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Robust Stability of Differential-Algebraic Equations under Parametric Uncertainty

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Abstract—This paper considers linear differential-algebraic equations (DAEs) representing a system of ordinary differential equations with an identically singular matrix at the derivative in the domain of its definition. The matrix coefficients of DAEs are assumed to depend on the uncertain parameters belonging to a given admissible set. For the parametric family under consideration, structural forms with separate differential and algebraic parts are built. As is demonstrated below, the robust stability of the DAE family is equivalent to the robust stability of its differential subsystem. For the structure of perturbations, sufficient conditions are established under which the separation of DAEs into the algebraic and differential components preserves the original type of functional dependence on the uncertain parameters. Sufficient conditions for robust stability are obtained by constructing a quadratic Lyapunov function.

Keywords: differential-algebraic equations, parametric uncertainty, arbitrarily high unsolvability index, robust stability

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1. INTRODUCTION

Consider a system of differential-algebraic equations (DAEs) of the form

$$A(\gamma)x'(t) + B(\gamma)x(t) = 0, \quad t \in T = [0, +\infty), \tag{1.1}$$

where $\gamma = \mathrm{col}\,(\gamma_1, \ldots, \gamma_l)$ denotes an uncertain vector parameter belonging to a given admissible set $\Gamma = \Big\{\gamma \in \mathbf{R}^l : \ \|\gamma\|_{\mathbf{R}^l} \leqslant a\Big\}$; $A(\gamma)$ and $B(\gamma)$ are known real matrices of dimensions $(n \times n)$; finally, $x(t)$ is the desired n-dimensional function. By assumption, $\det A(\gamma) \equiv 0$.

A crucial characteristic of DAEs is the unsolvability index, which reflects the complexity of the internal structure of the system. The higher value this index takes, the more difficult it will be to divide DAEs into differential and algebraic components. Without such separation, it is impossible to analyze the stability of nonstationary systems. See Subsection 2.1 for the exact definition of the index used in this paper. The closest concept to this definition is the differentiation index [1], introduced for unperturbed nonstationary systems of DAEs. If the unsolvability index exists, it is equal to the differentiation index.

This paper discusses the issue of asymptotic stability for the parametric family (1.1). Robust stability analysis is much more complicated for DAEs than for systems of ordinary differential equations resolvable with respect to the derivative; for example, see [2, pp. 186–225]. The explanation is that even in the simplest case of index 1, an arbitrarily small perturbation of the coefficients may violate the internal structure of the system and, consequently, change the properties and type of the general solution [3, p. 61].

There are relatively few publications on the robust stability of DAEs; in particular, we refer to [4–14]. The papers [4, 5] are considered to be the pioneering studies on this subject. Most authors investigated systems with the perturbed matrix coefficient at $x(t)$ [5–9]. Only a few works supposed perturbations of the matrix at the derivative of the desired vector function [4, 10–12]. Separate research was devoted to robust stability and estimation of the stability radius of nonstationary DAEs of index 1; see [6, 7, 13].

Robust stability is also analyzed when studying the admissibility of linear DAEs; for example, we mention [15–21]. In addition to stability, admissibility implies that the system possesses the properties of regularity and either causality (for discrete-time systems) or impulse freeness (in the continuous-time case).

Presently, it is still topical to investigate the robust stability of parametric families of the form (1.1) of an arbitrarily high unsolvability index with perturbations in all matrix coefficients.

For the family (1.1) with the vector and scalar parameters, we prove below the existence of structural forms with separated algebraic and differential subsystems as well as propose algorithms for building these forms. In our previous works (e.g., [22, 23]), the dimension of the solution space and the structure of the general solution of perturbed DAEs were ensured the same as those of the nominal system by imposing some finite relations on the perturbations. In this paper, the structural forms are defined simultaneously for the entire family (1.1), and their existence is established without additional constraints on the perturbations.

Unfortunately, the reduction of the DAEs (1.1) to a certain structural form generally complicates the original type of functional dependence on uncertain parameters. Under the sufficient conditions derived below, the system is separated into algebraic and differential parts while preserving the original type of this dependence in the differential subsystem.

Under the assumptions accepted here, the stability of system (1.1) is equivalent to the stability of its differential part, which represents a parametric family as well. Sufficient conditions for robust stability follow from the required existence of a general quadratic Lyapunov function for the differential subsystem.

2. SUFFICIENT CONDITIONS FOR ROBUST STABILITY

2.1. The Structural Form for DAEs with Parametric Uncertainty

For system (1.1) we define the following matrices:

$$\begin{aligned}\mathcal{B}_r\left[B(\gamma)\right] &= \operatorname{col}\left(B(\gamma), O, \dots, O\right),\\ \mathcal{A}_r\left[A(\gamma), B(\gamma)\right] &= \operatorname{col}\left(A(\gamma), B(\gamma), O, \dots, O\right)\end{aligned} \tag{2.1}$$

of dimensions $(n(r+1) \times n)$,

$$\Lambda_r\left[A(\gamma), B(\gamma)\right] = \begin{pmatrix} O & O & \dots & O & O \\ A(\gamma) & O & \dots & O & O \\ B(\gamma) & A(\gamma) & \dots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \dots & A(\gamma) & O \\ O & O & \dots & B(\gamma) & A(\gamma) \end{pmatrix} \tag{2.2}$$

of dimensions $(n(r+1) \times nr)$, and

$$\mathcal{D}_r\left[A(\gamma), B(\gamma)\right] = \left(\ \mathcal{B}_r\left[B(\gamma)\right] \,\middle|\, \mathcal{A}_r\left[A(\gamma), B(\gamma)\right] \,\middle\|\, \Lambda_r\left[A(\gamma), B(\gamma)\right]\ \right) \tag{2.3}$$

of dimensions $(n(r+1) \times n(r+2))$.

Assume that

$$\operatorname{rank} \Lambda_r \left[A(\gamma), B(\gamma)\right] = c = \text{const} \quad \forall \gamma \in \Gamma \tag{2.4}$$

for some r $(0 \leqslant r \leqslant n)$ and the matrix $\mathcal{D}_r\left[A(\gamma), B(\gamma)\right]$ $\forall\gamma \in \Gamma$ has a nonsingular minor of order $n(r+1)$ containing c columns of the matrix $\Lambda_r\left[A(\gamma), B(\gamma)\right]$ and all columns of the matrix $\mathcal{A}_r\left[A(\gamma), B(\gamma)\right]$. It will be called *the resolving minor*.

