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NONLINEAR SYSTEMS

Pattern Bifurcation in a Nonlocal Erosion Equation

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Abstract—This paper considers a periodic boundary value problem for a nonlinear partial differential equation with a deviating spatial variable. It is called the nonlocal erosion equation and was proposed as a model for the formation of dynamic patterns on the semiconductor surface. As is demonstrated below, the formation of a spatially inhomogeneous relief is a self-organization process. An inhomogeneous relief appears due to local bifurcations in the neighborhood of homogeneous equilibria when they change their stability. The analysis of this problem is based on modern methods of the theory of infinite-dimensional dynamic systems, including such branches as the theory of invariant manifolds, the apparatus of normal forms, and asymptotic methods for studying dynamic systems.

Keywords: nonlocal erosion equation, attractors, stability, bifurcations, normal forms, asymptotics

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1. INTRODUCTION

Since the 1980s, describing the formation of an inhomogeneous (e.g., wavelike) relief on the surface of semiconductor materials bombarded by an ion flow has always been a topical problem of micro- and nanoelectronics. The nature of a wavelike relief continues to cause much discussion; for example, see [1–6]. According to experimental evidence, a wavelike nanorelief is formed on the surface of semiconductors and dielectrics in a certain range of ion incidence angles. Of course, it depends on the beam intensity, ion type, and the material of the sample subjected to ion bombardment. The process of nanostructure formation on the silicon surface has been most intensively investigated at the experimental level.

Almost immediately, mathematical models were proposed to explain the phenomenon of inhomogeneous micro- and nano-relief formation. Two approaches were used: stochastic and deterministic.

In terms of applications, a more attractive approach is to treat such a process as dynamic. The best-known model was introduced by Bradley and Harper [7]. This model involves one version of the well-known Kuramoto–Sivashinsky equation supplemented with boundary conditions, natural from a physical point of view. In principle, different variations and modifications of this model gave a sufficiently convincing description of the process of inhomogeneous (wavelike) relief formation. At the same time, this model suffers from several drawbacks in its initial formulation. One of them is that in many cases, the corresponding boundary value problem revealed the possible formation of an inhomogeneous relief with the leading role of the first possible mode; for example, see [8–11]. In many cases, such a conclusion contradicts the results of experiments.

A possible informal modification is a model known as the nonlocal erosion equation [8–11]. It covers several nonlocal effects, first of all, the fact that the points of ion penetration (entry) into the semiconductor material and its exit do not necessarily coincide. This led to the emergence of a mathematical model with a partial differential equation in which some terms contain an unknown function with a deviating spatial variable.

This paper considers a nonlocal erosion equation augmented with periodic boundary conditions. Other problem statements for the nonlocal erosion equation can be found in [11–16].

Below, we study a periodic boundary value problem (BVP) for the nonlocal erosion equation

$$u_\tau = du_{yy} - c_1 w_y + c_2 w_y^2 + c_3 w_y^3, \tag{1.1}$$

$$u(\tau, y + 2l) = u(\tau, y), \tag{1.2}$$

where $u = u(\tau, y)$, $w = u(\tau, y - h_0)$, and h_0 is a positive constant to consider nonlocal effects (this constant is set proportional to the average distance between the entry and exit points of the ion from the incident beam); $d > 0$, $c_1 > 0$, c_2, and $c_3 \in \mathbb{R}$ are constants characterizing the bombardment conditions. For example, d is the diffusion coefficient of the target material, and the coefficient c_1 specifies the intensity (energy) of the ion beam. The deviation $h_0 > 0$ is the main parameter distinguishing this model from others (e.g., from the Bradley–Harper model). The constant h_0 is proportional to the inclination angle between the beam direction and the normal to the surface, which is considered flat before the bombardment. From the very beginning, let us emphasize the following aspect. In principle, the relief deviation from an equilibrium, $u(\tau, y)$, must depend on the second spatial coordinate y_1: $u = u(\tau, y, y_1)$. But in most experiments, the dependence on y_1 is rather weak and, therefore, the approximation $u = u(\tau, y)$ is considered acceptable.

With the changes of variables

$$\tau = \frac{l}{\pi c_1} t, \quad y = \frac{l}{\pi} x,$$

the BVP (1.1), (1.2) can be written as

$$u_t = au_{xx} - w_x + b_2 w_x^2 + b_3 w_x^3, \tag{1.3}$$

$$u(t, x + 2\pi) = u(t, x), \tag{1.4}$$

where $u = u(t, x)$, $w = u(t, x - h)$, and

$$h = \frac{h_0 \pi}{l}, \quad a = \frac{d\pi}{lc_1}, \quad b_2 = \frac{c_2 \pi}{c_1 l}, \quad b_3 = \frac{c_3 \pi^2}{l^2 c_1}.$$

Note that the BVP (1.3), (1.4) has the solution $u(t, x) = \alpha$, where $\alpha \in \mathbb{R}$. If the BVP (1.3), (1.4) is supplemented by the boundary conditions

$$u(0, x) = f(x) \ \ (w(0, x) = f(x - h)), \tag{1.5}$$

where $f(x) \in \mathbb{H}_2^1$, the resulting initial boundary value problem will be locally well-posed. This outcome follows from the results obtained in [17, 18]. Moreover, the initial boundary value problem (1.3)–(1.5) generates a local smooth semiflow $T^t : f(x) \to u(t, x),\ t \in (0, \delta), \delta > 0$.

Recall that $f(x) \in \mathbb{H}_2^1$ if:

1) $f(x + 2\pi) = f(x)$.

2) For $x \in [0, 2\pi]$, the inclusion $f(x) \in \mathbb{W}_2^1[0, 2\pi]$ holds, where $\mathbb{W}_2^1[0, 2\pi]$ is the space of functions $f(x)$ such that $f(x) \in \mathbb{L}_2(0, 2\pi)$ and their generalized derivative $f'(x) \in \mathbb{L}_2(0, 2\pi)$ (for example, see [19]).

The BVP (1.3), (1.4) has one peculiarity as follows. Let $u(t, x)$ be any solution of this problem; then $\alpha + u(t, x)$ is also its solution. Below, we will investigate the structure of the neighborhood of all solutions $u(t, x) = \alpha$ (spatially homogeneous equilibria). In particular, it is supposed to study the formation mechanism of local attractors containing spatially inhomogeneous solutions.

