavior of the system under consideration will depend on the value of the density function.

The presentation has the following distinctive features:

(1) In contrast to [1–8], the density function is not necessarily multiplied by the entire right-hand side of the system.

(2) In contrast to [10–14], the density function may be present implicitly on the right-hand side of the system.

(3) In contrast to [10–14], the density function can ensure system solutions in an unbounded set with forbidden domains and the boundaries of these sets can be defined by continuous (under some assumptions, even discontinuous) functions in all arguments.

The remainder of this paper is organized as follows. Section 2 presents motivating examples as well as the definitions of a density function and a density system. Some properties of these systems are also demonstrated. In Section 3, the theoretical results are applied to design control laws for plants with known and unknown parameters. In addition, the control schemes are numerically simulated to confirm theoretical conclusions.

We adopt the following *notations*: $\mathbb{R}^n$ is the n-dimensional Euclidean space with the norm $|\cdot|$; $\mathbb{R}_+$ ($\mathbb{R}_-$) is the set of positive (negative, respectively) real numbers; $p = d/dt$ indicates the differentiation operator; finally, λ stands for the complex variable.

2. MOTIVATING EXAMPLES. DEFINITIONS

Prior to introducing the main definitions, we consider two examples as follows.

Example 1. As is well known, the solutions of the system

$$\dot{x} = -x, \quad x \in \mathbb{R}, \tag{1}$$

asymptotically vanish. Multiplying the right-hand side of this system by a function $\rho(x,t)$: $\mathbb{R} \times [0,+\infty) \to \mathbb{R}$ that is continuous in t and locally Lipschitz in x, we write (1) as

$$\dot{x} = -\rho(x,t)x. \tag{2}$$

Obviously, the behavior of system (2) depends on the properties of $\rho(x,t)$. The function $\rho(x,t)$ can be used to define some properties and constraints in the space (x,t), thereby affecting the quality of transients of the original system (1) and changing them qualitatively. In this context, $\rho(x,t)$ will be called a *density function*. Here are several examples of this function and the corresponding behavior of the new system (2).

1. The density function $\rho(x,t) = \alpha > 0$ allows preserving the single equilibrium $x = 0$, takes the same positive value for any x and t, and hence has no qualitative effect on the exponential stability of the trajectories of the original system (1) (see Fig. 1 on the left) except the rate of convergence of the solution of (2) to the equilibrium depending on the value α. Indeed, choosing the Lyapunov function $V = 0.5x^2$ yields $\dot{V} = -\alpha x^2 < 0$ in the domain $D_S = \mathbb{R} \setminus \{0\}$.

2. Consider the density function $\rho(x,t) = \frac{\alpha}{w(t)-|x(t)|}$ with a continuous function $w(t) > 0$. The function $\rho(x,t)$ takes positive values in the domain $D_S = \{x \in \mathbb{R} : -w < x < w\}$, and $\rho(x,t) \to +\infty$ as $|x - w| \to 0$ in D_S. These properties ensure the uniform asymptotic stability of the equilibrium $x = 0$ under the initial conditions $x(0) \in (-w(0), w(0))$. In addition, the system trajectories will never leave this domain (see Fig. 1 on the right). Choosing the quadratic function $V = 0.5x^2$ yields $\dot{V} = -\frac{\alpha}{w-|x|}x^2 < 0$ in the domain $x \in D_S \setminus \{0\}$, which confirms the conclusions drawn.

3. Consider the density function $\rho(x,t) = \alpha[x(t) - w(t)]\arctan(x/\epsilon)$ with a continuous function $w(t)$ and a sufficiently large positive number ϵ. The function $\rho(x,t)$ takes positive values in the domain $D_S = \{x \in \mathbb{R} : x \in (-\infty, 0) \cup (w, +\infty)$ for $w > 0$ and $x \in (-\infty, w) \cup (0, +\infty)$ for $w < 0\}$ and negative values in the domain $D_U = \{x \in \mathbb{R} : x \in (0, w)$ for $w > 0$ and $x \in (w, 0)$ for $w < 0\}$, which

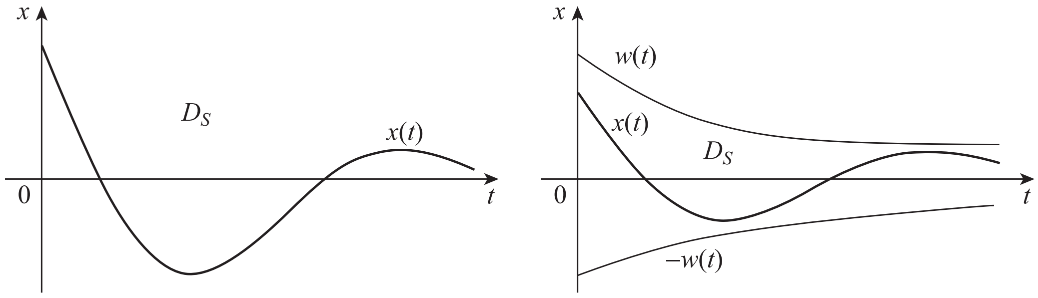

Fig. 1. Transients in system (2) with the density functions $\rho(x,t) = \alpha$ (left) and $\rho(x,t) = \frac{\alpha}{w(t)-|x(t)|}$ (right).

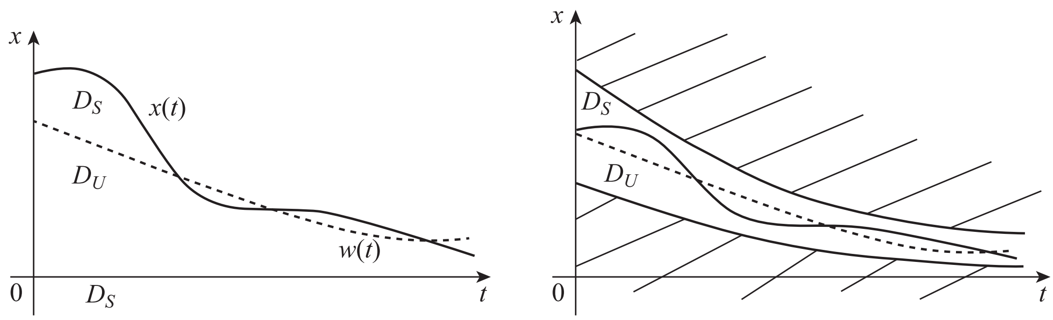

Fig. 2. Transients in system (2) with the density functions $\rho(x,t) = \alpha[x(t) - w(t)]\arctan(x/\epsilon)$ (left) and $\rho(x,t) = \alpha \ln \frac{\overline{w}(t)-x(t)}{x(t)-\underline{w}(t)}$ (right).

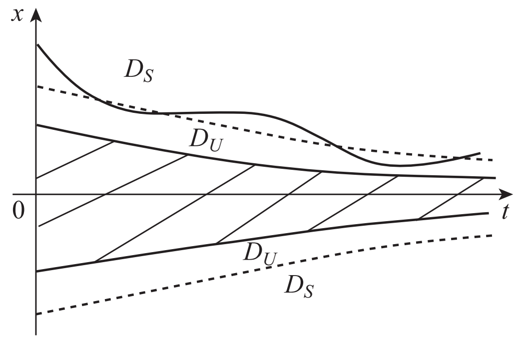

Fig. 3. Transients in system (2) with the density function $\rho(x,t) = \alpha \ln(x(t) - g(t))$.

ensures tracking of the trajectory $w(t)$ by $x(t)$ (see Fig. 2 on the left). Choosing the quadratic function $V = 0.5x^2$ yields $\dot{V} = \alpha[x - w]\arctan(x/\epsilon)\, x^2 < 0$ for $x \in D_S$ and $\dot{V} > 0$ for $x \in D_U$.