We denote by $\mathcal{M}_r\left[A(\gamma), B(\gamma)\right]$ a square submatrix of the matrix $\mathcal{D}_r\left[A(\gamma), B(\gamma)\right]$ that has order $n(r+1)$ and the resolving minor as its determinant.

Definition 1. *The unsolvability index* of the parametric family (1.1) is the smallest value r for which condition (2.4) holds and there exists the resolving minor in the matrix $\mathcal{D}_r\left[A(\gamma), B(\gamma)\right]$.

According to Lemma 1 below, the definition of the unsolvability index implies the permanent internal structure of system (1.1) for all $\gamma \in \Gamma$.

Let us introduce the notation

$$\left(A_1(\gamma) \;\; A_2(\gamma) \right) = A(\gamma)Q, \quad \left(B_1(\gamma) \;\; B_2(\gamma) \right) = B(\gamma)Q, \tag{2.5}$$

where Q is a column permutation matrix such that all columns of the matrix

$$\mathcal{B}_{2,r}(\gamma) = \operatorname{col}\left(B_2(\gamma), O, \ldots, O\right) \tag{2.6}$$

belong to the resolving minor and those of $\operatorname{col}\left(B_1(\gamma), O, \ldots, O\right)$ do not. The blocks $B_2(\gamma)$ and $A_2(\gamma)$ from (2.5), (2.6) have dimensions $n \times d$, where $d = nr - c$. The construction of the matrix Q was described in [22]. Thus, d is the number of columns of the matrix $\mathcal{B}_r[B(\gamma)]$ that belong to the resolving minor of the matrix (2.3).

Lemma 1. *Assume that:*

1) $A(\gamma),\ B(\gamma) \in \mathbf{C}^1(\Gamma)$.

2) *Condition* (2.4) *holds.*

3) *The matrix* $\mathcal{D}_r\left[A(\gamma), B(\gamma)\right]$ *has the resolving minor.*

Then there exists an operator

$$\mathcal{R}_\gamma = R_0(\gamma) + R_1(\gamma)\frac{d}{dt} + \ldots + R_r(\gamma)\left(\frac{d}{dt}\right)^r, \tag{2.7}$$

where $R_j(\gamma) \in \mathbf{C}^1(\Gamma)$ $(j = \overline{0,r})$ *are matrices of dimensions* $(n \times n)$, *such that*

$$\mathcal{R}_\gamma\left[A(\gamma)Q\xi'(t) + B(\gamma)Q\xi(t)\right] = \begin{pmatrix} O & O \\ E_{n-d} & O \end{pmatrix}\xi'(t) + \begin{pmatrix} J_1(\gamma) & E_d \\ J_2(\gamma) & O \end{pmatrix}\xi(t) \tag{2.8}$$

for all $t \in T$ *and* $\gamma \in \Gamma$ *and any* n*-dimensional vector function* $\xi(t) \in \mathbf{C}^{r+1}(T)$. *Here,* E_d *stands for an identity matrix of order* d; Q *is the permutation matrix given by* (2.5); *finally,* $J_1(\gamma),\ J_2(\gamma) \in \mathbf{C}^1(\Gamma)$ *are some matrices of compatible dimensions.*

In addition,

$$\left(R_0(\gamma) \;\; R_1(\gamma) \;\; \ldots \;\; R_r(\gamma) \right) = \left(E_n \;\; O \;\; \ldots \;\; O \right) \mathcal{M}_r^{-1}\left[A(\gamma), B(\gamma)\right]. \tag{2.9}$$

Lemma 2. *Consider the DAEs* (1.1) *and assume that:*

1) *All the hypotheses of Lemma 1 are satisfied.*

2) *The matrix* $\mathcal{D}_{r+1}\left[A(\gamma), B(\gamma)\right]$ *has an invertible submatrix* $\mathcal{M}_{r+1}\left[A(\gamma), B(\gamma)\right]$ *of order* $n(r+2)$ *for all* $\gamma \in \Gamma$ *that includes the matrix* $\mathcal{M}_r\left[A(\gamma), B(\gamma)\right]$ *and also* n *columns of the matrix* $\Lambda_{r+1}\left[A(\gamma), B(\gamma)\right]$.

Then the operator $\mathcal{R}_\gamma$ *possesses the left inverse operator* $\mathcal{L}_\gamma = L_0(\gamma) + L_1(\gamma)\frac{d}{dt}$, *where* $L_0(\gamma),\ L_1(\gamma) \in \mathbf{C}^1(\Gamma)$ *are matrices of dimensions* $(n \times n)$.

The proofs of these lemmas are provided in the Appendix.

2.2. Robust Stability Conditions

Due to Lemma 1, the operator $\mathcal{R}_\gamma$ reduces the family (1.1) to the form

$$\tilde{A}(\gamma)\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \tilde{B}(\gamma)\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 0, \tag{2.10}$$

where

$$Q\mathrm{col}\,(x_1(t), x_2(t)) = x(t), \tag{2.11}$$

Q is the row permutation matrix (see (2.5) and (2.6)), and

$$\tilde{A}(\gamma) = \begin{pmatrix} O & O \\ E_{n-d} & O \end{pmatrix} = (R_0(\gamma)A(\gamma) + R_1(\gamma)B(\gamma))\,Q, \tag{2.12}$$

$$\tilde{B}(\gamma) = \begin{pmatrix} J_1(\gamma) & E_d \\ J_2(\gamma) & O \end{pmatrix} = R_0(\gamma)B(\gamma)Q. \tag{2.13}$$

The component $x_1(t)$ of the solution of the DAEs (2.10) satisfies the equation

$$x_1'(t) + J_2(\gamma)x_1(t) = 0. \tag{2.14}$$

In turn, $x_2(t) = -J_1(\gamma)x_1(t)$. The matrix $J_1(\gamma)$ is constant for each fixed value $\gamma \in \Gamma$. Therefore, the family (2.10), (2.12), (2.13) is asymptotically stable if and only if the same property holds for system (2.14).