2. SOME PRELIMINARIES

Consider the nonlinear BVP (1.3), (1.4). We denote by

$$M(u) = \frac{1}{2\pi}\int_0^{2\pi} u(t,x)dx$$

the space mean of the function $u(t,x)$. Representing the solution $u(t,x)$ as a Fourier series with respect to the spatial variable x gives

$$u(t,x) = u_0(t) + \sum_{n\neq 0} u_n(t)\exp(inx),$$

where

$$u_0(t) = M(u), \quad u_n(t) = \frac{1}{2\pi}\int_0^{2\pi} u(t,x)\exp(-inx)dx.$$

Hence, any solution solution of the nonlinear BVP (1.3), (1.4) can be written as

$$u(t,x) = u_0(t) + v(t,x),\ v(t,x) = \sum_{n\neq 0} u_n(t)\exp(inx),\ \ M(v) = 0.$$

Therefore, the BVP (1.3), (1.4) takes the form

$$u_{0t}(t) = b_2 M(w_x^2) + b_3 M(w_x^3), \tag{2.1}$$
$$v_t = Av + F_2(w_x) + F_3(w_x), \tag{2.2}$$
$$v(t,x+2\pi) = v(t,x),\ M(v) = 0. \tag{2.3}$$

Equations (2.1) and (2.2) employ the following notations:

$$Av = av_{xx} - w_x, \quad w = v(t,x-h),$$
$$F_2(w_x) = b_2 w_x^2 - b_2 M(w_x^2), \quad F_3(w_x) = b_3 w_x^3 - b_3 M(w_x^3).$$

When forming the right-hand side of the differential equation (2.2), we take into account that $Au_0 = 0$ and the right-hand side of the original partial differential equation (1.3) is independent of $u_0(t)$.

The BVP (1.3), (1.4) can be analyzed in two stages. The first stage consists in studying the BVP (2.2), (2.3). After that, the second stage is to reconstruct $u_0(t)$ using equation (2.1). Note that the function $u_0(t)$ is reconstructed within an arbitrary constant without additional conditions from equation (2.1).

Thus, the main point in investigating the BVP (1.3), (1.4) is to study the auxiliary nonlinear BVP (2.2), (2.3). We emphasize that it has the unique spatially homogeneous equilibrium $v(t,x) = 0$.

3. STABILITY OF THE TRIVIAL SOLUTION OF THE AUXILIARY NONLINEAR BVP

To analyze the stability of the trivial equilibrium of the nonlinear BVP (2.2), (2.3), we first examine its linearized version, i.e., the linear BVP

$$v_t = Av,\ Av = av_{xx} - w_x, \tag{3.1}$$
$$v(t,x+2\pi) = v(t,x),\ M(v) = 0,\ w = v(t,x-h). \tag{3.2}$$

Consider the linear differential operator (LDO)

$$Ap = A(a,h)p = ap_{xx}(x) - p_x(x-h),$$

where a sufficiently smooth function $p(x)$ satisfies the periodic boundary conditions $p(x+2\pi) = p(x)$ and has the zero mean. This operator possesses a countable set of eigenvalues

$$\lambda_n = \lambda_n(a,h) = -an^2 - in\exp(-inh), \quad n = \pm 1, \pm 2, \ldots;$$

the corresponding eigenfunctions $\{\exp(inx)\}$ form a complete orthogonal system of functions in the space $\mathbb{L}_{2,0}(0,2\pi)$, i.e., $f(x) \in \mathbb{L}_{2,0}(0,2\pi)$ if $f(x) \in \mathbb{L}_2(0,2\pi)$ and $M(f) = 0$.

The following result is true.

Lemma 1. *If*

$$Re\lambda_n(a,h) < 0$$

for given values a and h, then all solutions of the linear BVP (3.1), (3.2) *are asymptotically stable in the metric of the initial condition space of the BVP* (3.1), (3.2).

A natural choice of the initial condition space is the functional space $\mathbb{H}^1_{2,0}$: $f(x) \in \mathbb{H}^1_{2,0}$ if $f(x) \in \mathbb{H}^1_2$ and $M(f) = 0$. Indeed, let us consider the initial boundary value problem (3.1), (3.2), (1.5) for $f(x) \in \mathbb{H}^1_{2,0}$. Its explicit-form solution is given by

$$v(t,x) = \sum_{n \neq 0} f_n \exp(\lambda_n t)\exp(inx), \tag{3.3}$$

where λ_n denote the eigenvalues of the LDO A and $\{f_n\}$ are the Fourier coefficients of the function $f(x)$ ($f_0 = 0$ since $M(f) = 0$).

It is straightforward to verify that:

1) $v(t,x) \to f(x)$ in the metric of $\mathbb{H}^1_2$ as $t \to +0$.

2) For $t \geqslant t_0 > 0$, the solution (3.3) is an infinitely differentiable function.

Property 2) is immediate from the following result, which can be checked in a fairly standard way: for $t \geqslant t_0 > 0$ the series on the right-hand side of (3.3) converges uniformly together with its partial derivatives of any order. This result is proved using the fact that

$$\lim_{|n|\to\infty} \frac{\lambda_n(a,h)}{n^2} = -a.$$

Note also that $\lim_{|n|\to\infty}(Im(\lambda_n(a,h)))/n^2 = 0$. Hence, $|Im\lambda_n(a,h)| \leqslant K|Re\lambda_n(a,h)|$ if $|n| \geqslant n_0$ ($n_0 \in \mathbb{N}$, the set of natural numbers) and K is some positive constant.

In particular, these results allow stating that the LDO A is the generator of the analytic semigroup of linear bounded operators, and the linear BVP (3.1), (3.2) can be included in the class of abstract parabolic equations in the sense of the definitions from [17, 18, 20].

If $Re\lambda_m(a,h) > 0$ for some $n = m$, the solutions of the linear BVP (3.1), (3.2) are, of course, unstable.

The considerations presented above lead to another result as follows.