4. Consider the density function $\rho(x,t) = -\alpha \ln \frac{\overline{w}(t)-x(t)}{x(t)-\underline{w}(t)}$ with continuous functions $\overline{w}(t) > \underline{w}(t) > 0$. Let us denote $w = 0.5[\overline{w} + \underline{w}]$, where $\rho(x,t) = 0$ for $x = w$ and any t. The function $\rho(x,t)$ takes positive values in the domain $D_S = \{x \in \mathbb{R} : w < x < \overline{w}\}$ and negative values in the domain $D_U = \{x \in \mathbb{R} : \underline{w} < x < w\}$, which ensures tracking of the trajectory $w(t)$ by $x(t)$. In the shaded domain, system (2) has no solutions (see Fig. 2 on the right). In addition, the system trajectories will never leave the domain $D_S \cup D_U$ since $|\rho(x,t)| \to +\infty$ as x approaches the boundaries $\underline{w}$ and $\overline{w}$. Choosing the quadratic function $V = 0.5x^2$ yields $\dot{V} = \alpha \ln \frac{\overline{w}-x}{x-\underline{w}} x^2 < 0$ for $x \in D_S$ and $\dot{V} > 0$ for $x \in D_U$, which confirms tracking of the trajectory $w(t)$ by $x(t)$.

5. Consider now $\rho(x,t) = \alpha \ln(x(t) - g(t))$ with a continuous function $g(t) > 0$. Let us denote $w = 1 + g$, where $\rho(x,t) = 0$ for $x = w$ and any t. The function $\rho(x,t)$ takes positive values in the domain $D_S = \{x \in \mathbb{R}_+ \colon w < x < +\infty\}$ and negative values in the domain $D_U = \{x \in \mathbb{R}_+ \colon g < x < w\}$, which ensures tracking of the trajectory $w(t)$ by $x(t)$ (see Fig. 3). In addition, the system trajectories will never enter the shaded domain: when approaching the boundary $g(t)$, we have $\rho(x,t) \to -\infty$ and, consequently, $x(t)$ moves along the surface (see Fig. 3). Choosing the quadratic function $V = 0.5x^2$ yields $\dot{V} = -\alpha \ln(x - g)x^2 < 0$ for $x \in D_S$ and $\dot{V} > 0$ for $x \in D_U$.

Remark 1. Here, we study the possibility of analyzing dynamic systems with a discontinuous right-hand side in t and x, including the density function $\rho(x,t)$.

Consider first a nonautonomous system of the general form $\dot{x} = f(x,t)$ with $x \in \mathbb{R}^n$. Let the function $f(x,t)$ be defined in some open domain D of the variables (x,t). The function $f(x,t)$ is said to satisfy the Carathéodory condition [15] if it is continuous in x for almost all t and piecewise continuous in t for all x (assuming measurability in t is sufficient) and, for any compact set $G \subset D$, there exists a nonnegative integrable function $m(t)$ such that $|f(x,t)| \leqslant m(t)$ for all $(x,t) \in G$.

If the function $f(x,t)$ satisfies the Carathéodory condition, then by Theorems 1.1.1.1 and 1.1.4 of [15], for any initial conditions from the domain D, there exists a locally absolutely continuous solution $x(t)$ of the system $\dot{x} = f(x,t)$. The equation $\dot{x}(t) = f(x(t),t)$ holds for almost all t. The derivative of $x(t)$ may not exist for those t at which the function $f(x,t)$ suffers a jump in t. Furthermore, either the solution $x(t)$ is defined on $[0, +\infty)$, or for some finite t_0, the solution $x(t)$ tends to the boundary of the domain D as $t \to t_0$. If the function $f(x,t)$ is locally Lipschitz in x, then by Theorem 1.1.2 of [15] the solution is unique.

Consider now system (2) from Example 1. If the function $\rho(x,t)$ satisfies the Carathéodory condition, then this system has a locally absolutely continuous solution; if the function $\rho(x,t)$ is also locally Lipschitz in x, then this solution is unique.

We proceed to case 2 of Example 1. As the domain D, we define the set of all those (x,t) not belonging to the closures of the graphs of the functions $w(t)$ and $-w(t)$. For example, let $w(t) = 2$ for $t \in [0,1]$ and $w(t) = 1$ for $t > 1$; in this case, then not only the graph of $w(t)$ but also the point $(1,1)$ must be excluded from the domain D. The function $\rho(x,t)$ satisfies the Carathéodory condition and is locally Lipschitz in x. Therefore, the equation $\dot{x} = -\rho(x,t)x$ has a unique locally absolutely continuous solution that is either defined on the entire axis or, for some finite t_0, the distance from $(x(t),t)$ to the boundary of D will vanish as $t \to t_0$.

Consider the Lyapunov function $V(x) = 0.5x^2$. We take the solution $x(t)$ and examine the function $V(x(t))$ that is locally absolutely continuous. By the differentiability theorem of a complex function, $\dot{V} = -\rho(x(t),t)x(t)^2 \leqslant 0$ for almost all t. Due to absolute continuity, $V(t) - V(0) = \int_0^t \dot{V}(s)ds \leqslant 0$ for any t. (An absolutely continuous function can be reconstructed through the integral over its derivative; see the proof of this result in [16].) The inequality $V(t) \leqslant V(0)$ implies that the equilibrium is stable. Its asymptotic stability can be established using LaSalle's theorem for nonautonomous systems (Theorem 1 of [17]) if the set $\{x : |x| < w(t)\}$ contains a pipe of constant nonzero width and the function $w(t)$ is not infinitely increasing. If $w(t) \to 0$ as $t \to \infty$, then asymptotic stability simply follows from the fact that the solution remains in the domain D_S.

Here, however, a reserve concerning the piecewise continuity of $w(t)$ is required. It may happen that the solutions from the domain D_S will leave this domain. For example, let $w(t) = 2$ for $t \in [0, t_0]$ and $w(t) = 1$ for $t > t_0$. If t_0 is small enough and the initial condition $x(0)$ is close to 2 or -2, as $t \to t_0$, the trajectory $x(t)$ will simply smash into the wall formed by the jump of the function $w(t)$. The existence theorem guarantees that this solution is continuable further, but the stability analysis is not applicable in this case. Such a solution will jump out of the domain D_S and increase infinitely.

Under a continuous function $w(t)$, it is easy to show that the solutions with initial conditions from D_S will not leave the domain D_S. However, for a piecewise continuous function $w(t)$, this property generally fails. If the function $w(t)$ has many jumps, it may turn out that some solutions starting in D_S will jump out of this domain when smashing into the walls formed by jumps of the function $w(t)$. If the solution remains in the domain D_S, it will tend to the equilibrium.

When considering differential equations with a discontinuous right-hand side in x (or in t as well [18, 19]), it is necessary to understand the solutions of such systems in the Filippov sense [15]. In case 2 of Example 1, we can then consider the function $\rho(x,t) = \alpha[x(t)-w(t)]\text{sgn}(x)$, where $\text{sgn}(\cdot)$ is the sign function. The stability of discontinuous nonautonomous systems was studied in detail, e.g., in [18–21].

Thus, it is possible to examine below dynamic systems with a discontinuous right-hand side, including discontinuous functions $\rho(x,t)$. However, such systems complicate the analysis due to justifying the choice of initial conditions, the frequency and magnitude of jumps of the function, etc. Recall that this paper is devoted to studying the behavior of dynamic systems depending on the properties of the density function. For the sake of simplicity, all theoretical results will be therefore formulated for dynamic systems with a right-hand side continuous in t and locally Lipschitz in x. Still, for illustration purposes, some examples may contain discontinuous right-hand sides.

Well then, Example 1 has demonstrated how a density function $\rho(x,t)$ defined in the space (x,t) can qualitatively affect the transients of the original system (1). The next example shows that the density function need not be multiplied by the entire right-hand side, as in Example 1 (see (1) and (2)) but can be explicitly or implicitly present on the right-hand side of the system.