Under the hypotheses of Lemma 2, the operator $\mathcal{R}_\gamma$ has the left inverse operator. Moreover, the solutions of systems (1.1) and (2.10), (2.12), (2.13) are related through the row permutation matrix Q (see (2.11)). Based on these facts, we conclude that the DAE family (1.1) is asymptotically stable if and only if system (2.14) is asymptotically stable.

Let the entire family (2.14) have a general quadratic Lyapunov function

$$W(x_1) = x_1^\top V x_1 \tag{2.15}$$

with a positive definite time derivative along the trajectories of system (2.14). Here, the matrix V of dimensions $(n-d) \times (n-d)$ is symmetric and positive definite. As is known [2, p. 198; 21, p. 210], the family (2.14) is asymptotically stable if there exists a solution of the system of linear matrix inequalities (LMIs)

$$J_2(\gamma)^\top V + VJ_2(\gamma) > 0, \quad \gamma \in \Gamma. \tag{2.16}$$

In this case, the DAE family (1.1) is asymptotically stable as well. Thus, we arrive at the following result.

Theorem 1. *Under the hypotheses of Lemma 2, assume the existence of a symmetric and positive definite constant matrix V satisfying the LMIs* (2.16).

Then the parametric DAE family (1.1) *is asymptotically stable.*

We formulate another useful fact based on a known result from perturbation theory [2, p. 198].

Theorem 2. *Under the hypotheses Lemma 2, assume that all eigenvalues $\lambda_i(0)$ of the matrix $J_2(0)$ in system* (2.10)–(2.13) *are different and let α_i and β_i be the corresponding right and left eigenvectors:*

$$J_2(0)\alpha_i = \lambda_i(0)\alpha_i, \quad \beta_i^* J_2(0) = \lambda_i(0)\beta_i^*,$$
$$\|\alpha_i\|_{\mathbf{R}^{n-d}} = \|\beta_i\|_{\mathbf{R}^{n-d}} = 1, \quad i = \overline{1, n-d}.$$

Then the eigenvalues of the matrix $J_2(\gamma)$ *can be written as*

$$\lambda_i(\gamma) = \lambda_i(0) + \sum_{j=1}^{l} \frac{\beta_i^* \Theta_j \alpha_i}{\beta_i^* \alpha_i} \gamma_j + o(\gamma),$$

where

$$\Theta_j = \frac{\partial J_2(\gamma)}{\partial \gamma_j}\Big|_{\gamma=0}.$$

Example. Consider the DAEs

$$\begin{pmatrix} 1 & -\gamma_2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -\varphi(\gamma_3) & 0 & 0 & \varphi(\gamma_3) \\ 0 & 0 & \psi(\gamma_3) & 0 \end{pmatrix} x'(t) + \begin{pmatrix} 2 & 0 & 0 & \gamma_1 - 1 \\ 0 & 2-\gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} x(t) = 0, \tag{2.17}$$

where

$$\Gamma = \left\{ \mathrm{col}\,(\gamma_1, \gamma_2, \gamma_3) : \ \sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2} \leqslant 1/2 \right\}.$$

The functions $\varphi(\gamma_3)$ and $\psi(\gamma_3)$ are infinitely differentiable with respect to the parameter $\gamma_3 \in [-1/2,\ 1/2]$ and given by the rule $\psi(\gamma_3) = 0$ if $\varphi(\gamma_3) \neq 0$. Since $\phi(\gamma_3)$ and $\psi(\gamma_3)$ vanish either simultaneously or in turn, the matrix at the derivative has variable rank in the domain Γ.

The matrix $\mathcal{D}_2\,[A(\gamma),\ B(\gamma)]$ has the resolving minor indicated by the dashed line:

$$\left(\begin{array}{cc:cc|cccc||cccc|cc:cc}
2 & 0 & 0 & f_1 & 1 & -\gamma_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & f_2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -\varphi & 0 & 0 & \varphi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & \psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 2 & 0 & 0 & f_1 & 1 & -\gamma_2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & f_2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\varphi & 0 & 0 & \varphi & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & \psi & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & f_1 & 1 & -\gamma_2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_2 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\varphi & 0 & 0 & \varphi \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & \psi & 0
\end{array}\right).$$

Here, the dependence of φ and ψ on the parameter γ_3 is omitted, $f_1 = \gamma_1 - 1$, and $f_2 = 2 - \gamma_2$. The matrix $\Lambda_2[A(\gamma), B(\gamma)]$ is located to the right of the double vertical line and has rank 6.

We construct the matrix $\mathcal{D}_3\,[A(\gamma),\ B(\gamma)]$ by supplementing $\mathcal{D}_2$ with four zero columns on the right and four rows at the bottom:

$$\left(\begin{array}{ccc|ccc||ccc|cccc|cccc}
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 2 & 0 & 0 & f_1 & \boxed{1} & -\gamma_2 & 1 & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & f_2 & 0 & 0 & 0 & \boxed{1} & 0 & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \boxed{1} & 0 & -\varphi & 0 & 0 & \varphi \\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & -1 & 0 & 0 & \boxed{1} & 0 & 0 & \psi & 0
\end{array}\right).$$

There is the submatrix $\mathcal{M}_3\,[A(\gamma),\ B(\gamma)]$ in $\mathcal{D}_3$: it includes all columns corresponding to the resolving minor and four more columns with the framed units. Thus, all the hypotheses of Lemma 2 are satisfied, $d = 2$, and $Q = E_4$.

The operator

$$\mathcal{R}_\gamma = R_0(\gamma) + R_1(\gamma)\frac{d}{dt} + R_2(\gamma)\left(\frac{d}{dt}\right)^2,$$

where

$$R_0(\gamma) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & \gamma_2 & 0 & 1-\gamma_1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad R_1(\gamma) = \begin{pmatrix} 0 & 0 & 0 & -\varphi(\gamma_3) \\ 0 & 0 & -\psi(\gamma_3) & 0 \\ 0 & 0 & (\gamma_1-1)\psi(\gamma_3)-1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_2(\gamma) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varphi(\gamma_3) \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

transforms system (2.17) into

$$\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) x'(t) + \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ \hline 1+\gamma_1 & \gamma_2\,(2-\gamma_2) & 0 & 0 \\ 0 & 2-\gamma_2 & 0 & 0 \end{array}\right) x(t) = 0.$$

In addition, $J_2(\gamma) = \begin{pmatrix} 1+\gamma_1 & \gamma_2\,(2-\gamma_2) \\ 0 & 2-\gamma_2 \end{pmatrix}$.

