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LINEAR SYSTEMS

Interval Observers for Continuous-Time Systems with Parametric Uncertainties

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Abstract—In this paper, interval observers are designed for linear dynamic systems described by continuous-time models with exogenous disturbances, measurement noises, and parametric uncertainties. Jordan canonical form-based relations are presented for an interval observer that estimates the set of admissible values of a given linear function of the system state vector. The theoretical results are illustrated by a practical example.

Keywords: linear systems, interval observers, Jordan canonical form, estimation

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1. INTRODUCTION AND PROBLEM STATEMENT

The problem of constructing interval observers has received much attention in recent years. Solutions for different classes of systems—discrete- and continuous-time, delayed, and singular—can be found in [1–11], including practical applications of such observers in various fields. The corresponding results were thoroughly overviewed in [10, 11]. As a rule, the interval observers considered therein estimate the set of admissible values of the complete state vector. In practice, however, it may be of interest to get an interval estimate of admissible values only for some linear function of this vector. The corresponding interval observer may turn out to be much simpler than the observer for the full state vector, and the interval width may be noticeably smaller due to the possibility of minimizing the effect of exogenous disturbances. Moreover, when estimating a given linear function, the dynamics of the observer can be represented in canonical form, which simplifies the process of solving the problem and expands the class of systems with the possibility of constructing an interval observer for them.

In recent years, the Jordan canonical form (JCF) [4, 10] has been actively used to design interval observers. (Previously, it was employed to analyze the self-correction property of faults [12].) Under an appropriate choice of the eigenvalues, the system dynamics matrix implemented in the JCF ensures observer stability and is a Metzler matrix, i.e., its off-diagonal elements are nonnegative. Due to these properties, at each time instant, the interval observer produces an estimate of the set of admissible values of the system state vector with uncertainties. According to the analysis results, in addition to stabilization, the JCF simplifies the procedure of ensuring the observer's insensitivity to disturbances and, in some cases, reduces its dimension due to no need for stabilization by special means.

This paper is a logical continuation of the works [13, 14], devoted to the problem of constructing interval observers for systems described by linear continuous-time models with exogenous disturbances and measurement noises. As was demonstrated in [14], the JCF-based interval observer design allows reducing the dimension of observers and, in some cases, the interval width as well. In this paper, for systems described by continuous-time models with exogenous disturbances, measurement noises, and parametric uncertainties, we construct interval observers estimating the set of admissible values of a given linear function of the state vector.

Consider a class of systems with the linear model

$$\begin{aligned} \dot{x}(t) &= (F + \Delta F(\mu(t)))x(t) + Gu(t) + L\rho(t), \\ y(t) &= Hx(t) + v(t), \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^l$ denote the state, control, and output vectors, respectively; F and G are constant matrices describing the linear dynamics; H and L are given constant matrices; $\rho(t) \in \mathbb{R}^p$ is an unknown bounded time-varying function describing exogenous disturbances of the system, $\|\rho(t)\| \leqslant \rho_*$; $v(t) \in \mathbb{R}^l$ is an unknown bounded time-varying function describing measurement noises, $\|v(t)\| \leqslant v_*$; finally, $\mu(t) \in \Pi \subset \mathbb{R}^s$ is a the bounded vector of variable parameters. By analogy with [10], the values of the vector $\mu(t)$ are unmeasurable, the set of admissible values Π is known, and the matrix function $\Delta F(\mu)$ is bounded for all $\mu(t) \in \Pi$, i.e., $\underline{\Delta F} \leqslant \Delta F \leqslant \overline{\Delta F}$, where $\underline{\Delta F}$ and $\overline{\Delta F}$ are given. As in the paper [2], for arbitrary vectors w_1, w_2 and matrices A_1, A_2, the relations $w_1 \leqslant w_2$ and $A_1 \leqslant A_2$ are understood elementwise.