Example 2. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 - \rho_1(x,t)x_1, \\ \dot{x}_2 &= -x_1 - \rho_2(x,t)x_2, \end{aligned} \tag{3}$$

where $\rho_1(x,t)$ and $\rho_2(x,t)$ are continuous functions in t and x on $\mathbb{R}^2 \times [0,+\infty)$. We choose the quadratic function

$$V = 0.5(x_1^2 + x_2^2). \tag{4}$$

Taking the total time derivative of this function along the solutions of (3) yields

$$\dot{V} = -\rho_1 x_1^2 - \rho_2 x_2^2. \tag{5}$$

1. Let $\rho_1 = \rho_2 = \rho = \ln \frac{\overline{g}-|x_1|^\beta-|x_2|^\beta}{|x_1|^\beta+|x_2|^\beta-\underline{g}}$, where $\beta > 0$ (for $0 < \beta < 1$, see Remark 1) and $\overline{g}(t) > \underline{g}(t) > 0$ are continuous functions. We have $\rho = 0$ for $|x_1|^\beta + |x_2|^\beta = g$, where $g = 0.5(\underline{g} + \overline{g})$. Then $\dot{V} < 0$ for $\rho(x,t) > 0$, i.e., in the domain $D_S = \{x \in \mathbb{R}^2 : g < |x_1|^\beta + |x_2|^\beta < \overline{g}\}$, and $\dot{V} > 0$ for $\rho(x,t) < 0$, i.e., in the domain $D_U = \{x \in \mathbb{R}^2 : \overline{g} < |x_1|^\beta + |x_2|^\beta < g\}$. In this case, the density function $\rho(x,t)$ is explicitly present in system (3) but is not multiplied by the entire right-hand side, as in Example 1. Figure 4 shows the simulation results for $\beta = 1$, $\overline{g} = 3$, $\underline{g} = 2$ (left) and $\beta = 0.6$, $\overline{g} = 3^{0.6}$, $\underline{g} = 1$ (right) under $x(0) = col\{0,\ 2,5\}$.

Hereinafter:

- The gray domains in the figures mean that the density function is chosen so that there are no solutions of the system in these domains. (At the boundary of such a domain, the density value increases to infinity.)
- The point curve corresponds to the zero value of the density function and, accordingly, this curve is the boundary separating the stable D_S and unstable D_U domains.

2. Let $\rho_1 = 1 - x_1^2$ and $\rho_2 = -x_1^2$. Then $\dot{V} = -\rho(x,t)x_1^2 < 0$ for $\rho(x,t) = x_1^2 + x_2^2 - 1 > 0$ in the domain $D_S = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 > 1 \text{ and } x_1 \neq 0\}$, and $\dot{V} > 0$ for $\rho(x,t) < 0$ in the domain

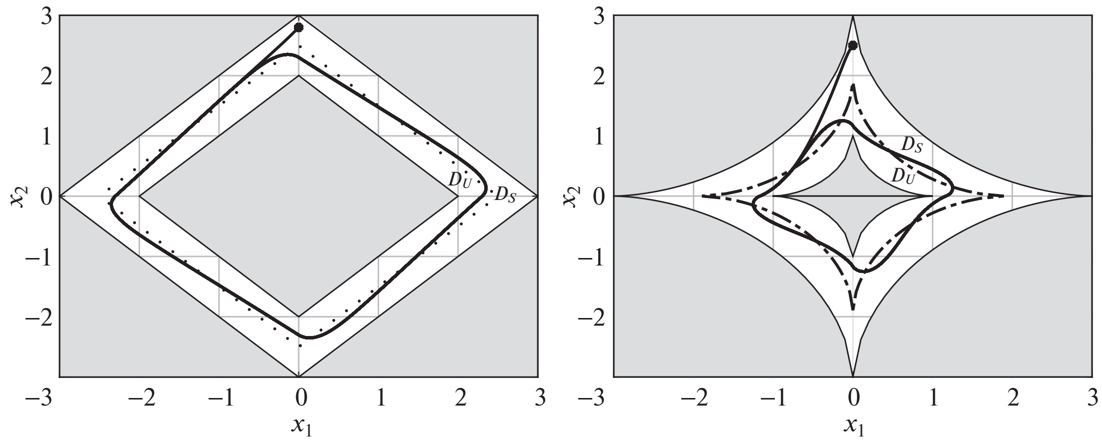

Fig. 4. The phase trajectory of system (3) with the density functions $\rho(x,t) = \ln \frac{3-|x_1|-|x_2|}{|x_1|+|x_2|-2}$ (left) and $\rho(x,t) = \ln \frac{3^{0.6}-|x_1|^{0.6}-|x_2|^{0.6}}{|x_1|^{0.6}+|x_2|^{0.6}-1}$ (right).

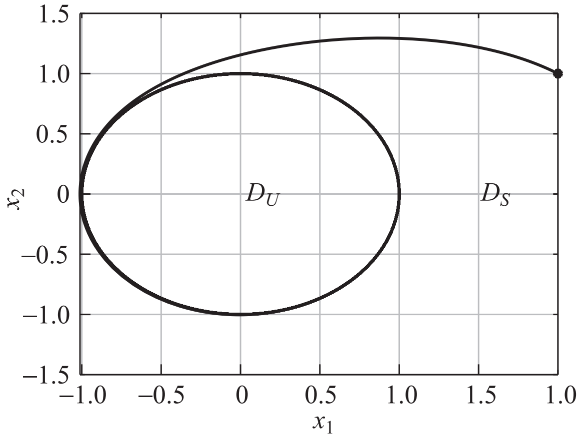

Fig. 5. The phase trajectory of system (3) with the density function $\rho(x,t) = x_1^2 + x_2^2 - 1$.

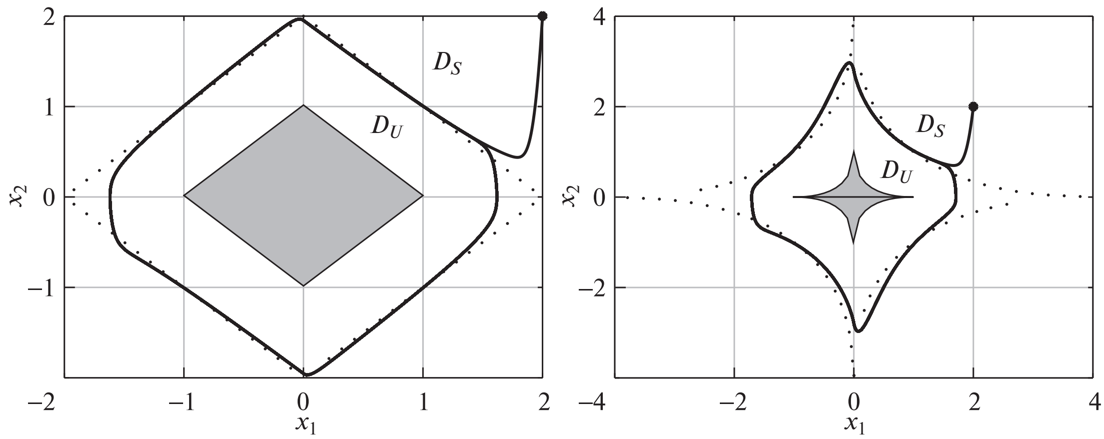

Fig. 6. The phase trajectory of system (3) with the density functions $\rho(x,t) = 20\ln(|x_1| + |x_2| - 1)$ (left) and $\rho(x,t) = 20\ln(|x_1|^{0.5} + |x_2|^{0.5} - 1)$ (right).

$D_U = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1 \text{ and } x_1 \neq 0\}$. In this case, unlike the previous one, the density function $\rho(x,t)$ is implicitly present in (3). However, in contrast to the previous case, the value of the density function does not affect system (3) everywhere (i.e., $\dot{V} = 0$ for $x_1 = 0$ regardless of the value of ρ). Figure 5 demonstrates the simulation results under $x(0) = col\{2, 1\}$.

3. Let $\rho_1 = \alpha \ln(|x_1|^\beta + |x_2|^\beta - 1)$, $\alpha > 0$, $\beta > 0$ (for $0 < \beta < 1$, see Remark 1), and $\rho_2 = 0$. Then $\dot{V} = -\rho(x,t)x_1^2 < 0$ for $\rho(x,t) = \alpha \ln(|x_1|^\beta + |x_2|^\beta - 1) > 0$ in the domain $D_S = \{x \in \mathbb{R}^2 : |x_1|^\beta + |x_2|^\beta > 2 \text{ and } x_1 \neq 0\}$, and $\dot{V} > 0$ for $\rho(x,t) < 0$ in the domain $D_U = \{x \in \mathbb{R}^2 : 1 < |x_1|^\beta + |x_2|^\beta < 2 \text{ and } x_1 \neq 0\}$. In this case, the density function $\rho(x,t)$ is present in just one of equations (3), in contrast to cases 1 and 2. However, as in case 2, the density function has no effect on system (3) everywhere. Figure 6 demonstrates the simulation results for $\beta = 1$ (left) and $\beta = 0.5$ (right) under $\alpha = 20$ and $x(0) = col\{2, 2\}$.