Let $V = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ in the matrix inequality (2.16). Then

$$J_2(\gamma)^\top V + V J_2(\gamma) = \begin{pmatrix} 4\,(1+\gamma_1) & 2\gamma_2\,(2-\gamma_2) \\ 2\gamma_2\,(2-\gamma_2) & 4\,(2-\gamma_2) \end{pmatrix}.$$

This matrix is positive definite due to the positive definiteness of all its principal minors for any γ_1 and γ_2 from the segment $[-1/2, 1/2]$. In particular, the determinant of this matrix is $4(2-\gamma_2)\{4(1+\gamma_1) - \gamma_2^2(2-\gamma_2)\}$. Obviously, $2-\gamma_2 > 0\ \ \forall \gamma_2 \in [-1/2, 1/2]$. The expression in curly brackets achieves a minimum of $11/8 > 0$ for $\gamma_1 = -1/2$ and $\gamma_2 = -1/2$.

By Theorem 1, this means that the family (2.17) is asymptotically stable.

3. ROBUST STABILITY CONDITIONS FOR DAES WITH A SCALAR PARAMETER

Consider the DAEs

$$A(\gamma_0)x'(t) + B(\gamma_0)x(t) = 0, \quad t \in T, \tag{3.1}$$

where $\gamma_0 \in G_0$ denotes a scalar parameter and $G_0 = \{\gamma_0 \in \mathbf{R} : \ |\gamma_0| \leqslant a\}$ is a given admissible set.

Theorem 3. *Assume that:*

1) $A(\gamma_0), B(\gamma_0) \in \mathbf{C}^A(G_0)$.

2) $\operatorname{rank} \Lambda_r[A(\gamma_0), B(\gamma_0)] = \text{const} \ \ \forall \gamma_0 \in G_0$.

3) *The matrix* $\mathcal{D}_r[A(\gamma_0), B(\gamma_0)]$ *has a resolving minor.*

Then there exist matrices $P(\gamma_0)$, $S(\gamma_0) \in \mathbf{C}^A(G_0)$ *invertible for all* $\gamma_0 \in G_0$ *such that, with multiplication on the left by* $P(\gamma_0)$ *and the change of variable*

$$x(t) = S(\gamma_0)\operatorname{col}(z_1(t), z_2(t)),$$

system (3.1) *is transformed into*

$$\begin{pmatrix} E_{n-d} & O \\ O & N(\gamma_0) \end{pmatrix} \begin{pmatrix} z_1'(t) \\ z_2'(t) \end{pmatrix} + \begin{pmatrix} J(\gamma_0) & O \\ O & E_d \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = 0, \quad t \in T, \tag{3.2}$$

where $N(\gamma_0)$ *is an upper triangular matrix with* r *square zero blocks on the principal diagonal,* $N^r(\gamma_0) \equiv O$, *and* $J(\gamma_0)$ *is some matrix of dimensions* $(n-d)\times(n-d)$.

The proof of this theorem is omitted. It involves the step-by-step zeroing of linearly dependent rows in the matrix at $x'(t)$ and the differentiation operator applied to the corresponding rows of the matrix at $x(t)$.

The structural form (3.2) is an analog of the strong standard canonical form introduced in [25] for the system $A(t)x'(t) + B(t)x(t) = f(t)$ under the analytical solvability assumption.

Under the hypotheses of Theorem 3, the solutions of systems (3.1) and (3.2) are related by the identity

$$x(t) = S(\gamma_0)\mathrm{col}\,(z_1(t), z_2(t)) \tag{3.3}$$

with the matrix $S(\gamma_0)$ invertible for all $\gamma_0 \in G_0$.

Consider the DAEs (3.2). Due to the structure of the matrix $N(\gamma_0)$, the subsystem

$$N(\gamma_0)z_2'(t) + z_2(t) = 0$$

has only the trivial solution $z_2(t) \equiv 0$ for all $\gamma_0 \in G_0$.

In turn, the component $z_1(t)$ satisfies the subsystem

$$z_1'(t) + J(\gamma_0)z_1(t) = 0. \tag{3.4}$$

Therefore, the DAEs (3.2) are asymptotically stable if and only if the same property holds for system (3.4).

For each fixed value γ_0, the matrix $S(\gamma_0)$ in (3.3) is constant. Hence, the DAE family (3.1) possesses asymptotic stability if and only if system (3.4) does so.

Let

$$W_0(z_1) = z_1^\top V_0 z_1$$

be a general Lyapunov function for system (3.4) with a symmetric and positive definite matrix V_0 of dimensions $(n-d) \times (n-d)$ that satisfies the matrix inequality

$$J^\top(\gamma_0)V_0 + V_0 J(\gamma_0) > 0, \quad \gamma_0 \in G_0. \tag{3.5}$$

In this case, the family (3.4) is asymptotically stable, and hence the parametric family (3.1) possesses the same property.

Thus, the following result holds.

Theorem 4. *Under the hypotheses of Theorem 3, the DAE family* (3.1) *is asymptotically stable if there exists a constant, symmetric, and positive definite matrix* V_0 *that satisfies the matrix inequality* (3.5) *for all* $\gamma_0 \in G_0$.

4. CONDITIONS FOR PRESERVING THE TYPE OF FUNCTIONAL DEPENDENCE ON UNCERTAIN PARAMETERS

In the previous sections, we have considered the reduction of parametric families of DAEs to some structural forms. As it has been demonstrated above, under certain conditions, the robust stability of the original family is equivalent to the robust stability of its differential subsystem. Unfortunately, in the process of such transformations, the original type of the functional dependence of the system coefficients on the uncertain parameters generally becomes more complicated.