It is required to construct a minimum-dimension interval observer that forms the lower $\underline{z}(t)$ and upper $\overline{z}(t)$ bounds of the variable $z(t) = Mx(t)$ with a known matrix M such that $\underline{z}(t) \leqslant z(t) \leqslant \overline{z}(t)$ for all $t \geqslant 0$. The problems of considering measurement noises and exogenous disturbances for continuous-time systems were discussed in detail in [13]. Therefore, this paper focuses on parametric uncertainties, first assuming that $v(t) = 0$ and $\rho(t) = 0$.

2. MODEL BUILDING

This problem is solved using the minimum-dimension model of system (1.1) that estimates the variable $z(t)$ and some variable $y_*(t)$ determined during the solution process:

$$\begin{aligned} \dot{x}_*(t) &= (F_* + \Delta F_*(t) - PH_*)x_*(t) + (J_* + J')y(t) + Py_*(t) + G_*u(t), \\ y_*(t) &= H_*x_*(t), \\ z(t) &= H_z x_*(t) + Qy(t), \end{aligned} \tag{2.1}$$

where $x_*(t) \in \mathbb{R}^k$ and $k < n$ denotes the model dimension; F_*, G_*, J_*, H_*, H_z, and Q are the matrices to be found; the matrices ΔF_* and J' are defined below; finally, the choice of the matrix P is explained in Section 3. In [4], the stable observer designed previously was reduced to the JCF for the interval estimation of the vector $x(t)$. In contrast, in this paper, the matrix F_* will be immediately found in the JCF. Assuming that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ of this matrix are different and negative, we obtain the diagonal matrix

$$F_* = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & 0 & \ldots & \lambda_k \end{pmatrix}. \tag{2.2}$$

Let us show the correctness of this assumption. According to [13], the matrix F_* can always be implemented in the identification canonical form

$$F_* = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

By introducing the residual signal feedback with the matrix K, it is reduced to the form $F_* - KH_*$; by choosing (different and negative) eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ and by determining the corresponding elements of the matrix K, the matrix $F_* - KH_*$ can be made stable. Since the eigenvalues are assumed different, the corresponding eigenvectors $v_1, v_2, \dots, v_k$ form a linearly independent system, so the matrix $T^{-1} = (v_1 \ v_2 \ \dots \ v_k)$ will be nonsingular. The matrix T has the form

$$T = \begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_k \end{pmatrix},$$

where the vectors $w_1, w_2, \dots, w_k$ satisfy the condition $w_i v_i = 1$, $i = 1, \dots, k$, $w_i v_j = 0$, $i \neq j$. By the definition of eigenvalues and eigenvectors, $(F_* - KH_*)v_1 = \lambda_1 v_1$. Multiplying on the left by the matrix T, we transform this equation into $T(F_* - KH_*)T^{-1}Tv_1 = \lambda_1 Tv_1$. Due to the form of T, $Tv_1 = (1 \ 0 \ \dots \ 0)^T$ and, consequently,

$$T(F_* - KH_*)T^{-1}(1 \ 0 \ \dots \ 0)^T = (\lambda_1 \ 0 \ \dots \ 0)^T.$$

Therefore, the first column of the matrix $T(F_* - KH_*)T^{-1}$ is $(\lambda_1 \ 0 \ \dots \ 0)^T$. Considering the equation $(F_* - KH_*)v_i = \lambda_i v_i$, $i = 2, 3, \dots, k$, by analogy, we finally arrive at the matrix (2.2).