Consider now the dynamic system

$$\dot{x} = f(x,t), \tag{6}$$

where $t \geqslant 0$, $x \in D \subset \mathbb{R}^n$ denotes the state vector, and $f : D \times [0, +\infty) \to \mathbb{R}^n$ is a function continuous in t and locally Lipschitz in x on $D \times [0, +\infty)$. The possibility of studying systems (6) with a discontinuous right-hand side has been discussed in Remark 1.

Definition 1. System (6) is called a density system with a density function $\rho(x,t) : D \times [0, +\infty) \to \mathbb{R}$ if there exists a continuously differentiable function $V(x,t) : D \times [0, +\infty) \to \mathbb{R}$ such that:

(a) $w_1(x) \leqslant V(x,t) \leqslant w_2(x)$,

(b) $\dot{V} \leqslant \rho(x,t)W_1(x) \leqslant 0$ or $\dot{V} \geqslant \rho(x,t)W_2(x) \geqslant 0$

for any $t \geqslant 0$ and $x \in D$. Here, $\rho(x,t)$ is a function continuous in t and locally Lipschitz in x, $w_1(x)$ and $w_2(x)$ are positive definite functions, and $W_1(x)$ and $W_2(x)$ are continuous nonzero (except the equilibrium) functions in D.

Definition 2. If the functions $W_1(x)$ and $W_2(x)$ in Definition 1 are continuous in D, then system (6) is called a weak density system.

Definition 3. If $\dot{V} \leqslant \rho(x,t)W_1(x) <$ or $\dot{V} \geqslant \rho(x,t)W_2(x) > 0$ in condition (b) of Definition 1, then system (6) is called a strict density system.

Definition 4. If $\dot{V} \leqslant \rho(x,t)W_1(x) \leqslant 0$ in the domain $D_S \times [0, +\infty)$, then the density function $\rho(x,t)$ and the domain D_S are said to be stable. If $\dot{V} \geqslant \rho(x,t)W_2(x) > 0$ in the domain $D_U \times [0, +\infty)$, then the density function $\rho(x,t)$ and the domain D_U are said to be unstable.

Proposition 1. *Assume that system* (6) *is a strict density system in the domains* D_S *and* D_U. *If for each* t, $V(x_s,t) - V(x_u,t) > 0$, *where* $x_s \in D_S$ *and* $x_u \in D_U$, *then the system trajectories are attracted to the separation boundary of the domains* D_S *and* D_U. *If system* (6) *satisfies the condition* $V(x_s,t) - V(x_u,t) < 0$ *for each* t, *then the system trajectories move away from the separation boundary of the domains* D_S *and* D_U.

Proof. Let $V(x_s,t) - V(x_u,t) > 0$ for each $t \geqslant 0$, where $x_s \in D_S$ and $x_u \in D_U$. Since the system is a strict density system, by Definition 3 we have $\dot{V} \leqslant \rho(x,t)W_1(x) < 0$ in the domain D_S and $\dot{V} \geqslant \rho(x,t)W_2(x) > 0$ in the domain D_U. Hence, the separation boundary of the domains D_S and D_U is a set attracting the system trajectories.

Now let $V(x_s,t) - V(x_u,t) < 0$ for each $t \geqslant 0$, where $x_s \in D_S$ and $x_u \in D_U$. According to Definition 3, we have $\dot{V} \leqslant \rho(x,t)W_1(x) < 0$ in the domain D_S and $\dot{V} \geqslant -\rho(x,t)W_2(x) > 0$ in the domain D_U. Hence, the separation boundary of the domains D_S and D_U is a set left by the system trajectories.

Remark 2. In Proposition 1 and its proof, the attraction of trajectories to some set covers the cases where trajectories approach this set over time or belong to some neighborhood of this set.

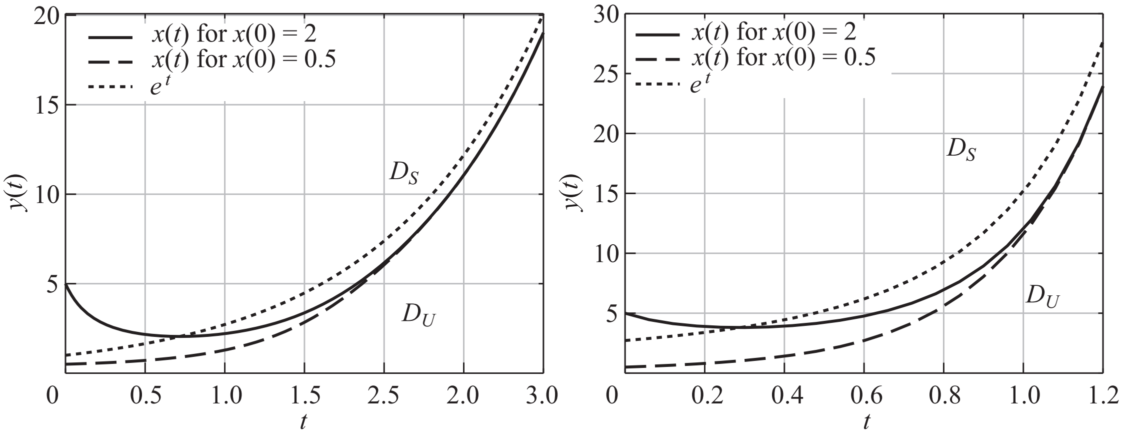

Fig. 7. $x(t)$ tracks the unbounded signals $w(t) = e^t$ (left) and $w(t) = e^{e^t}$ (right).

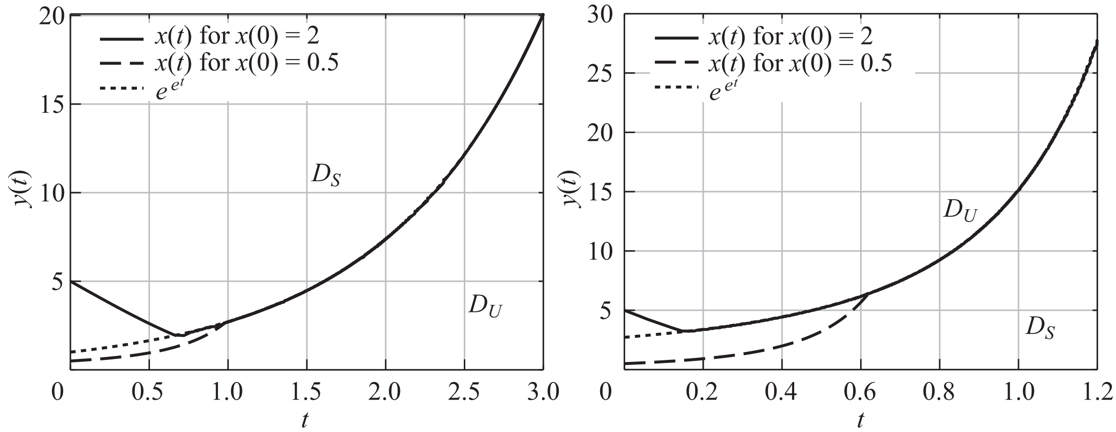

Fig. 8. $x(t)$ tracks the unbounded signals $w(t) = e^t$ (left) and $w(t) = e^{e^t}$ (right).

Note that the neighborhood size can remain constant or increase with time. It depends on the value of the space density. Here are some limiting cases:

- If in the neighborhood of the boundary separating the domains D_S and D_U the value of the density function decreases to zero, the system trajectories will not approach this boundary. Therefore, they can be in the neighborhood of this boundary or move far from it.
- If in the neighborhood of the boundary separating the domains D_S and D_U the value of the density function increases infinitely, the system trajectories will approach this boundary.