Consider the family of DAEs with an affine uncertainty:

$$\left(A_0+\sum_{j=1}^{l}\gamma_jA_j\right)x'(t)+\left(B_0+\sum_{j=1}^{l}\gamma_jB_j\right)x(t)=0,\quad t\in T, \tag{4.1}$$

where A_j and B_j $(j=\overline{0,l})$ are given real matrices of dimensions $(n\times n)$; $\det A_0=0$ and $\gamma=\operatorname{col}(\gamma_1,\dots,\gamma_l)\in\Gamma_a=\{\gamma\in\mathbf{R}^l:\,|\gamma_j|\leqslant a,\ j=\overline{1,l}\}$.

This section presents sufficient conditions on the structure of the coefficients A_j and B_j $(j=\overline{1,l})$ under which the robust stability of the family (4.1) is equivalent to the robust stability of a system of ordinary differential equations resolvable with respect to the derivative, with affine uncertainty.

4.1. Conditions Based on the Canonical Kronecker–Weierstrass Form

Definition 2. A matrix pencil μA_0+B_0 is said to be *regular* if there exists $\mu\in\mathbf{R}$ such that $\det(\mu A_0+B_0)\neq 0$.

Lemma 3 [26, p. 313]. *Assume that a matrix pencil μA_0+B_0 is regular. Then there exist invertible matrices P and S of dimensions $(n\times n)$ such that*

$$PA_0S=\begin{pmatrix}E_{n-d} & O\\ O & N\end{pmatrix},\quad PB_0S=\begin{pmatrix}J_0 & O\\ O & E_d\end{pmatrix}, \tag{4.2}$$

where J_0 is some square matrix of order $n-d$ and N is an upper triangular matrix with r square zero blocks on the principal diagonal such that $N^r=O$.

The system $PA_0Sz'(t)+PB_0Sz(t)=0$, $z(t)=S^{-1}x(t)$, with property (4.2) is called *the canonical Kronecker–Weierstrass form* for the DAEs $A_0x'(t)+B_0x(t)=0$.

Let the matrix pencil μA_0+B_0 in (4.1) be regular. Then, by Lemma 4, there exist invertible matrices P and S with property (4.2). Assume that

$$PA_jS=\begin{pmatrix}O & O\\ A_{j,1} & O\end{pmatrix},\quad PB_jS=\begin{pmatrix}B_{j,1} & O\\ B_{j,2} & O\end{pmatrix},\quad j=\overline{1,l}, \tag{4.3}$$

where $A_{j,1}$, $B_{j,1}$, and $B_{j,2}$ are some matrices with possibly nonzero elements. Note that $A_{j,1}$ and $B_{j,2}$ have dimensions $d\times(n-d)$, whereas the square block $B_{j,1}$ is of order $(n-d)$.

We multiply (4.1) on the left by the matrix P and change the variable: $x(t)=S\operatorname{col}(x_1(t),x_2(t))$. In view of (4.2), (4.3), the resulting system has the form

$$\begin{pmatrix}E_{n-d} & O\\ \sum\limits_{j=1}^{l}\gamma_jA_{j,1} & N\end{pmatrix}\begin{pmatrix}x_1'(t)\\ x_2'(t)\end{pmatrix}+\begin{pmatrix}J_0+\sum\limits_{j=1}^{l}\gamma_jB_{j,1} & O\\ \sum\limits_{j=1}^{l}\gamma_jB_{j,2} & E_d\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}=0. \tag{4.4}$$

Denoting

$$Y_1(\gamma)=J_0+\sum_{j=1}^{l}\gamma_jB_{j,1}, \tag{4.5}$$

$$Y_2(\gamma)=\sum_{j=1}^{l}\gamma_j\left(B_{j,2}-A_{j,1}Y_1(\gamma)\right), \tag{4.6}$$

from the first equation of system (4.4) we find

$$x_1'(t) = -Y_1(\gamma)x_1(t) \tag{4.7}$$

and consequently,

$$\left(\frac{d}{dt}\right)^i x_1(t) = (-1)^i Y_1^i(\gamma)x_1(t), \quad i = \overline{2,l}. \tag{4.8}$$

Considering (4.5)–(4.7), the second equation of system (4.4) can be written as

$$Nx_2'(t) + x_2(t) + Y_2(\gamma)x_1(t) = 0. \tag{4.9}$$

Applying the operator

$$E_d + \sum_{k=1}^{r-1}(-1)^k N^k \left(\frac{d}{dt}\right)^k \tag{4.10}$$

to (4.9) and using formulas (4.7) and (4.8), we obtain

$$x_2(t) = \left(Y_2(\gamma) + \sum_{k=1}^{r-1} N^k Y_2(\gamma)Y_1^k(\gamma)\right) x_1(t). \tag{4.11}$$

Obviously, the inverse of the operator (4.10) has the form $E_d + N\frac{d}{dt}$.

Following the same considerations as in the previous sections, we can demonstrate that the DAE family (4.1) is asymptotically stable for all $\gamma \in \Gamma_a$ if and only if the same property holds for the family (4.7) or, equivalently, for the DAEs

$$x_1'(t) + \left(J_0 + \sum_{j=1}^{l} \gamma_j B_{j,1}\right) x_1(t) = 0. \tag{4.12}$$

In system (4.12), the affine structure of uncertainty is preserved, being violated in equation (4.11).

Remark. If the dependence on the parameters in the original family were not affine (e.g., multilinear or polynomial), conditions (4.3) would ensure the same type of uncertainty in an equation analogous to (4.12).

The family (4.12) is asymptotically stable for all values $\gamma \in \Gamma_a$ if there exists a general Lyapunov function (2.15) with a positive definite time derivative along the trajectories of system (4.12).

Theorem 5. *Consider system* (4.1) *under the assumptions that the matrix pencil* $\mu A_0 + B_0$ *is regular and equalities* (4.3) *hold. If there exists a symmetric and positive definite matrix* V *of dimensions* $(n-d) \times (n-d)$ *that satisfies the inequality*

$$Y_1^\top(\gamma)V + VY_1(\gamma) > 0 \tag{4.13}$$

for all $\gamma \in \Gamma_a$, *then the family* (4.1) *is asymptotically stable for all* $\gamma \in \Gamma_a$. *In this case, the matrix* $Y_1(\gamma)$ *is calculated by formula* (4.5).