In the absence of noises, disturbances, and uncertainties, let $x_*(t) = \Phi x(t)$ and $y_*(t) = R_* x(t)$, where the matrix Φ satisfies the well-known conditions [15]

$$\Phi F = F_* \Phi + J_* H, \quad H_* \Phi = R_* H, \quad \Phi G = G_*, \tag{2.3}$$

and an auxiliary condition due to the variable $z(t)$. We derive it below. From $z(t) = Mx(t)$ and (2.1) it follows that

$$M = H_z \Phi + QH = (H_z \ Q) \begin{pmatrix} \Phi \\ H \end{pmatrix}. \tag{2.4}$$

This equation has a solution if

$$\operatorname{rank} \begin{pmatrix} \Phi \\ H \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \Phi \\ H \\ M \end{pmatrix}. \tag{2.5}$$

Note that the condition $H_* \Phi = R_* H$ can be written as the equation

$$(H_* \ \ -R_*) \begin{pmatrix} \Phi \\ H \end{pmatrix} = 0, \tag{2.6}$$

which has a solution if

$$\operatorname{rank} \begin{pmatrix} \Phi \\ H \end{pmatrix} < \operatorname{rank}(\Phi) + \operatorname{rank}(H). \tag{2.7}$$

For the matrix (2.2), the first equation in (2.3) splits into k independent equations:

$$\Phi_i F = \lambda_i \Phi_i + J_{*i} H, \quad i = 1, \ldots, k.$$

They can be written as

$$(\Phi_i \quad -J_{*i}) \begin{pmatrix} F - \lambda_i I_n \\ H \end{pmatrix} = 0, \quad i = 1, \ldots, k, \tag{2.8}$$

where Φ_i and J_{*i} stand for the ith rows of the matrices Φ and J_*, respectively, and I_n denotes an identity matrix of dimensions $n \times n$.

When solving equation (2.8), the values $\lambda_i < 0$ must be set so that the resulting matrix Φ with the minimum number of rows satisfies conditions (2.5) and (2.7). After that, the matrices H_z and Q are determined from (2.4), the matrices H_* and R_* from (2.6), and the matrix G_* from (2.3). The uncertainty $\Delta F_*(t)$ figuring in (2.1) is determined from $\Delta F(\mu(t))$ as follows. By analogy with (2.3), formulas (1.1) and (2.1) imply

$$\Phi(F + \Delta F(\mu(t))) = (F_* + \Delta F_*(t))\Phi + J_* H + J' H.$$

Since the matrix F_* satisfies the condition $\Phi F = F_* \Phi + J_* H$, we obtain

$$\Phi \Delta F(\mu(t)) = \Delta F_*(t)\Phi + J' H,$$

or

$$\Phi \Delta F(\mu)) = (\Delta F_* \;\; J') \begin{pmatrix} \Phi \\ H \end{pmatrix}. \tag{2.9}$$

This equation has a solution if

$$\operatorname{rank} \begin{pmatrix} \Phi \\ H \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \Phi \\ H \\ \Phi \Delta F(\mu)) \end{pmatrix}. \tag{2.10}$$

After calculating the matrix Φ, this condition is checked and, if it holds, $\Delta F_*(t)$ and J' are determined from equation (2.9). The matrices $\underline{\Delta F}_*$ and $\overline{\Delta F}_*$ are also determined from (2.9). If condition (2.10) fails, another solution of equation (2.8) is sought. For simplicity, we consider the case where J' is independent of $\Delta F(\mu)$.

3. INTERVAL OBSERVER DESIGN

3.1. The First Variant

By analogy with [2, 10], we consider two variants of constructing interval observers, namely, a simple one with constraints on the original system and a more complicated one without them. Let us begin with the first variant, assuming that $\underline{\Delta F}_* \geqslant 0$ and $x_*(t) \geqslant 0$ for all $t \geqslant 0$. In this case, the observer based on model (2.1) is designed in the form

$$\begin{aligned} \dot{\underline{x}}_*(t) &= (F_* + \underline{\Delta F}_* - \underline{P} H_*)\underline{x}_*(t) + \underline{P} y_*(t) + (J_* + J')y(t) + G_* u(t), \\ \dot{\overline{x}}_*(t) &= (F_* + \overline{\Delta F}_* - \overline{P} H_*)\overline{x}_*(t) + \overline{P} y_*(t) + (J_* + J')y(t) + G_* u(t), \\ \underline{z}(t) &= H_z \underline{x}_*(t) + Q y(t), \\ \overline{z}(t) &= H_z \overline{x}_*(t) + Q y(t), \\ \underline{x}_*(0) &= \underline{x}_{*0}, \quad \overline{x}_*(0) = \overline{x}_{*0}. \end{aligned} \tag{3.1}$$