To illustrate Proposition 1 and Remark 2, we give another example as follows.

Example 3. Consider system (2) again but with $x \in \mathbb{R}_+$.

Choosing the quadratic function $V = 0.5x^2$ yields $\dot{V} = -\rho(x,t)x^2$. Hence, $\dot{V} < 0$ in the domain $D_S = \{x, t \in \mathbb{R}_+ : \rho(x,t) > 0\}$ and $\dot{V} > 0$ in the domain $D_U = \{x, t \in \mathbb{R}_+ : \rho(x,t) < 0\}$.

By Definition 3, system (2) is a strict density system for $x \in \mathbb{R}_+$. Analyzing Proposition 1, we fix an arbitrary time instant $t = t_1$. Then $V(x_s(t_1)) - V(x_u(t_1)) > 0$, and this inequality is obviously valid for any fixed t. According to Proposition 1, the system trajectories will be attracted to the separation boundary of the domains D_S and D_U.

Let the density function be $\rho(x,t) = x - w$, where $w(t) = e^t$. In this case, $\lim_{t\to\infty}(w(t) - x(t)) =$ const (see Fig. 7 on the left). If $w(t) = e^{e^t}$, the difference between $w(t)$ and $x(t)$ increases with time (see Fig. 7 on the right). This fact can be explained as follows: $w(t)$ is an unbounded function,

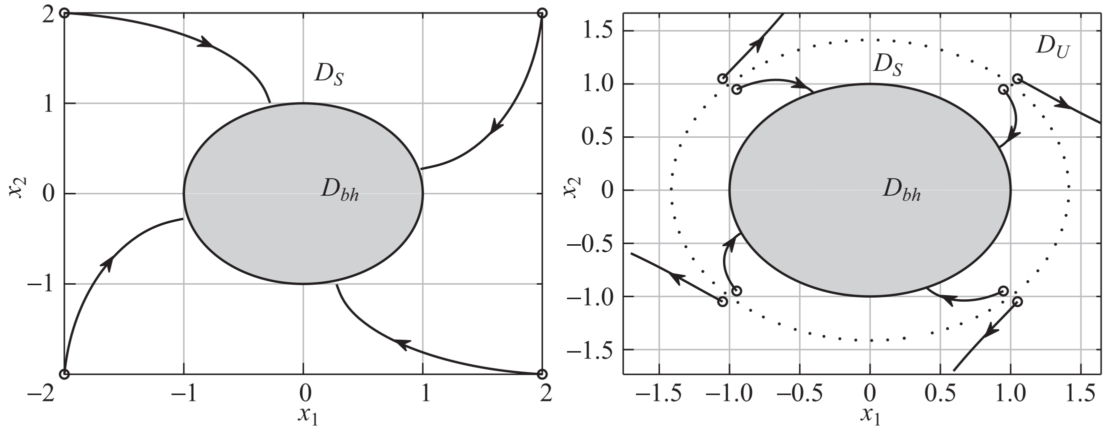

Fig. 9. The phase portrait of system (3) with the density functions $\rho(x) = e^{(x_1^2+x_2^2-1)^{-0.98}}$ (left) and $\rho(x) = -\ln(x_1^2 + x_2^2 - 1)$ (right).

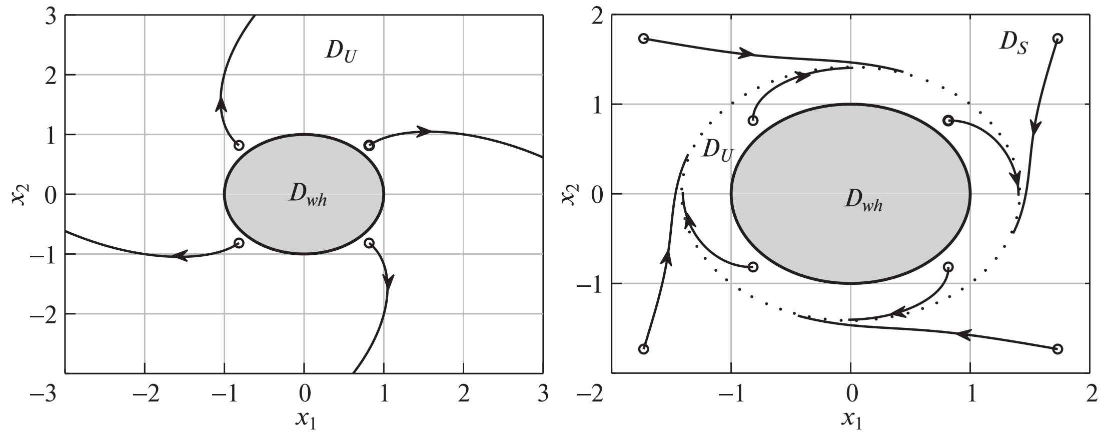

Fig. 10. The phase portrait of system (3) with the density functions $\rho(x) = -e^{(x_1^2+x_2^2-1)^{-0.98}}$ (left) and $\rho(x) = \ln(x_1^2 + x_2^2 - 1)$ (right).

whereas the density $|\rho(x,t)|$ decreases for x approaching w. As a result, $x(t)$ "tries to approach" $w(t)$ but fails due to the low density of the space in the neighborhood of $w(t)$ and a high rate of change of $w(t)$.

Let the density function be $\rho = w \arctan\frac{x-w}{\epsilon}$ (or $\rho = w\,\mathrm{sgn}(x - w)$ due to Remark 1), $\epsilon > 0$ be a sufficiently large number, and $w(t) = e^t$ or $w(t) = e^{e^t}$. In this case, the density of the space in the neighborhood of $w(t)$ grows with increasing $w(t)$, which ensures the approach of x to w as $t \to \infty$ (see Fig. 8).

When studying density systems, we will also distinguish special domains. They have been considered earlier as gray domains in the figures. Here is a rigorous definition.

Definition 5. If $\dot{V} \leqslant \rho(x,t)W_1(x) < 0$ in the neighborhood of a domain $D_{bh} \times [0, +\infty)$, there are no solutions of (6) in this domain, and the value of the density function grows infinitely when approaching it, then the domain D_{bh} is said to be absolutely stable.

Definition 6. If $\dot{V} \geqslant \rho(x,t)W_2(x) > 0$ in the neighborhood of a domain $D_{wh} \times [0, +\infty)$, there are no solutions of (6) in this domain, and the value of the density function grows infinitely when approaching it, then the domain D_{wh} is said to be absolutely unstable.

Example 4. Consider system (3) with $\rho_1(x,t) = \rho_2(x,t) = \rho(x)$. Choosing the quadratic function (4) yields $\dot{V} = -\rho(x)(x_1^2 + x_2^2)$.

If $\rho(x) = e^{(x_1^2+x_2^2-1)^{-0.98}}$, then all trajectories tend to the domain $D_{bh} = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leqslant 1\}$ (see Fig. 9 on the left). That is, from any initial conditions, the system trajectories will be attracted to D_{bh}, where the value of the density function grows to infinity when approaching the boundary of this domain. If $\rho(x) = -\ln(x_1^2 + x_2^2 - 1)$, the trajectories with initial conditions from the domain $x_1^2 + x_2^2 \geqslant \sqrt{2}$ will remain in this domain, but all trajectories with initial conditions from the domain $1 \leqslant x_1^2 + x_2^2 \leqslant \sqrt{2}$ cannot leave this region and will be attracted to the domain $D_{bh} = \{x \in \mathbb{R}^2 : x_1^2+x_2^2 \leqslant 1\}$ (see Fig. 9 on the right), where the density function grows to infinity when approaching the boundary of D_{bh}.

If $\rho(x) = -e^{(x_1^2+x_2^2-1)^{-0.98}}$, all trajectories will move away from the domain $D_{wh} = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leqslant 1\}$ (see Fig. 10 on the left) without approaching its boundary, where the density function will grow to minus infinity when approaching the boundary of D_{wh}. If $\rho(x) = \ln(x_1^2 + x_2^2 - 1)$, the trajectories with initial conditions from the domain $x_1^2 + x_2^2 \geqslant \sqrt{2}$ will remain in this domain, but all trajectories with initial conditions from the domain $1 \leqslant x_1^2 + x_2^2 \leqslant \sqrt{2}$ will move away from the domain $D_{wh} = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leqslant 1\}$ (see Fig. 10 on the right).