Lemma 2. *Assume that*

$$Re\lambda_n \leqslant -\gamma_0 < 0$$

for all $n \in \mathbb{Z}_*$ (the set of integers $n \neq 0$). Then the trivial solution of the nonlinear BVP (2.2), (2.3) *is asymptotically stable. At the same time, if there exists an integer $m \in \mathbb{Z}_*$ such that $Re\lambda_m > 0$, then this solution is unstable.*

Note that the conditions $Re\lambda_n \leqslant 0$, $Re\lambda_m = 0$ for some $m \in Z_*$ select a critical case in the stability problem of the trivial solution of the BVP (2.2), (2.3).

The remainder of this section focuses on the following question: under what conditions is the critical case implemented in the stability problem of the trivial solution of the BVP (2.2), (2.3). Let us emphasize that, for $h = 0$,

$$\lambda_n(a, 0) = -an^2 - in$$

and, consequently, $Re\lambda_n(a, 0) < 0$ for all $n \in \mathbb{Z}_*$ ($\mathbb{Z}_* = \mathbb{Z}\backslash\{0\}$). Therefore, the critical case is possible only if $h > 0$ ($h \geqslant 0$ by the problem statement). For all $a > 0$ we will determine the least positive value $h = h_*(a)$ implementing the critical case.

First, it is necessary to find h_n satisfying

$$Re\lambda_n(a, h_n) = 0.$$

Such values h_n should be obtained as the solutions of the equation $-an^2 - n\sin nh = 0$ or

$$\sin nh = -an. \tag{3.4}$$

Equation (3.4) has solutions if $|an| \leqslant 1$. Without loss of generality, assume that $n \in \mathbb{N}$ (the set of natural numbers) since replacing n with $-n$ does not change equation (3.4).

Thus, $an \leqslant 1$. Then equation (3.4) possesses two groups of solutions:

1) $h_n(m) = \frac{1}{n}(2\pi m - \arcsin(an)), m \in \mathbb{Z}$,

2) $h_n(k) = \frac{1}{n}(\pi + \arcsin(an) + 2\pi k), k \in \mathbb{Z}$.

In the first group of solutions of the trigonometric equation, the least positive root is $h_n(1) = (2\pi - \arcsin(na))/n$; in the second group of solutions, $h_n(0) = (\pi + \arcsin(na))/n$. Furthermore, we have the inequality

$$\frac{1}{n}(\pi + \arcsin(na)) \leqslant \frac{1}{n}(2\pi - \arcsin(na)) \tag{3.5}$$

for any natural number n (of course, if $na \leqslant 1$). Inequality (3.5) is equivalent to

$$2\arcsin(na) \leqslant \pi \quad \text{or} \quad \arcsin(na) \leqslant \frac{\pi}{2}.$$

The resulting conclusion is that h_* should be determined as the least element of the sequence

$$d_n = d_n(a) = \frac{1}{n}(\pi + \arcsin(an)) \text{ if } n \leqslant \frac{1}{a}.$$

We underline that $d_n = h_n(0)$. Clearly, in principle, the least value d_n can be chosen by linear search. For example, if $a = 1$, this sequence contains one element $d_1 = 3\pi/2$ and $h_* = 3\pi/2$. In the case $a = 1/2$, we obtain $d_1 = 7\pi/6$ and $d_2 = 3\pi/4$; hence, $h_* = 3\pi/4$. At the same time, the number of elements in the sequence $d_n(a)$ grows when decreasing a. Therefore, linear search should be improved to select h_* faster.

For example, suppose that for all $a \geqslant a_0$, $d_k(a) \leqslant d_{k-1}(a)$; in other words, the sequence $d_k(a)$ decreases with increasing k. (An appropriate positive constant a_0 will be specified below.) This property of the sequence $d_k(a)$ can be checked by analyzing the inequalities

$$g_k(a) = (k-1)\arcsin(ka) - k\arcsin((k-1)a) \leqslant \pi;$$

for $k = 1, 2, 3$, they are trivial and hold for all admissible a. As is easily established,

$$\frac{dg_k(a)}{da} = k(k-1)\left(\frac{1}{\sqrt{1-(ak)^2}} - \frac{1}{\sqrt{1-a^2(k-1)^2}}\right) > 0, \quad ak \in [0, 1).$$

Hence, $d_k(a) \leqslant d_{k-1}(a)$ for all admissible a if this inequality is valid for the maximum possible value $a\,(a = 1/k)$. The issue under consideration is thereby reduced to checking the inequalities

$$\frac{1}{k-1}\left(\pi + \arcsin\frac{k-1}{k}\right) \geqslant \frac{3\pi}{2k} \quad \text{or} \quad \arcsin\frac{k-1}{k} \geqslant \frac{\pi(k-3)}{2k}.$$

As it turns out, the latter inequalities hold for $k = 1, \ldots, 10$.

Well, let $a > 1/11$. Then the minimum value h_* is $h(a) = (\pi + \arcsin(ka))/k$, where $k = [1/a]$, since this choice of a makes the sequence d_k decreasing. For the other numbers k, i.e., $k \geqslant 11$ $(a \leqslant 1/11)$, the choice procedure of h_* can be alternatively simplified as follows. Consider the auxiliary function

$$B(z) = \frac{1}{z}(\pi + \arcsin(z)),$$

which is defined for $z \in (0, 1]$. It is straightforward to verify that this function decreases for $z \in (0, z_*)$ and increases for $z \in (z_*, 1)$; naturally, z_* is its minimum point, i.e., $B'(z_*) = 0$. We determine the corresponding value z_* as the least positive root of the equation $B'(z) = 0$. As it turns out, $z = z_* \approx 0.9761$ and $d_n(a) = aB(na)$. Therefore, $h_* = \min\{d_m(a), d_{m+1}(a)\}$, where $ma \leqslant z_*$ and $(m+1)a > z_*$.

The possibility $d_m = d_{m+1}$ will be eliminated from consideration as a special (exceptional) case not discussed here. This case leads to another bifurcation problem with a codimension of 2. Further analysis will be restricted to the general case $d_m \neq d_{m+1}$.