We introduce the estimation errors

$$\begin{aligned} \underline{e}_*(t) &= x_*(t) - \underline{x}_*(t), & \overline{e}_*(t) &= \overline{x}_*(t) - x_*(t), \\ \underline{e}_z(t) &= z(t) - \underline{z}(t), & \overline{e}_z(t) &= \overline{z}(t) - z(t). \end{aligned} \tag{3.2}$$

Due to (2.1) and (3.1), these errors satisfy the differential equations

$$\begin{aligned} \dot{\underline{e}}_*(t) &= (F_* - \underline{P}H_*)\underline{e}_*(t) + \Delta F_* x_*(t) - \underline{\Delta F}_* \underline{x}_*(t), \\ \dot{\overline{e}}_*(t) &= (F_* - \overline{P}H^*)\overline{e}_*(t) + \overline{\Delta F}_* \overline{x}_*(t) - \Delta F_* x_*(t). \end{aligned} \tag{3.3}$$

Equations (3.1) and (3.3) lead to the following requirements for the matrices $\underline{P}$ and $\overline{P}$: they are chosen so that $F_* + \underline{\Delta F}_* - \underline{P}H_*$ and $F_* + \overline{\Delta F}_* - \overline{P}H_*$ are stable matrices whereas $F_* - \underline{P}H_*$ and $F_* - \overline{P}H_*$ are Metzler matrices. Some recommendations on determining these matrices can be found in [2]. The matrix P in (2.1) is set equal to $\underline{P}$ for the first equation in (3.1) and to $\overline{P}$ for the second one.

Theorem 1. *Assume that* $0 \leqslant \underline{\Delta F}_* \leqslant \Delta F_* \leqslant \overline{\Delta F}_*$, $H_z \geqslant 0$, $x_*(t) \geqslant 0$ *for all* $t \geqslant 0$, $\underline{x}_*(0) \leqslant x_*(0) \leqslant \overline{x}_*(0)$, *and there exist matrices* $\underline{P}$ *and* $\overline{P}$ *with the properties specified above. Then the interval observer* (3.3) *satisfies the relation* $\underline{z}(t) \leqslant z(t) \leqslant \overline{z}(t)$.

Proof. If $x_*(t) \geqslant \underline{x}_*(t)$, we obtain $\underline{\Delta F}_*(x_*(t) - \underline{x}_*(t)) \geqslant 0$ and $\underline{\Delta F}_* x_*(t) \geqslant \underline{\Delta F}_* \underline{x}_*(t)$ because $\underline{\Delta F}_* \geqslant 0$. Since $\Delta F_* \geqslant \underline{\Delta F}_*$ and $x_*(t) \geqslant 0$, it follows that $(\Delta F_* - \underline{\Delta F}_*)x_*(t) \geqslant 0$ and $\Delta F_* x_*(t) \geqslant \underline{\Delta F}_* x_*(t)$. Due to the considerations above, we therefore have $\Delta F_* x_*(t) \geqslant \underline{\Delta F}_* \underline{x}_*(t)$, i.e., $\Delta F_* x_*(t) - \underline{\Delta F}_* \underline{x}_*(t) \geqslant 0$ in (3.3). By the assumptions made, for $t = 0$, $x_*(0) \geqslant \underline{x}_*(0)$, i.e., $\underline{e}_*(0) \geqslant 0$, and the matrix $F_* - \underline{P}H_*$ in (3.3) is Metzler; hence, $\underline{e}_*(t) \geqslant 0$ for all $t \geqslant 0$ by induction [2]. Since $z(t) = H_z x_*(t) + Qy(t)$, from (3.2) it follows that