Remark 3. Let us explain the physical meaning of the systems under consideration. If the density function is explicitly present on the right-hand side of the system equation, e.g., in the form $\dot{x} = \rho(x,t)f(x,t)$, then the value of the space density directly affects the phase flow velocity. For example, for $\rho(x,t) = 1$, we have the original system $\dot{x} = f(x,t)$. If $\rho(x,t) > 0$, the presence of the density function possibly does not qualitatively affect the equilibria, but it may affect the phase portrait. Under $0 < \rho(x,t) < 1$, the value of the phase velocity vector decreases because the space density does so. In the case $\rho(x,t) > 1$, on the contrary, the value of the phase velocity vector increases because the space density does so. When the density function changes its sign, the phase portrait varies qualitatively.

Condition (b) of Definition 1 can be interpreted as the rate of change of the phase volume given by the function $V(x,t)$ considering the space density.

We now provide some models of real processes:

- The pendulum equation has the form $\dot{x}_1 = x_2$, $\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$, where x_1 is the deviation angle of the pendulum from the vertical axis, x_2 is the angular velocity of the pendulum, g is the acceleration of gravity, l is the pendulum length, and k is the friction coefficient [22]. Choosing the Lyapunov function $V = \frac{g}{l}(1 - \cos x_1) + 0.5x_2^2$ (the total energy of the system) yields $\dot{V} = -\frac{k}{m}x_2^2$. Introducing the density function $\rho(x,t) = k$, we obtain persistent oscillations in the case of no friction ($\rho(x,t) = 0$) and damped oscillations otherwise ($\rho(x,t) \neq 0$).

- The types of reproduction models in [25] can be written as $\dot{x} = \rho(x)x$, where x denotes the biological population size. For $\rho(x) = k > 0$, we have the normal reproduction model; for $\rho(x) = kx$, the explosion model; for $\rho(x) = 1 - x$, the logistic curve model.

- Absolutely stable and absolutely unstable domains from Definitions 5 and 6 can be found as the simplest models of black holes and white holes, respectively, [26], which possess high density and gravity. Therefore, density systems can also be treated as *gravitational systems*, where $\rho(x,t)$ is a *gravity function*. In other words, the behavior of systems can be affected not by the density of the space but by the gravitational field produced by a dense body. In this regard, the control design principles based on the density function in the following sections can be considered control using the gravity function. All the mathematical descriptions and conclusions remain valid regardless of the name of the function $\rho(x,t)$ and the corresponding systems.

Remark 4. As has been noted in the Introduction, the concepts of density functions and density systems proposed in Section 2 and their properties generalize the results of [1–8, 10–14]. Let us discuss these issues in detail.

- In [1–8], the stability of an original system $\dot{x} = f(x)$ was analyzed by introducing a new system of the form $\dot{x} = \rho(x)f(x)$, which was (in turn) investigated using either divergent methods or the method of Lyapunov functions. In this section, we analyze systems of the form $\dot{x} = f(x)$ using the dependencies $\dot{V} \leqslant \rho(x,t)W_1(x)$ or $\dot{V} \geqslant \rho(x,t)W_1(x)$ (see Definition 1 and Example 2), which does not require multiplying the entire right-hand side of the system by the density function.
- In [10–14], the stability of systems $\dot{x} = f(x, \rho(x,t))$ with the density function explicitly appearing on the right-hand side was analyzed. The results proposed in this paper allow for the implicit presence of the density function on the right-hand side of the system (see case 2 in Example 2).
- In [10–14], the solutions of a system were placed into a given bounded set without forbidden domains by a special choice of the density function. Note that the boundaries of such sets were specified by functions continuously differentiable in t and x. In this paper, we allow for unbounded sets (see case 5 in Example 1 and case 3 in Example 2) with forbidden domains (see cases 4 and 5 in Example 1 as well as cases 1 and 3 in Example 2). Moreover, the boundaries of these sets can be specified by functions continuous in t and locally Lipschitz in x. If the solutions of systems are understood in the Filippov sense, then we can consider systems even with discontinuous right-hand sides; see Remark 1.

Thus, the new concepts of a density function and a density system introduced above provide a novel look at a certain class of dynamic systems, which is broader than those considered in [1–8, 10–14]. Also, the evolution of dynamic systems can now be considered and influenced by the density of the space.

The results obtained in this section can be used to analyze dynamic systems and, moreover, to design control laws. The next section will be devoted to this issue.

3. DENSITY CONTROL

We present several examples of designing control laws to obtain closed loop systems described by density systems.

3.1. Plants with Known Parameters

Consider a plant of the form

$$Q(p)y(t) = R(p)u(t), \tag{7}$$

where $y \in \mathbb{R}$ and $u \in \mathbb{R}$ denote the output and control signals, respectively, $Q(p)$ and $R(p)$ are linear differential operators with known constant coefficients, and $R(\lambda)$ is a Hurwitz polynomial.

If the relative degree of the plant (7) is 1 (i.e., $\deg Q(p) - \deg R(p) = 1$), then the control law

$$u(t) = -\frac{Q(p)}{pR(p)}\rho(y,t)y(t) \tag{8}$$

transforms system (7) into

$$\dot{y}(t) = -\rho(y,t)y(t), \tag{9}$$

which is structurally a density system. In particular, some density functions $\rho(y,t)$ have been defined in Example 1.

If the relative degree of the plant (7) exceeds 1 (i.e., $\deg Q(p) - \deg R(p) = \gamma > 1$), we write the control law (8) as

$$u(t) = -\frac{Q(p)}{pR(p)(\mu p+1)^{\gamma-1}}\rho(y,t)y(t) \tag{10}$$

where $\mu > 0$ is a sufficiently small number. The resulting system takes the form

$$\dot{y}(t) = -\frac{1}{(\mu p+1)^{\gamma-1}}\rho(y,t)y(t). \tag{11}$$

For $\mu = 0$, system (11) has the density system structure (9). If the solutions of the density system (9) with an appropriately chosen density function $\rho(y,t)$ are asymptotically stable, then [22, 24] there exists a sufficiently small number $\overline{\mu} > 0$ such that, for $0 < \mu < \overline{\mu}$, the solutions of system (11) are sufficiently close to the solution of system (9).

3.2. Plants with Unknown Parameters

Consider the plant (7) with the unknown parameters of the operators $Q(p)$ and $R(p)$ but a known value k. Assume that the relative degree of this plant is 1. All the results obtained can be extended to the plants with a relative degree above 1, e.g., using the schemes of [23]. In this paper, we focus on the plants with a relative degree of 1 only to avoid cumbersome considerations for overcoming the high relative degree problem.

Let the operators $Q(p)$ and $R(p)$ be written as $Q(p) = Q_m(p) + \Delta Q(p)$ and $R(p) = R_m(p) + \Delta R(p)$, where $Q_m(\lambda)$ and $R_m(\lambda)$ are arbitrary Hurwitz polynomials of degrees n and $n-1$, respectively, and the polynomials $\Delta Q(p)$ and $\Delta R(p)$ have degrees $n-1$ and $n-2$, respectively. Choosing $Q_m(\lambda)/R_m(\lambda) = \lambda + a$ with a known value $a > 0$ and taking the integer part of $\frac{\Delta Q(\lambda)}{Q_m(\lambda)} = k_{0y} + \frac{\Delta\tilde{Q}(\lambda)}{R_m(\lambda)}$, we transform (7) into

$$\dot{y}(t) = -ay(t) + k\left(u(t) + \frac{\Delta R(p)}{R_m(p)}u - \frac{\Delta\tilde{Q}(p)}{R_m(p)}y - k_{0y}y\right). \tag{12}$$

We introduce $c_0 = col\{c_{0y}, c_{0u}, k_{0y}\}$ as the vector of unknown parameters, where $\Delta\tilde{Q}(p) = c_{0y}^{\mathrm{T}}[1\ p\ \ldots\ p^{n-2}]$ and $\Delta R(p) = c_{0u}^{\mathrm{T}}[1\ p\ \ldots\ p^{n-2}]$, and the regression vector $w = col\{V_y, V_u, y\}$ constructed using the filters

$$\begin{aligned}\dot{V}_y &= FV_y + by,\\ \dot{V}_u &= FV_u + bu.\end{aligned} \tag{13}$$

Here, F is the Frobenius matrix with the characteristic polynomial $R_m(\lambda)$ and $b = col\{0,\ldots,0,1\}$.