Since the affine structure of the dependence on the uncertain parameters is preserved in (4.12), it suffices to solve inequalities (4.13) only at a finite number of points for which $|\gamma_j| = a$ $(j = \overline{1,l})$ [2, p. 199].

For the same reason, Theorem 2 can be used to estimate the stability radius of system (4.12).

Suppose that all eigenvalues λ_k of the matrix J_0 have positive real parts: $\mathrm{Re}\,(\lambda_k) > 0$, $k = \overline{1, n-d}$. We denote by λ the eigenvalue of this matrix with the least real part. Let α and β be the corresponding right and left eigenvectors. By Theorem 2, λ turns into the eigenvalue $\lambda(\gamma)$ of the matrix $J_0 + \sum\limits_{j=1}^{l} \gamma_j B_{j,1}$:

$$\lambda(\gamma) \approx \lambda + \sum_{j=1}^{l} \frac{\beta^* B_{j,1}\alpha}{\beta^*\alpha}\gamma_j$$

for small γ. Therefore, for

$$\bar{a} = \mathrm{Re}\,\lambda / \sum_{j=1}^{l} \left|\mathrm{Re}\,\frac{\beta^* B_{j,1}\alpha}{\beta^*\alpha}\right|$$

at least one eigenvalue of the matrix (4.5) has zero real part. In other words, the value $\bar{a}$ provides an estimate for the stability radius of system (4.12) [2, p. 198].

4.2. Conditions Based on the Differential Operator

Note that for a regular pencil $\mu A_0 + B_0$, the construction of matrices P and S with the property (4.2) is generally a nontrivial problem. Therefore, we will obtain other, more constructive, conditions for preserving the type of functional dependence on the parameters for the family (4.1). In this case, some constraints will be imposed on the coefficients of the system under consideration, different from those adopted in Section 4.1.

Assume that the matrices A_0 and B_0 in system (4.1) satisfy the following conditions:

A1) There is a resolving minor in the matrix $\mathcal{D}_r[A_0, B_0]$.

A2) $\mathrm{rank}\,\Lambda_{r+1}[A_0, B_0] = \mathrm{rank}\,\Lambda_r[A_0, B_0] + n$.

Then, by Lemma 2, there exist an operator

$$\mathcal{R} = R_0 + R_1\frac{d}{dt} + \ldots + R_r\left(\frac{d}{dt}\right)^r$$

such that

$$\mathcal{R}\left[A_0 x'(t) + B_0 x(t)\right] = \begin{pmatrix} O & O \\ E_{n-d} & O \end{pmatrix}\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \begin{pmatrix} J_1 & E_d \\ J_2 & O \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

where $Q\mathrm{col}\,(x_1(t), x_2(t)) = x(t)$ and Q is the corresponding row permutation matrix (see (2.5) and (2.6)). In addition, $\mathcal{R}$ has the left inverse operator $\mathcal{L} = L_0 + L_1\frac{d}{dt}$ and

$$(\, R_0\ R_1\ \ldots\ R_r\,) = (\, E_n\ O\ \ldots\ O\,)\, M_r^{-1}[A_0, B_0],$$

where the determinant of the matrix $M_r[A_0, B_0]$ is the resolving minor.

Applying the operator $\mathcal{R}$ to (4.1) yields the system

$$\begin{pmatrix} O & O \\ E_{n-d} & O \end{pmatrix}\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \begin{pmatrix} J_1 & E_d \\ J_2 & O \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \sum_{i=0}^{r+1}\left[\sum_{j=1}^{l}\gamma_j R_{i,j}\right]\left(\frac{d}{dt}\right)^i\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 0, \tag{4.14}$$

where

$$R_{0,j} = R_0 B_j Q;\quad R_{i,j} = (R_{i-1}A_j + R_i B_j)\,Q,\quad i = \overline{1,r};\\ R_{r+1,j} = R_r A_j Q,\quad j = \overline{1,l}. \tag{4.15}$$

Suppose that the matrix coefficients (4.15) have the structure

$$R_{0,j} = \begin{pmatrix} B_{j,1} & O \\ B_{j,2} & O \end{pmatrix}, \quad R_{i,j} = \begin{pmatrix} A_j^{[i]} & O \\ O & O \end{pmatrix}, \quad j = \overline{1,l},\ i = \overline{1,r+1}, \tag{4.16}$$

where $B_{j,1}$, $B_{j,2}$, and $A_j^{[i]}$ are some matrices: $B_{j,1}$ and $A_j^{[i]}$ have dimensions $d \times (n-d)$, and the square block $B_{j,2}$ is of order $(n-d)$.

Due to (4.16), system (4.14) takes the form

$$\begin{pmatrix} O & O \\ E_{n-d} & O \end{pmatrix} \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \begin{pmatrix} J_1 + \sum\limits_{j=1}^{l} \gamma_j B_{j,1} & E_d \\ J_2 + \sum\limits_{j=1}^{l} \gamma_j B_{j,2} & O \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$$+ \begin{pmatrix} \sum\limits_{i=1}^{r+1} \left[\sum\limits_{j=1}^{l} \gamma_j A_j^{[i]} \right] \left(\frac{d}{dt} \right)^i x_1(t) & O \\ O & O \end{pmatrix} = 0.$$

Following the same considerations as in the previous case, under Assumptions A1 and A2, we can demonstrate that system (4.1) is asymptotically stable for each $\gamma \in \Gamma_a$ if and only if the same property holds for the subsystem

$$x_1'(t) + \left(J_2 + \sum_{j=1}^{l} \gamma_j B_{j,2} \right) x_1(t) = 0.$$

Thus, the following analog of Theorem 5 is true.

Theorem 6. *Consider system* (4.1) *under Assumptions A1 and A2 and equalities* (4.16)*. If there exists a symmetric and positive definite matrix V of dimensions $(n-d) \times (n-d)$ that satisfies the inequality*

$$\left(J_2 + \sum_{j=1}^{l} \gamma_j B_{j,2} \right)^{\top} V + V \left(J_2 + \sum_{j=1}^{l} \gamma_j B_{j,2} \right) > 0$$

for all $\gamma \in \Gamma_a$, then the family (4.1) *is asymptotically stable for all $\gamma \in \Gamma_a$.*

5. CONCLUSIONS

In this paper, the robust stability of differential-algebraic equations (DAEs) with norm-bounded vector and scalar uncertain parameters has been analyzed by proposing algorithms for building structural forms with separated differential and algebraic subsystems (Lemma 1 and Theorem 3, respectively).