$$\underline{e}_z(t) = H_z x_*(t) + Qy(t) - (H_z \underline{x}_*(t) + Qy(t)) = H_z \underline{e}_*(t),$$

which yields $\underline{e}_z(t) \geqslant 0$ due to $\underline{e}_*(t) \geqslant 0$ and the assumption $H_z \geqslant 0$. The inequality $\overline{e}_z(t) \geqslant 0$ is established by analogy. Obviously, the latter inequalities are equivalent to the desired result. The stable matrices $F_* + \underline{\Delta F}_* - \underline{P}H_*$ and $F_* + \overline{\Delta F}_* - \overline{P}H_*$ ensure that the bounds $\underline{x}_*(t)$ and $\overline{x}_*(t)$ (and hence, $\underline{z}(t)$ and $\overline{z}(t)$) are finite. The proof of Theorem 1 is complete.

Remark 1. In the case $H_z \leqslant 0$, the bounds $\underline{z}(t)$ and $\overline{z}(t)$ are given by

$$\underline{z}(t) = H_z \overline{x}_*(t) + Qy(t), \quad \overline{z}(t) = H_z \underline{x}_*(t) + Qy(t).$$

Indeed, in this case,

$$\underline{e}_z(t) = z(t) - \underline{z}(t) = H_z x_*(t) + Qy(t) - (H_z \overline{x}_*(t) + Qy(t)) = -H_z \overline{e}_*(t),$$

which implies $\underline{e}_z(t) \geqslant 0$ due to $\overline{e}_*(t) \geqslant 0$ and $H_z \leqslant 0$. The corresponding inequality for the bound $\overline{z}(t)$ is proved by analogy.

If the matrix H_z is oscillating, the final result will remain the same but with more complicated formulas for the upper and lower bounds.

Thus, an interval observer estimating the variable $z(t) = Mx(t)$ is constructed in the following steps: find solutions of equation (2.8) that give with minimum k a matrix Φ satisfying conditions (2.5) and (2.7); calculate the matrices J_*, G_*, R_*, and H_* and choose matrices $\underline{P}$ and $\overline{P}$ with the properties specified above. Such a choice is not always possible; in this case, the second observer design variant should be used.

3.2. The Second Variant

Consider a more complex interval observer without constraints on the original system. The presentation is preceded by an auxiliary result.

Lemma 1. *Assume that* $\underline{A} \leqslant A \leqslant \overline{A}$ *for some matrices* $\underline{A}$, A, *and* $\overline{A}$ *of dimensions* $k \times k$ *and* $\underline{x}_* \leqslant x_* \leqslant \overline{x}_*$ *for some k-dimensional vectors* $\underline{x}_*$, x_*, *and* $\overline{x}_*$. *Then*

$$\underline{A}^+\underline{x}_*^+ - \overline{A}^+\underline{x}_*^- - \underline{A}^-\overline{x}_*^+ + \overline{A}^-\overline{x}_*^- \leqslant Ax \leqslant \overline{A}^+\overline{x}_*^+ - \underline{A}^+\overline{x}_*^- - \overline{A}^-\underline{x}_*^+ + \underline{A}^-\underline{x}_*^-,$$

where $A^+ = \max(0, A)$ *and* $A^- = A^+ - A$, *and the analogous relations hold for the vector* x_*.

The proof was given in [1].