With these notations, equation (12) can be written as

$$\dot{y}(t) = -ay(t) + k[u(t) - c_0^{\mathrm{T}}w(t)]. \tag{14}$$

Let the control law be given by

$$u(t) = c^{\mathrm{T}}(t)w(t) + \frac{a}{k}y(t) + \rho(y,t). \tag{15}$$

Substituting (15) into (14) yields the closed loop system

$$\dot{y}(t) = \rho(y,t) + k(c(t) - c_0)^{\mathrm{T}}w(t). \tag{16}$$

Theorem 1. *The control law* (15) *with the adaptation algorithm*

$$\dot{c} = -\alpha y w \tag{17}$$

transforms the plant (7) *into a density system. If we have a stable density* $\rho(y,t)$ *with a stable limit set in the neighborhood of zero as* $t \to \infty$, *then all signals in the closed loop system are bounded.*

Proof. Let us choose the quadratic function

$$V = \frac{1}{2}y^2 + \frac{k}{2\alpha}(c(t) - c_0)^{\mathrm{T}}(c(t) - c_0). \tag{18}$$

Calculating the total time derivative of (18) along the solutions of (16), (17), we write the result as

$$\dot{V} = \rho(y,t)y. \tag{19}$$

This is a density system.

If we have a stable density $\rho(y,t)$ with a stable limit set in the neighborhood of zero as $t \to \infty$, then $\rho(y,t)$ is chosen so that $\rho(y,t)y < 0$. Hence, $\lim\limits_{t\to\infty} y(t) = 0$. From (16) it then follows that $\lim\limits_{t\to\infty}(c(t) - c_0)^{\mathrm{T}}(t)w(t) = 0$. The ultimate boundedness of $V_y(t)$ is immediate from the first equation of (13), the ultimate boundedness of $y(t)$, and the Hurwitz property of the matrix F. Substituting (15) into the second equation of (13) yields

$$\begin{aligned}\dot{V}_u &= FV_u + bc_0^{\mathrm{T}}w + b(c - c_0)^{\mathrm{T}}w + b\frac{a}{k}y(t) + b\rho(y,t)\\ &= (F + bc_{0u})V_u + b\left[c_{0y}^{\mathrm{T}}V_y + k_{0y}y + b(c - c_0)^{\mathrm{T}}w + \frac{a}{k}y(t) + \rho(y,t)\right].\end{aligned}$$

The matrix $F + bc_{0u}$ has a Hurwitz characteristic polynomial $R(\lambda)$ by the problem statement. So, under bounded terms in square brackets, the function $V_u(t)$ is ultimately bounded. Then the regression vector $w(t)$ is also ultimately bounded. The condition $\lim\limits_{t\to\infty} y(t) = 0$, the ultimate boundedness of $w(t)$, and (17) imply $\lim\limits_{t\to\infty}\dot{c}(t) = 0$. Hence, $c(t)$ is an ultimately bounded function. From (15) it then follows that the control law is bounded. As a result, all signals in the closed loop system are bounded.

Example 5. Consider the plant (7) with unknown parameters of the operators $Q(p) = (p-1)^3$ and $R(p) = (p+1)^2$, $k = 1$ (known value), and the unknown initial conditions $p^2y(0) = 1$, $py(0) = 1$, and $y(0) = 4$.

We define $F = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ for the filters (13). Also, we choose $\alpha = 0.1$ in the adaptation algorithm (17) and $a = 1$ in the control law (15).

Consider different density functions $\rho(y,t)$ in (15).

1. For $\rho(y,t) = -\alpha y$, the closed loop system (16) has the equilibrium $y = 0$. Substituting $\rho(y,t)$ into (19) yields $\dot{V} = -\alpha y^2 \leqslant 0$ in the domain $D_S = \mathbb{R}$. This is an active stabilization problem, described in detail in [23]. Figure 11 shows the transient for $\alpha = 1$, $p^2y(0) = py(0) = 0$, and $y(0) = 4$ (see the curve intersecting the grey domain).

2. For $\rho(y,t) = \alpha \ln\frac{g-y}{g+y}$, where $g(t) > 0$, the closed loop system (16) has the equilibrium $y = 0$. Substituting $\rho(y,t)$ into (19) yields $\dot{V} = \alpha\ln\frac{g-y}{g+y}y < 0$ in the domain $D_S = \{y \in \mathbb{R} : -g < y < g\}$. Moreover, $\rho(y,t) \to -\infty$ as $y \to g$ and $\rho(y,t) \to +\infty$ as $y \to -g$. This is a stabilization problem with the symmetric constraints $-g$ and g. Figure 11 demonstrates the transients for $\alpha = 1$ (the curve inside the pipe with dashed boundaries), $p^2y(0) = py(0) = 0$, $y(0) = 4$, and $g(t) =$

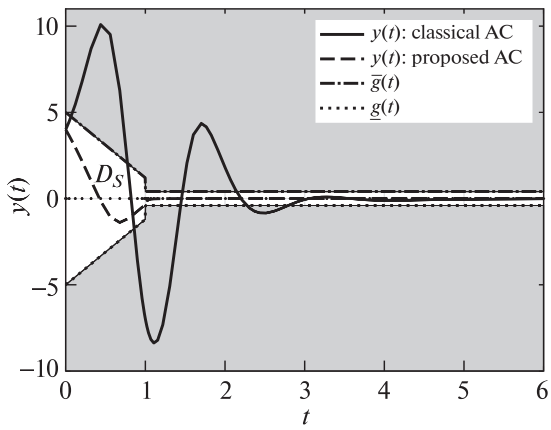

Fig. 11. Transients in the adaptive control system with the density functions $\rho(y,t) = -\alpha y$ [23] (the curve intersecting the grey domain) and $\rho(y,t) = \alpha \ln \frac{g-y}{g+y}$ (the curve inside the pipe with dashed boundaries).

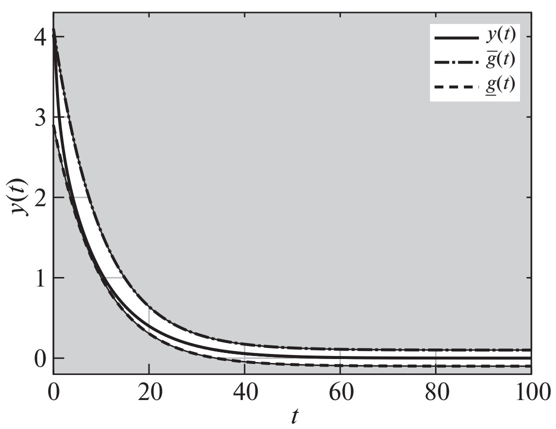

Fig. 12. Transients in the adaptive control system with the density function $\rho(y,t) = -\alpha \ln \frac{\overline{g}-y}{y-\underline{g}}$.

$\begin{cases} -4.6t+5, & t \leqslant 1 \\ 0.4, & t > 1. \end{cases}$ Clearly, in contrast to the classical adaptive control scheme [23] (the curve corresponding to $\rho(y,t) = -\alpha y$), the density function $\rho(y,t) = \alpha \ln \frac{g-y}{g+y}$ ensures a transient inside the pipe at any time instant.