Consider now the LDO depending on a small parameter, i.e.,

$$A(\varepsilon)y = ay'' - y'(x - h(\varepsilon)),$$

where $h(\varepsilon) = h_*(1 + \nu\varepsilon)$, $\nu = \pm 1$ or $0, \varepsilon \in (0, \varepsilon_0)$, $h_* = (\pi + \arcsin(ma))/m$. The definitional domain of this operator consists of sufficiently smooth functions satisfying the condition $y(x + 2\pi) = y(x)$, $M(y) = 0$. For such $h = h(\varepsilon)$, the LDO has a countable set of eigenvalues

$$\lambda_k(\varepsilon) = -ak^2 - ik\exp(-ikh(\varepsilon)), \quad k = \pm 1, \pm 2, \ldots.$$

In addition, given $k \neq \pm m$,

$$Re\lambda_k \leqslant -\gamma_0 < 0,$$

$\lambda_{\pm m}(\varepsilon) = -am^2 \mp im\exp(-i(\pi + \mu_m)(1 + \nu\varepsilon))$, where $\mu_m = \arcsin(ma)$, and consequently,

$$\lambda_{\pm m}(0) = \pm i\sigma_m, \quad \sigma_m = m\cos(\mu_m) = m\sqrt{1 - (ma)^2}.$$

In other words, for $(ma)^2 \neq 1$, the LDO A has a pair of simple pure imaginary eigenvalues; for $(ma)^2 = 1$, it has double zero eigenvalue. Therefore, the case $(ma)^2 = 1$ needs deeper analysis. The next section will be restricted to the general case $\sigma_m \neq 0$.

Note also that

$$\lambda'_m(\varepsilon)|_{\varepsilon=0} = \tau'_m + i\sigma'_m,$$

where $\tau'_m = \nu m(\pi + \arcsin(ma))\sqrt{1 - (ma)^2} \neq 0$ and $\sigma'_m = -\nu(\pi + \arcsin(ma))m^2 a \neq 0$. For $\nu = 1$, we obtain the inequality $\tau'_m > 0$, i.e., stability is lost when exceeding the critical value $h = h_*$; if $\nu = -1$, then $\tau'_m < 0$ and the trivial solution of the BVP (2.2), (2.3) remains stable. Finally, $\nu = 0$ corresponds to the critical case of a pair of pure imaginary eigenvalues.

4. LOCAL BIFURCATIONS

This section is devoted to the nonlinear BVP (2.2), (2.3) for $h = h(\varepsilon) = h_*(1 + \nu\varepsilon)$. With such a choice of h, the BVP (2.2), (2.3) can be written as

$$v_t = A(\varepsilon)v + F_2(w_x, \varepsilon) + F_3(w_x, \varepsilon), \tag{4.1}$$

$$v(t, x + 2\pi) = v(x), \ M(v) = 0, \tag{4.2}$$

where

$$w = v(t, x - h(\varepsilon)), \quad A(\varepsilon)v = av_{xx}(t, x) - v_x(t, x - h(\varepsilon)),$$
$$F_j(w_x, \varepsilon) = F_j(w_x), \quad j = 2, 3.$$

The functions $F_j(w_x)$ have been introduced above. Recall that the deviation value h_* has been chosen in Section 3. As a result, an almost critical case is implemented for the stability spectrum of the trivial solution of the BVP (4.1), (4.2) (the spectrum of the LDO $A(\varepsilon)$). The BVP (4.1), (4.2) has a two-dimensional invariant manifold $M_2(\varepsilon)$ attracting all solutions from a sufficiently small neighborhood $Q(r_0)$ of the trivial solution of this BVP. In addition, the radius r_0 of the ball in the space $\mathbb{H}_2^1$ is sufficiently small but independent of ε; for example, see [21–23]. As is well known (e.g., see [23]), bifurcations can be analyzed by studying a system of two differential equations, commonly called the normal form (NF) according to the terminology originally proposed by A. Poincaré [23]. In the general case, such an equation can be written in the complex-valued form

$$z' = (\tau_m' + i\sigma_m')z + (l_1 + il_2)z|z|^2, \tag{4.3}$$

where $z = z(s)$, $s = \varepsilon t$ denotes "slow" time, the prime indicates the derivative with respect to s, and $l_1, l_2 \in \mathbb{R}$. By an a priori assumption, $l_1 \neq 0$: the first Lyapunov value is nonzero. In equation (4.3), all vanishing terms as $\varepsilon \to 0$ are discarded. Equation (4.3) represents the principal part of the NF or the truncated NF. Under $l_1 \neq 0$, equation (4.3) plays a determinative role in the analysis of the BVP (2.2), (2.3), (4.1), (4.2).

The solutions of the BVP (4.1), (4.2) belonging to $M_2(\varepsilon)$ will be found as the sum

$$v(t, x, z, \overline{z}, \varepsilon) = \varepsilon^{1/2}v_1(t, x, z, \overline{z}) + \varepsilon v_2(t, x, z, \overline{z}) + \varepsilon^{3/2}v_3(t, x, z, \overline{z}) + O(\varepsilon^2). \tag{4.4}$$

In addition, of course,

$$w(t, x, z, \overline{z}, \varepsilon) = \varepsilon^{1/2}w_1(t, x, z, \overline{z}) + \varepsilon w_2(t, x, z, \overline{z}) + \varepsilon^{3/2}w_3(t, x, z, \overline{z}) + O(\varepsilon^2), \tag{4.5}$$

where

$$w(t, x, z, \overline{z}, \varepsilon) = v(t, x - h_*(\varepsilon), z, \overline{z}, \varepsilon),$$
$$w_j(t, x, z, \overline{z}) = v_j(t, x - h_*, z, \overline{z}), \quad j = 1, 2, 3.$$

Finally,

$$v_1(t, x) = zq + \overline{zq}, \quad q = q(t, x) = \exp(i\sigma_m t)\exp(imx).$$

The functions v_2 and v_3 are $v_2(t, x, z, \overline{z}), v_3(t, x, z, \overline{z}) \in \Phi$. The symbol Φ denotes the class of functions defined above.

We have $\varphi = \varphi(t, x, z, \overline{z}) \in \Phi$ if this function satisfies the following conditions:

1) It smoothly depends on the variables for all t, $x \in \mathbb{R}$, and $|z| < \delta$, where δ is some positive constant.

2) $\varphi(t, x, 0, 0) = 0$.

3) It has periods of $2\pi/\sigma_m$ and 2π in the variables t and x, respectively.

4) a) $M(\varphi) = 0$ for all $t, z, \overline{z}$ under consideration.

b) $M_\pm(\varphi) = \dfrac{\sigma_m}{(2\pi)^2} \int\limits_0^{2\pi} \left(\int\limits_0^{2\pi/\sigma_m} \varphi q_\mp dt \right) dx = 0,\ q_+ = q,\ q_- = \overline{q}.$

To analyze the BVP (4.1), (4.2), we formulate an auxiliary statement, often called the solvability conditions in branches of differential equations. Let $A_0 = A(0)$.