The approach based on a linear differential operator has a constructive nature: the coefficients of this operator are determined by inverting the matrix whose determinant is the resolving minor. Moreover, the system obtained by such a transformation is equivalent to the original system in the sense of solutions (Lemma 2).

When investigating the stability of DAEs, the main difficulty is that even in the simplest cases, the internal structure of the system (and, consequently, the form of the general solution) may change under an arbitrarily small perturbation of the coefficients. As a result, the structure and properties of the unperturbed system may lose any significance for analysis.

A distinctive feature of the results presented above is that the stability analysis involves no information about the internal structure of the nominal system. The structural forms are built for the entire family (Subsections 2.1 and 3.1). For this reason, there is no need to introduce additional structural constraints on the perturbations that would ensure the coinciding internal structures of the nominal and perturbed DAEs.

It has been demonstrated that the stability of the parametric family (1.1) is equivalent to the robust stability of its differential subsystem, which also depends on the uncertain parameters. The sufficient condition for robust stability follows from the existence of a general quadratic Lyapunov function (2.15) for the differential subsystem (Theorems 1 and 4).

On the other hand, reducing the system to one or another structural form may appreciably complicate the type of functional dependence on the uncertain parameters of this subsystem compared to system (1.1). In this regard, for DAEs with affine uncertainty, sufficient conditions have been established under which the differential subsystem is also an affine family (Section 4). This approach is also applicable to systems with other types of parametric uncertainty.

APPENDIX

Proof of Lemma 1. Let Q_Λ be a column permutation matrix such that

$$\Lambda_r\left[A(\gamma), B(\gamma)\right] Q_\Lambda = \left(\Lambda_{r,1}(\gamma)\ \Lambda_{r,2}(\gamma)\right),$$

where the block $\Lambda_{r,1}(\gamma)$ belongs to the resolving minor and consists of c columns, $\operatorname{rank}\Lambda_{r,1}(\gamma) = c$ $\forall\gamma\in\Gamma$.

Consider the matrix $\mathcal{D}_r\left[A(\gamma), B(\gamma)\right]$. Multiplying it on the right by the column permutation matrix $Q_r = \operatorname{diag}\{Q, Q, Q_\Lambda\}$ [1] and on the left by $\mathcal{M}_r^{-1}\left[A(\gamma), B(\gamma)\right]$ yields

$$\mathcal{M}_r^{-1}\left[A(\gamma), B(\gamma)\right]\mathcal{D}_r\left[A(\gamma), B(\gamma)\right] Q_r = \left(\begin{array}{cc|cc||cc} J_1(\gamma) & E_d & O & O & O & \Phi_1(\gamma) \\ J_2(\gamma) & O & E_{n-d} & O & O & \Phi_2(\gamma) \\ \hline J_3(\gamma) & O & O & E_d & O & \Phi_3(\gamma) \\ J_4(\gamma) & O & O & O & E_c & \Phi_4(\gamma) \end{array}\right),$$

where $J_i(\gamma)$ and $\Phi_i(\gamma)$ $(i = \{1, 2, 3, 4\})$ are some matrices of compatible dimensions. Due to (2.4),

$$\Phi_1(\gamma) \equiv O,\ \ \Phi_2(\gamma) \equiv O,\ \ \Phi_3(\gamma) \equiv O,\ \ \gamma\in\Gamma.$$

In this case,

$$\begin{aligned} &\left(E_n\ \ O\ \ \ldots\ \ O\right)\mathcal{M}_r^{-1}\left[A(\gamma), B(\gamma)\right]\mathcal{D}_r\left[A(\gamma), B(\gamma)\right] Q_r \operatorname{col}\left(\xi(t), \ldots, \xi^{(r+1)}(t)\right) \\ &= \left(\begin{array}{cc|cc||ccc} J_1(\gamma) & E_d & O & O & O & \ldots & O \\ J_2(\gamma) & O & E_{n-d} & O & O & \ldots & O \end{array}\right)\operatorname{col}\left(\xi(t), \ldots, \xi^{(r+1)}(t)\right), \end{aligned} \tag{A.1}$$

which obviously implies formulas (2.9) and (2.8).

Proof of Lemma 2. Note that identity (A.1) remains in force for

$$Q_r = \operatorname{diag}\{Q, \ldots, Q\}.$$

[1] The matrix Q_r is quasi-diagonal: the blocks listed in curly brackets stand on the principal diagonal and all other elements are zero.

Let $\xi(t)\in\mathbf{C}^{r+2}(T)$ be an arbitrary n-dimensional vector function. In view of the representation (2.9), differentiating (A.1) with respect to the variable t gives

$$\begin{aligned}&\left(O\ \ R_0(\gamma)\ \ R_1(\gamma)\ \ \ldots\ \ R_r(\gamma)\right)\mathcal{D}_{r+1}\left[A(\gamma),B(\gamma)\right]Q_{r+1}\mathrm{col}\left(\xi(t),\ldots,\xi^{(r+2)}(t)\right)\\ &=\left(\begin{array}{cc|cc|cc|ccc} O & O & J_1(\gamma) & E_d & O & O & O & \ldots & O\\ O & O & J_2(\gamma) & O & E_{n-d} & O & O & \ldots & O\end{array}\right)\mathrm{col}\left(\xi(t),\ldots,\xi^{(r+2)}(t)\right).\end{aligned}\tag{A.2}$$

From (A.1) and (A.2) it follows that

$$\begin{aligned}&\begin{pmatrix} R_0(\gamma) & R_1(\gamma) & \ldots & R_r(\gamma) & O\\ O & R_0(\gamma) & \ldots & R_{r-1}(\gamma) & R_r(\gamma)\end{pmatrix}\mathcal{D}_{r+1}\left[A(\gamma),B(\gamma)\right]Q_{r+1}\\ &=\left(\begin{array}{cc|cc|cc} J_1(\gamma) & E_d & O & O & O & O\\ J_2(\gamma) & O & E_{n-d} & O & O & O\\ \hline O & O & J_1(\gamma) & E_d & O & O\\ O & O & J_2(\gamma) & O & E_{n-d} & O\end{array}\right).\end{aligned}\tag{A.3}$$