3. Consider $\rho(y,t) = \alpha \ln \frac{\overline{g}-y}{y-\underline{g}}$, where $\overline{g}(t) > \underline{g}(t)$ for all t. Let us denote $y = w = \frac{\overline{g}+\underline{g}}{2}$. Then $\rho(y,t) = 0$ for $y = w$ and any t. Substituting $\rho(y,t)$ into (19) yields $\dot{V} = \alpha \ln \frac{\overline{g}-y}{y-\underline{g}} y < 0$ in the domain $D_S = \{y \in \mathbb{R}_+ : w < y < \overline{g}\} \cup \Big\{y \in \mathbb{R}_- : \underline{g} < y < w\Big\}$ and $\dot{V} > 0$ in the domain $D_U = \Big\{y \in \mathbb{R}_+ : \underline{g} < y < w\Big\} \cup \{y \in \mathbb{R}_- : w < y < \overline{g}\}$. In addition, $\rho(y,t) \to -\infty$ as $y \to \overline{g}$ and $\rho(y,t) \to +\infty$ as $y \to \underline{g}$ if $y \in \mathbb{R}_+$; $\rho(y,t) \to +\infty$ as $y \to \overline{g}$ and $\rho(y,t) \to -\infty$ as $y \to \underline{g}$ if $y \in \mathbb{R}_-$. This is a stabilization problem with asymmetric constraints $\overline{g}$ and $\underline{g}$. Figure 12 shows the transient for $\alpha = 5$, $\overline{g} = 4e^{-0.1t} + 0.1$, $\underline{g} = 3e^{-0.1t} - 0.1$, $p^2 y(0) = py(0) = 0$, and $y(0) = 4$.

4. Let $\rho(y,t) = -\alpha(y - y_m)$. Then $\rho(y,t) = 0$ for $y = y_m$ and any t. Substituting $\rho(y,t)$ into (19) yields $\dot{V} = -\alpha(y-y_m)y < 0$ in the domain $D_S = \{y \in \mathbb{R}_+ : y > y_m\} \cup \{y \in \mathbb{R}_- : y < y_m\}$ and $\dot{V} > 0$ in the domain $D_U = \{y \in \mathbb{R}_+ : y < y_m\} \cup \{y \in \mathbb{R}_- : y > y_m\}$. This is a tracking problem of y_m by y. Figure 13 demonstrates the transient for $\alpha = 50$, $y_m = e^{-0.1t}\sin(t)P(t)$, where

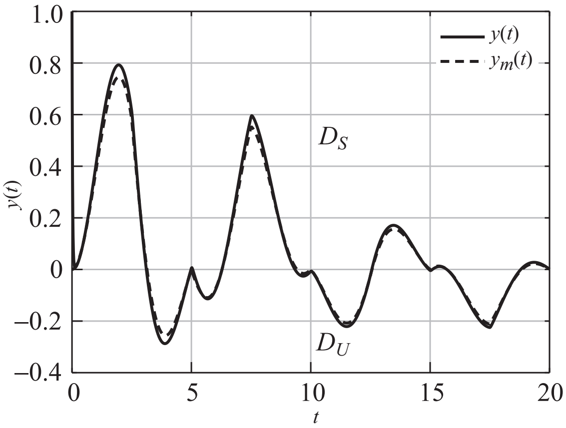

Fig. 13. Transients in the adaptive control system with the density function $\rho(y,t) = -\alpha(y - y_m)$.

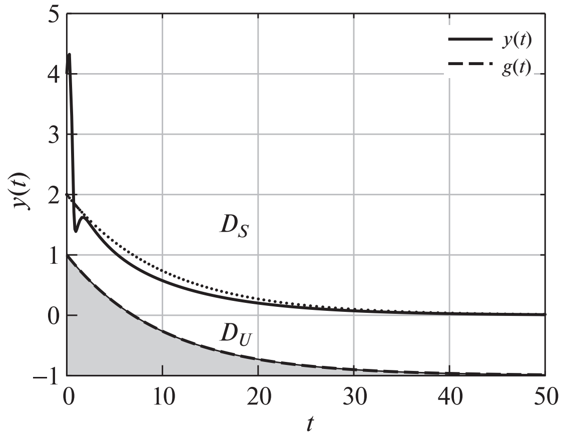

Fig. 14. Transients in the adaptive control system with the density function $\rho(y,t) = -\alpha \ln(y - g)$.

$P(t) \in [0, 1.25]$ is a generator of triangular pulses (with isosceles triangles) with a period of 5 s, $p^2 y(0) = py(0) = 0$, and $y(0) = 1$.

5. Consider $\rho(y,t) = -\alpha \ln(y - g)$, where $g(t) \geqslant -1$. Then $\rho(y,t) = 0$ for $y = g + 1$ and any t. Substituting $\rho(y,t)$ into (19) yields $\dot{V} = -\alpha \ln(y - g)y < 0$ in the domain $D_S = \{y \in \mathbb{R}_+ : y > g+1\}$ and $\dot{V} > 0$ in the domain $D_U = \{y \in \mathbb{R}_+ : y > g + 1\}$. In addition, $\rho(y,t) \to -\infty$ as $y \to g$. This is a descent problem of $x(t)$ long a surface with the boundary $g(t)$. Figure 14 shows the transient for $\alpha = 10$, $g = 2e^{-0.1t} - 1$, $p^2 y(0) = py(0) = 0$, and $y(0) = 4$.

3.3. Comparing the Control Laws Proposed with Some Existing Ones

In this subsection, we compare the control laws (8), (10) and (13), (15), (17) with the control laws obtained using the method of barrier Lyapunov functions [27, 28], the method of funnel control [10, 12], and control methods with prescribed performance [11, 13, 14, 29].

- By the method and type of defining the target (admissible) set (a desired domain for the transients of the output signal in a closed loop system), the methods of [10–14, 27–29] and the proposed approach differ as follows:
  - — In [27, 28], the boundaries of the admissible set were constant whereas the reference signal was required to be smooth enough.

— In [10–12], the boundaries of the target set were defined by continuously differentiable functions symmetric with respect to the time axis. Only bounded target sets were considered therein.

— In [13, 14, 29], the boundaries of the target set were defined by continuously differentiable asymmetric functions. Only bounded target sets were considered therein.

— Within the approach proposed in this paper, target sets can be defined by continuous (or discontinuous, see Remark 1), asymmetric functions (see case 5 in Example 5). The target set may be unbounded (see cases 2, 3, and 5 in Example 5). The reference signal may be defined by continuous (see case 4 in Example 5) or piecewise continuous (see Remark 1) functions.

• By the control law design and stability analysis of the closed loop system, the methods of [10–14, 27–29] and the proposed approach differ as follows:

— The methods [27, 28] involve special-form Lyapunov functions existing on a certain subset of the definitional domain of the system (an admissible set).

— The methods [10–12] involve special-form density functions.

— The approaches [11, 13, 14, 29] consider a nonlinear coordinate transformation reducing the original problem with constraints to an unconstrained problem. However, this transformation leads to the analysis of an extended system containing the variables of the original and new systems, which complicates the control law structure and the stability analysis of the closed loop system. Moreover, the nonlinear coordinate transformation leads to the study of the system $\dot{\varepsilon}(t) = \rho(\varepsilon, t) f(\varepsilon, t)$, where ε is a new variable and $\rho(\varepsilon, t) > 0$ is a density function depending on the derivative of the coordinate replacement function with respect to the variable ε. Hence, this system is a particular case of the systems considered in Section 2.

— The approach proposed in the paper involves no coordinate transformation, which eliminates the need to consider additional variables and additional dynamic systems. The sign of the density function can be arbitrary. The Lyapunov function may exist in the entire definitional domain of the system.

4. CONCLUSIONS

This paper has considered a class of dynamic systems called density systems. Such systems contain on the right-hand side a density function defining the properties of the space. It is possible to affect the behavior of the system under study by determining the properties of the density function. This conclusion has been used to design control laws. As has been demonstrated above, different definitions of the density function lead to classical control laws and new ones with new target requirements for the system. In particular, an adaptive control law ensuring transients in a given set has been constructed as one example; classical adaptive control provides only the ultimate boundedness of system trajectories. In this case, the parameters of the set are specified using the density function, which defines the density of the space under consideration. The simulation results have confirmed the theoretical outcomes.

Also, it has been demonstrated how some existing control algorithms can be modified using the density function to obtain new transient quality. In the future, the properties of density systems can also be employed to design more complex control algorithms, e.g., output-feedback control with any relative degree of the plant, control using observers, sliding mode control, etc.

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