Remark 1. Consider the linear inhomogeneous BVP

$$A_0 v = g(t,x),\ v = v(t,x),$$
$$v(t, x + 2\pi) = v(t,x),\ M(v) = 0.$$

Here, $g(t,x)$ is a sufficiently smooth function with periods of $2\pi/\sigma_m$ and 2π in the variables t and x, respectively. In addition, suppose that $M(g(t,x)) = 0$. Then this BVP has periodic solutions in the variable t if

$$M_\pm(g(t,x)) = 0.$$

The conditions $M_\pm(v) = 0$ highlight one suitable solution of the linear inhomogeneous BVP considered in Remark 1.

Substituting the sum (4.4) and the related sum (4.5) into the nonlinear BVP (4.1), (4.2) yields linear inhomogeneous BVPs for determining v_2 and v_3.

Extracting the terms at ε, we obtain the inhomogeneous BVP

$$v_{2t} - A_0 v_2 = \Phi_2(t, x, z, \overline{z}), \tag{4.6}$$
$$v_2(t, x + 2\pi) = v_2(t,x),\ M(v_2) = M_\pm(v_2) = 0. \tag{4.7}$$

When extracting the terms proportional to $\varepsilon^{3/2}$, a similar BVP has the form

$$v_{3t} - A_0 v_3 = \Phi_3(t, x, z, \overline{z}), \tag{4.8}$$
$$v_3(t, x + 2\pi) = v_3(t,x),\ M(v_3) = M_\pm(v_3) = 0, \tag{4.9}$$

where $A_0 v_j = a v_{jxx} - w_{jx},\ w_j = v_j(t, x - h_*),\ j = 2, 3,$

$$\Phi_2(t,x,z,\overline{z}) = b_2 w_{1x}^2 - b_2 M(w_{1x}^2),$$
$$\Phi_3(t,x,z,\overline{z}) = b_3 w_{1x}^3 - b_3 M(w_{1x}^3) + 2b_2(w_{1x}w_{2x} - M(w_{1x}w_{2x})) + A_1 u_1 - (z'q + \overline{z}'\overline{q}),$$
$$w_{1x} = im(Qzq - \overline{Q}\overline{zq}),\ Q = \exp(-imh_*).$$

Note that $A_1 = A'(\varepsilon)|_{\varepsilon=0}$ and

$$Q = Q_1 + iQ_2,\ Q_1 = -\sqrt{1 - (ma)^2},\ Q_2 = ma.$$

The solutions $v_2(t,x,z,\overline{z}) \in \Phi$ of the BVP (4.6), (4.7) can (and should) be found in the form

$$v_2(t,x,z,\overline{z}) = \eta_m z^2 q^2 + \overline{\eta}_m \overline{z}^2 \overline{q}^2.$$

In our case,

$$\Phi_2(t,x,z,\overline{z}) = b_2 m^2 (Q^2 z^2 q^2 - \overline{Q}^2 \overline{z}^2 \overline{q}^2),$$

and quite easy calculations give

$$\eta_m = -\frac{b_2 m^2 Q^2 \overline{p}_m}{|p_m|^2},\ p_{m_1} = 4mQ_2(1 - Q_1),\ p_{m_2} = 2m(Q_1^2 - Q_2^2 - Q_1).$$

Now, we pass to the BVP (4.8), (4.9). It has a solution $v_3(t, x, z, \overline{z}) \in \Phi$ under the solvability conditions (see Remark 1), i.e.,

$$M_{\pm}(\Phi_3) = 0. \tag{4.10}$$

Conditions (4.10) serve to determine the coefficients of the NF (4.3). As it turns out,

$$l_1 = l_1^{(2)} + l_1^{(3)}, \quad l_2 = l_2^{(2)} + l_2^{(3)},$$

where

$$l_1^{(3)} = -3b_3m^3Q_2, \quad l_2^{(3)} = 3b_3m^3Q_1,$$
$$l_1^{(2)} = -\frac{4b_2^2m^4}{p_{m_1}^2 + p_{m_2}^2}\Big(p_{m_1}Q_1(Q_1^2 - 3Q_2^2) + p_{m_2}Q_2(3Q_1^2 - Q_2^2)\Big),$$
$$l_2^{(2)} = -\frac{4b_2^2m^4}{p_{m_1}^2 + p_{m_2}^2}\Big(p_{m_1}Q_2(3Q_1^2 - Q_2^2) - p_{m_2}Q_1(Q_1^2 - 3Q_2^2)\Big),$$
$$\tau_m' = \nu m(\pi + \mu_m)\sqrt{1 - (ma)^2}, \ \sigma_m' = -\nu(\pi + \mu_m)m^2a, \ \mu_m = \arcsin(ma).$$

Note that $\tau_m' > 0$ if $\nu = 1$, and $\tau_m' < 0$ if $\nu = -1$. Thus, the coefficients of the NF (4.3) have been calculated explicitly.

Let us emphasize that, after clear transformations,

$$l_1^{(2)} = -\frac{8b_2^2m^6a}{p_{m_1}^2 + p_{m_2}^2}\Big(\sqrt{1 - (ma)^2}(1 + 4(ma)^2) + 1\Big).$$

Hence,

$$l_1 = -3b_3m^4a - \frac{8b_2^2m^6a}{p_{m_1}^2 + p_{m_2}^2}\Big(\sqrt{1 - (ma)^2}(1 + 4(ma)^2) + 1\Big).$$

According to this formula, $l_1 < 0$ for $b_3 > 0$. Obviously, the case $l_1 < 0$ is implemented "more frequently" compared to the one $l_1 > 0$.

To proceed, we investigate the NF (4.3) by letting

$$z(s) = \rho(s)\exp(i\varphi(s)).$$

Transition to the trigonometric form leads to the two real-valued differential equations

$$\rho' = \tau_m'\rho + l_1\rho^3, \tag{4.11}$$
$$\varphi' = \sigma_m' + l_2\rho^2. \tag{4.12}$$

Besides the trivial equilibrium $\rho = 0$, equation (4.11) may also have the nonzero one

$$\rho(s) = \xi = \sqrt{-\frac{\tau_m'}{l_1}},$$

which exists if $\tau_m'/l_1 < 0$.