According to assumption 2 of Lemma 2, the matrix

$$\mathcal{D}_{r+1}\left[A(\gamma),B(\gamma)\right]=\left(\ \mathcal{B}_{r+1}\left[B(\gamma)\right]\,\middle|\,\mathcal{A}_{r+1}\left[A(\gamma),B(\gamma)\right]\,\middle\|\,\Lambda_{r+1}\left[A(\gamma),B(\gamma)\right]\ \right)$$

has an invertible submatrix $\mathcal{M}_{r+1}\left[A(\gamma),B(\gamma)\right]$ of order $n(r+2)$ on Γ. In addition, d columns of the matrix $\mathcal{B}_{r+1}\left[B(\gamma)\right]$ are included in $\mathcal{M}_{r+1}\left[A(\gamma),B(\gamma)\right]$, namely, those of the matrix

$$\mathcal{B}_{2,r+1}(\gamma)=\mathrm{col}\left(B_2(\gamma),O,\ldots,O\right)$$

(see (2.5) and (2.6)); $(c+n)$ columns of the matrix $\Lambda_{r+1}\left[A(\gamma),B(\gamma)\right]$ are included in $\mathcal{M}_{r+1}$.

Consider a matrix that eliminates in $\mathcal{D}_{r+1}\left[A(\gamma),B(\gamma)\right]Q_{r+1}$ the columns not included in $\mathcal{M}_{r+1}\left[A(\gamma),B(\gamma)\right]$. Multiplying both sides of equality (A.3) on the right by this matrix yields

$$\begin{pmatrix} R_0(\gamma) & R_1(\gamma) & \ldots & R_r(\gamma) & O\\ O & R_0(\gamma) & \ldots & R_{r-1}(\gamma) & R_r(\gamma)\end{pmatrix}\mathcal{M}_{r+1}\left[A(\gamma),B(\gamma)\right]=\left(\mathcal{E}(\gamma)\ \ O\right),\tag{A.4}$$

where

$$\mathcal{E}(\gamma)=\left(\begin{array}{c|cc||c} E_d & O & O & O\\ O & E_{n-d} & O & O\\ \hline O & J_1(\gamma) & E_d & O\\ O & J_2(\gamma) & O & E_{n-d}\end{array}\right).\tag{A.5}$$

In addition, the block column

$$\mathrm{col}\left(O,O,O,E_{n-d}\right)\tag{A.6}$$

of the matrix on the right-hand side of identity (A.3) completely enters the matrix $\mathcal{E}(\gamma)$. Indeed, $\mathcal{D}_{r+1}\left[A(\gamma),B(\gamma)\right]$ and the matrix on the right-hand side of (A.3) have full row rank for all $\gamma\in\Gamma$; therefore, due to (A.3), the same property holds for the matrix

$$\begin{pmatrix} R_0(\gamma) & R_1(\gamma) & \ldots & R_r(\gamma) & O\\ O & R_0(\gamma) & \ldots & R_{r-1}(\gamma) & R_r(\gamma)\end{pmatrix}.\tag{A.7}$$

Assume on the contrary that the transformation of (A.3) into (A.4) eliminates some columns of the matrix (A.6). In this case, the matrix $\mathcal{E}(\gamma)$ will no longer be invertible, which contradicts the full row rank of the matrix (A.7).

From (A.4) it follows that

$$\begin{pmatrix} R_0(\gamma) & R_1(\gamma) & \dots & R_r(\gamma) & O \\ O & R_0(\gamma) & \dots & R_{r-1}(\gamma) & R_r(\gamma) \end{pmatrix} = (\, \mathcal{E}(\gamma) \;\; O \,)\, \mathcal{M}_{r+1}^{-1}\, [A(\gamma), B(\gamma)]\,. \tag{A.8}$$

The desired result of this lemma will be established by demonstrating that the system

$$(\, L_0(\gamma)\; L_1(\gamma) \,) \begin{pmatrix} R_0(\gamma) & R_1(\gamma) & \dots & R_r(\gamma) & O \\ O & R_0(\gamma) & \dots & R_{r-1}(\gamma) & R_r(\gamma) \end{pmatrix} = (\, E_n \; O \; \dots O \,) \tag{A.9}$$

has a solution $(\, L_0(\gamma)\; L_1(\gamma) \,) \in \mathbf{C}^1(\Gamma)$.

Considering the representation (A.8), a necessary and sufficient condition for the point-wise solvability of equation (A.9) can be written as

$$\operatorname{rank} \mathcal{E}(\gamma) = \operatorname{rank} \left(\frac{(\, \mathcal{E}(\gamma)\; O \,)}{(\, E_n \; O \; \dots O \,)\, \mathcal{M}_{r+1}\, [A(\gamma), B(\gamma)]} \right). \tag{A.10}$$

Obviously, see (A.5), $\operatorname{rank} \mathcal{E}(\gamma) = 2n$. By construction,

$$(\, E_n \; O \; \dots O \,)\, \mathcal{M}_{r+1}\, [A(\gamma), B(\gamma)] = \left(\; B_2(\gamma) \;\middle|\; A_1(\gamma) \quad A_2(\gamma) \;\middle\|\; O \;\right). \tag{A.11}$$

In view of (A.5) and (A.11), it is straightforward to verify (A.10) for all $\gamma \in \Gamma$.

The solution $L_0(\gamma),\ L_1(\gamma) \in \mathbf{C}^1(\Gamma)$ of system (A.9) is given by the formula

$$(\, L_0(\gamma)\; L_1(\gamma) \,) = (\, E_n \; O \; \dots O \,)\, \mathcal{M}_{r+1}\, [A(\gamma), B(\gamma)] \begin{pmatrix} \mathcal{E}^{-1}(\gamma) \\ O \end{pmatrix}.$$

The proof of Lemma 2 is complete.

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