The standard analysis using Lyapunov's theorem on the first (linear) approximation shows that the nonzero equilibrium $\rho(s) = \xi$ is asymptotically stable if $l_1 < 0$ ($\tau_m' > 0$ or, equivalently, $\nu = 1$) and unstable if $l_1 > 0$ ($\tau_m' < 0$ or, equivalently, $\nu = -1$). The trivial equilibrium of the differential equation (4.11) is asymptotically stable if $\tau_m' < 0$ and unstable if $\tau_m' > 0$.

As is easily observed, for $\rho(s) = \xi$, equation (4.12) has the solution

$$\varphi(s) = (\sigma_m' + l_2\xi^2)s + \varphi_0, \quad \varphi_0 \in \mathbb{R}.$$

The considerations presented above establish the following result.

Lemma 3. *The differential equation* (4.3) *has the limit cycle* C_0 *generated by the one-parameter family of periodic solutions*

$$z(s) = \xi \exp(i(\sigma'_m + l_2\xi^2)s + i\varphi_0), \tag{4.13}$$

where $\xi = \sqrt{-\tau'_m/l_1}$. *These periodic solutions are stable (orbitally asymptotically stable) if* $l_1 < 0$ $(\tau'_m > 0)$ *and unstable if* $l_1 > 0$ $(\tau'_m < 0)$.

Lemma 3 was proved in many publications. According to [11–16], we arrive at the following fact.

Theorem 1. *There exists a value* $\varepsilon_0 > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_0)$, *the BVP* (2.2), (2.3) *with* $h = h(\varepsilon) = h_*(1 + \nu\varepsilon)$ *has a limit cycle* $C(\varepsilon)$ *corresponding to* C_0 *that inherits the stability of* C_0. *The solutions forming* $C(\varepsilon)$ *satisfy the asymptotic representation*

$$v(t, x, \varepsilon, \gamma) = \varepsilon^{1/2}\xi\Big(\exp(imx + i(\sigma_m + \varepsilon\beta_m)t + i\gamma) + \exp(-imx - i(\sigma_m + \varepsilon\beta_m)t - i\gamma)\Big)$$
$$+ \varepsilon\xi^2\Big(\eta_m \exp(2imx + 2i(\sigma_m + \varepsilon\beta_m)t + 2i\gamma) + \overline{\eta}_m \exp(-2imx - 2i(\sigma_m + \varepsilon\beta_m)t - 2i\gamma)\Big) + O(\varepsilon^{3/2}),$$

where $\xi = \sqrt{-\tau'_m/l_1}$, $\beta_m = \sigma'_m - \tau'_m l_2/l_1$, γ *is an arbitrary real constant, and the constant* η_m *is chosen when constructing the NF (see the algorithm above).*

5. A SPECIAL CASE OF THE BIFURCATION PROBLEM

As has been mentioned in Section 3, the eigenvalues $\lambda_k(\varepsilon)$ of the LDO $A(\varepsilon)$ are given by

$$\lambda_k(\varepsilon) = -ak^2 - ik\exp(-ikh(\varepsilon)), \quad k = \pm 1, \pm 2, \ldots.$$

In the special case $a = 1/m$, it is easy to verify $Re\lambda_k(\varepsilon) \leqslant -\gamma < 0$ for $k \neq \pm m$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, where $0 < \varepsilon_0 << 1$.

For $a = 1/m$, where $m < 11$ is some natural number, the following assertions are true:

1) $h_* = 3\pi/(2m)$.

2) If $h(\varepsilon) = h_* + \varepsilon$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, then $\lambda_m(\varepsilon) = \tau_m(\varepsilon) + i\sigma_m(\varepsilon)$, where $\tau_m(\varepsilon) = -m(1 - \cos m\varepsilon)$ and $\sigma_m(\varepsilon) = -m\sin m\varepsilon$.

3) The analytical function $\tau_m(\varepsilon)$ is an even function of the variable ε.

4) The analytical function $\sigma_m(\varepsilon)$ is an odd function of the variable ε.

5) $\lambda_m(0) = \lambda_{-m}(0) = 0$. For $\varepsilon = 0$, the LDO A_0 has the double zero eigenvalue. The corresponding eigenfunctions are $\exp(\pm imx)$.

Hence, in the BVP (4.1), (4.2) with $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, there exists a two-dimensional invariant manifold $M_2(\varepsilon)$ in the neighborhood of the trivial equilibrium (as before, see Section 4). This manifold will be a local attractor for the solutions of the BVP (4.1), (4.2) with sufficiently small initial conditions. By analogy with the previous section, the dynamics of the solutions of the BVP (4.1), (4.2) can be analyzed by investigating the complex-valued differential equation (NF)

$$z' = (\tau(\varepsilon) + i\sigma(\varepsilon))z + \psi(z, \overline{z}, \varepsilon) \tag{5.1}$$

with a sufficiently smooth function $\psi(z, \overline{z}, \varepsilon)$ such that

$$\psi(0, 0, \varepsilon) = \frac{\partial\psi}{\partial z}|_{z=0} = \frac{\partial\psi}{\partial\overline{z}}|_{\overline{z}=0} = 0.$$

Once again, when analyzing the behavior of the solutions of the BVP (4.1), (4.2), we emphasize the determinative role of both equation (5.1) and its truncated version

$$z' = (\tau(\varepsilon) + i\sigma(\varepsilon))z + \psi_0(z, \overline{z}), \tag{5.2}$$

where $\psi_0(z,\overline{z}) = \psi(z,\overline{z},0)$ and a sufficiently smooth function $\psi_0(z,\overline{z})$ has an infinitesimal order above 1 at the zero point.

To find the principal part of the function $\psi_0(z,\overline{z})$, it suffices to consider the BVP (4.1), (4.2) with $\varepsilon = 0$ and construct the NF for it. The solutions of the BVP (4.1), (4.2) with $\varepsilon = 0$ will be obtained in the form

$$\begin{aligned} v(t,x,z,\overline{z}) = (qz + \overline{qz}) + p_2(x)z^2 + p_0(x)z\overline{z} + \overline{p}_2(x)\overline{z}^2 \\ + r_3(x)z^3 + r_1(x)z^2\overline{z} + \overline{r}_1(x)z\overline{z}^2 + \overline{r}_3(x)\overline{z}^3 + \ldots, \end{aligned} \tag{5.3}$$

where the ellipsis indicates the terms of a higher infinitesimal order in the variables $z, \overline{z}$. Finally, $q(x) = \exp(imx)$, the functions $p_2(x), p_0(x), r_1(x)$, and $r_3(x)$ have a period of 2π in the variable x and the zero means:

$$M(p_j) = 0,\ \ M_\pm(p_j) = 0,\ \ M(r_k) = 0,\ \ M_\pm(r_k) = 0,$$

where $j = 0,2$, $k = 1,3$, $q_+ = \exp(imx)$, $q_- = \exp(-imx)$, $q = q_+$, $\overline{q} = q_-$, and

$$M(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \varphi dx,\quad M_\pm(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \varphi q_\mp dx,\quad \varphi = \varphi(x).$$

Substituting the sum (5.3) into the BVP (4.1), (4.2) with $\varepsilon = 0$ and sequentially extracting the terms proportional to z^2, $z\overline{z}$, $\overline{z}^2$, z^3, $z^2\overline{z}$, $z\overline{z}^2$, and $\overline{z}^3$, we get a system of linear inhomogeneous equations. This system will be analyzed to determine the principal part of the complex-valued function $\psi_0(z,\overline{z})$. When forming the BVP, it is necessary to consider the formula

$$\psi_0(z,\overline{z}) = \psi_2 z^2 + \psi_0 z\overline{z} + \overline{\psi}_2\overline{z}^2 + \psi_3 z^3 + \psi_1 z^2\overline{z} + \overline{\psi}_1 z\overline{z}^2 + \overline{\psi}_3\overline{z}^3 + \ldots,$$

where the ellipsis indicates the terms of a higher infinitesimal order in the variables z and $\overline{z}$.

As a result, we arrive at the following inhomogeneous BVPs for determining the periodic functions with the zero means. For example, $p_2(x)$ and $p_0(x)$ are obtained from the two linear inhomogeneous BVPs

$$A_0 p_2(x) = -b_2 m^2 q^2 + \psi_2 q, \tag{5.4}$$
$$p_2(x+2\pi) = p_2(x),\ \ M(p_2) = M_\pm(p_2) = 0, \tag{5.5}$$
$$A_0 p_0(x) = \psi_0, \tag{5.6}$$
$$p_0(x+2\pi) = p_0(x),\ \ M(p_0) = M_\pm(p_0) = 0. \tag{5.7}$$

In the required class of functions, the solutions of the auxiliary linear inhomogeneous BVPs (5.4), (5.5) and (5.6), (5.7) should be found with $\psi_2 = 0$ and $\psi_0 = 0$, and consequently,

$$p_0(x) = 0,\ p_2(x) = \eta_2 Q^2 q^2,\ \overline{p}_2(x) = \overline{\eta}_2 \overline{Q}^2 \overline{q}^2,$$

where (in this case) $Q = \exp(-ih_* m)$, i.e., $Q = \exp(-i3\pi/2) = i$ and $Q^2 = -1$. Finally,

$$\eta_2 = \frac{b_2 m}{10}(2+i).$$

At the third step of the algorithm, we have two BVPs for determining $r_1(x)$ and $r_3(x)$:

$$A_0 r_3(x) = \psi_3 q + b_3 m^3 q^3 - 4b_2\eta_2 i m^2 q^3, \tag{5.8}$$
$$r_3(x+2\pi) = r_3(x),\ M(r_3) = M_\pm(r_3) = 0, \tag{5.9}$$
$$A_0 r_1(x) = \psi_1 q + 3b_3 m^3 q - 4b_2\eta_2 i m^2 q, \tag{5.10}$$
$$r_1(x+2\pi) = r_1(x),\ M(r_1) = M_\pm(r_1) = 0. \tag{5.11}$$

From the solvability conditions of the BVPs (5.8), (5.9) and (5.10), (5.11) it follows that $\psi_3 = 0$ and $\psi_1 = -3b_3m^3 + \frac{2}{5}b_2^2m^3(-1+2i)$.

Thus, the principal part of the NF (5.1) is

$$z' = \left(-\frac{\varepsilon^2}{2}m^3 - im^2\varepsilon\right) z + (l_1 + il_2)z|z|^2, \tag{5.12}$$

where $l_1 = -3b_3m^3 - \frac{2}{5}b_2^2m^3$ and $l_2 = \frac{4}{5}b_2^2m^3$.

For analysis, we write this differential equation in the trigonometric form and let

$$z(t) = \rho(t)\exp(i\varphi(t)). \tag{5.13}$$

Then the complex-valued equation (5.12) is replaced by the two real-valued differential equations

$$\rho' = -\frac{\varepsilon^2}{2}m^3\rho + l_1\rho^3, \tag{5.14}$$

$$\varphi' = -\varepsilon m^2 + l_2\rho^2. \tag{5.15}$$

By analogy with the previous section, we start the analysis of system (5.14), (5.15) with the differential equation (5.14) for the amplitude $\rho(t)$.

Lemma 4. *In addition to the trivial equilibrium S_0: $\rho = 0$, the differential equation* (5.14) *may have the equilibrium S_*: $\rho_* = \sqrt{\varepsilon^2m^3/(2l_1)}$. The equilibrium ρ_* of the differential equation* (5.14) *exists if $l_1 > 0$. This equilibrium is unstable. The asymptotically stable equilibrium is $\rho = 0$.*

The stability of these equilibria is analyzed using Lyapunov's theorem on the first (linear) approximation. Note that the equilibrium S_* is associated with the solution of the differential equation (5.15) of the form

$$\varphi_*(t) = \left(-\varepsilon m^2 + \frac{l_2\varepsilon^2m^3}{2l_1}\right)t + \varphi_0,$$

where φ_0 is an arbitrary real constant. Equality (5.13) allows finding the periodic solution of the NF (5.12) in the variable t:

$$z(t) = z(t,\varepsilon) = \sqrt{\frac{\varepsilon^2m^3}{2l_1}}\exp(i\varphi_*(t)).$$

According to [11–16], we obtain the following result.

Theorem 2. *Let $am = 1$ $(h_* = 3\pi/(2m))$, where $m = 1,\dots,10$. There exists a positive constant ε_0 such that, for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $h = h_* + \varepsilon$, and $\varepsilon \neq 0$, the BVP* (4.1), (4.2) *has the one-parameter family of unstable periodic solutions in the variable t :*

$$\begin{aligned} v_*(t,x,\varepsilon,\varphi_0) &= \rho_*(q_m(t,x,\varphi_0) + \overline{q}_m(t,x,\varphi_0)) \\ &+ \rho_*^2(\eta_2q_m^2(t,x,\varphi_0) + \overline{\eta}_2\overline{q}_m^2(t,x,\varphi_0)) + o(\varepsilon^2), \end{aligned} \tag{5.16}$$

where $q_m(t,x,\varphi_0) = \exp(i\omega(\varepsilon)t + imx + i\varphi_0)$ and $\omega(\varepsilon) = -\varepsilon m^2 + l_2m^3\varepsilon^2/(2l_1) + o(\varepsilon^2)$. The constants l_1 and l_2 have been specified above. The family of solutions (5.16) *exists if the first Lyapunov value is $l_1 > 0$ (see Lemma 4).*

In the case $l_1 < 0$, the trivial solution of the BVP (4.1), (4.2) *is asymptotically stable.*

Remark 2. The existence of periodic solutions in the special case of the BVP (4.1), (4.2) occurs rather rarely; if it does, they are unstable. Obviously, the case $l_1 > 0$ is rare as well. The dominating situation is when $l_1 > 0$. Then the differential equation (5.14), and hence the BVP (4.1), (4.2), has no small periodic solutions in the variable t.

6. THE MAIN RESULT

We revert to the analysis of the primary nonlinear BVP (1.3), (1.4) with $h = h(\varepsilon) = h_*(1 + \nu\varepsilon)$ provided that $am \neq 1$.

Let $v(t, x, \varepsilon, \gamma)$ be a periodic solution obtained for the BVP (2.2), (2.3). Then $u_0(t)$ is determined from equation (2.1): $v(t, x, \varepsilon, \gamma)$ is substituted into its right-hand side and the resulting equation is integrated. In this case, we have

$$u_0(t, \varepsilon, \gamma_0) = \Big(2b_2\xi^2 m^2\varepsilon + o(\varepsilon)\Big)t + \gamma_0,$$

where γ_0 is an arbitrary real constant and $\xi^2 = -(\tau_m'/l_1) > 0$.

Theorem 3. *There exists a positive constant ε_0 such that, for all $\varepsilon \in (0, \varepsilon_0)$ and $h = h_*(1 + \nu\varepsilon)$, the nonlinear BVP* (1.3), (1.4) *has the two-parameter family $V_2(\varepsilon, \gamma_0, \gamma)$ of solutions*

$$u(t, x, \varepsilon) = u_0(t, \varepsilon, \gamma_0) + v(t, x, \varepsilon, \gamma)$$

if the BVP (2.2), (2.3) *possesses the limit cycle $C(\varepsilon)$. This family forms the integral manifold of the nonlinear BVP* (1.3), (1.4).

The family $V_2(\varepsilon, \gamma_0, \gamma)$ is a local attractor if the limit cycle $C(\varepsilon)$ of the auxiliary BVP (2.2), (2.3) *is a local attractor. This family is unstable (saddle) if the same property holds for $C(\varepsilon)$.*

Assume that the special case is implemented: $am = 1$ and $h_* = 3\pi/(2m)$. Of course, the main result needs an appropriate correction. In this case, Theorem 2 implies the following result.

Theorem 4. *There exists a positive constant ε_0 such that, for $\varepsilon \in (-\varepsilon_0, 0) \cup (0, \varepsilon_0)$ and $h = h_* + \varepsilon$, the nonlinear BVP* (1.3), (1.4) *has the two-parameter family $V_*(\varepsilon, \gamma_0, \varphi_0)$ of solutions*

$$u(t, x, \varepsilon) = u_0(t, \varepsilon, \gamma_0) + v_*(t, x, \varepsilon, \varphi_0),$$

where $v_*(t, x, \varepsilon, \varphi_0)$ is the solution (5.16) *of the auxiliary BVP* (4.1), (4.2) *(see Theorem 2) and*

$$u_0(t, \varepsilon, \gamma_0) = \left(b_2\frac{\varepsilon^2 m^2}{l_{1,0}} + o(\varepsilon^2)\right)t + \gamma_0,$$

where γ_0 is an arbitrary constant and $l_{1,0} = -3b_3 - 2b_2^2/5$. Recall that in this case, the solution exists if $l_{1,0} > 0$.

Note that the solution family $V_*(\varepsilon, \gamma_0, \varphi_0)$ is always unstable.

7. CONCLUSIONS

In this paper, we have studied local bifurcations in a periodic boundary value problem for a nonlocal erosion equation. It represents a partial differential equation with a deviating spatial variable. It has been demonstrated that proper consideration of the deviating variable is an essential factor in bifurcation analysis. For the deviation value $h = 0$, an inhomogeneous relief is not formed. Increasing h to some threshold values causes nanorelief formation.

In most cases, such a relief is formed as the result of Andronov–Hopf bifurcations with an appropriate choice of $h \approx h_*$ and a. The special case $ak \approx 1$ leads to another type of bifurcations and unstable patterns.

The above analysis of nanorelief formation, a topical physical problem, has turned out quite effective due to applying modern methods of the theory of dynamic systems, namely, the methods of invariant manifolds and the theory of normal Poincaré forms, extended to the class of problems with an infinite-dimensional phase space. We emphasize that the use and development of the method of integral (invariant) manifolds is quite productive in the analysis of many problems of mathematical physics: in many cases, this method reduces an original infinite-dimensional problem to the analysis of a finite-dimensional dynamic system. Another approach to the analysis of infinite-dimensional dynamic systems was shown in [24, 25].

Note that the inclusion (consideration) of nonlocal terms in a partial differential equation often significantly changes the dynamics of its solution towards higher complexity and richness. For example, bifurcations may arise on higher modes, which has been repeatedly observed in experiments.

Moreover, considering nonlocal terms in mathematical models reveals new effects in nanoelectronics and in other nonlinear models of physics (for example, see [26–29]).

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