The interval observer based on model (2.1) with $P = 0$ is designed in the form

$$\begin{aligned}
\dot{\underline{x}}_*(t) &= F_*\underline{x}_*(t) + (\underline{\Delta F}_*^+\underline{x}_*^+ - \overline{\Delta F}_*^+\underline{x}_*^- - \underline{\Delta F}_*^-\overline{x}_*^+ + \overline{\Delta F}_*^-\overline{x}_*^-) + (J_* + J')y(t) + G_*u(t),\\
\dot{\overline{x}}_*(t) &= F_*\overline{x}_*(t) + (\overline{\Delta F}_*^+\overline{x}_*^+ - \underline{\Delta F}_*^+\overline{x}_*^- - \overline{\Delta F}_*^-\underline{x}_*^+ + \underline{\Delta F}_*^-\underline{x}_*^-) + (J_* + J')y(t) + G_*u(t),\\
\underline{z}(t) &= H_z\underline{x}_*(t) + Qy(t),\\
\overline{z}(t) &= H_z\overline{x}_*(t) + Qy(t),\\
\underline{x}_*(0) &= \underline{x}_{*0}, \quad \overline{x}_*(0) = \overline{x}_{*0}.
\end{aligned} \tag{3.4}$$

Here, the equations for the estimation errors take the form

$$\begin{aligned}
\dot{\underline{e}}_*(t) &= F_*\underline{e}_*(t) + \Delta F_* x_*(t) - (\underline{\Delta F}_*^+\underline{x}_*^+ - \overline{\Delta F}_*^+\underline{x}_*^- - \underline{\Delta F}_*^-\overline{x}^+ + \overline{\Delta F}_*^-\overline{x}_*^-),\\
\dot{\overline{e}}_*(t) &= F_*\overline{e}_*(t) + (\overline{\Delta F}_*^+\overline{x}_*^+ - \underline{\Delta F}_*^+\overline{x}_*^- - \overline{\Delta F}_*^-\underline{x}_*^+ + \underline{\Delta F}_*^-\underline{x}_*^-) - \Delta F_* x_*(t).
\end{aligned} \tag{3.5}$$

Theorem 2. *Assume that* $\underline{\Delta F}_* \leqslant \Delta F_* \leqslant \overline{\Delta F}_*$ *and* $\underline{x}_*(0) \leqslant x_*(0) \leqslant \overline{x}_*(0)$. *Then the interval observer* (3.4) *satisfies the relation* $\underline{z}(t) \leqslant z(t) \leqslant \overline{z}(t)$.

Proof. Due to (3.2), from $\underline{x}_*(0) \leqslant x_*(0) \leqslant \overline{x}_*(0)$ it follows that $\underline{e}_*(0) \geqslant 0$ and $\overline{e}_*(0) \geqslant 0$. Since F_* is a Metzler matrix, by (3.5) and the lemma we have $\underline{e}_*(t) \geqslant 0$ and $\overline{e}_*(t) \geqslant 0$ for all $t \geqslant 0$ [2]. The relation $\underline{z}(t) \leqslant z(t) \leqslant \overline{z}(t)$ is established by analogy with the second part of Theorem 1 and Remark 1. The proof of Theorem 2 is complete.

Remark 2. Direct comparison shows that the first variant imposes noticeably more constraints on the system than the second one but has a simpler structure. In particular, the second variant does not need the matrices H_* and R_*; hence, condition (2.7) can be ignored when implementing this variant, and the corresponding interval observer can be constructed for a wider class of systems. When solving a particular problem, one should therefore begin with the first variant, passing to the second variant only if the former is impossible to implement.

Remark 3. An essential role in Theorems 1 and 2 is played by the condition $\underline{x}_*(0) \leqslant x_*(0) \leqslant \overline{x}_*(0)$. Due to the stability of the matrices $F_* + \underline{\Delta F}_* - \underline{P}H_*$, $F_* + \overline{\Delta F}_* - \overline{P}H_*$, and F_*, the desired inequality $\underline{z}(t) \leqslant z(t) \leqslant \overline{z}(t)$ will hold for some $t > 0$ even without the condition $\underline{x}_*(0) \leqslant x_*(0) \leqslant \overline{x}_*(0)$: the effect of the initial conditions almost disappears after some time. Indeed, we demonstrate this fact for the second variant and the first equation in (3.5). Let us denote

$$v_0(t) = \Delta F_* x_*(t) - (\underline{\Delta F}_*^+\underline{x}_*^+ - \overline{\Delta F}_*^+\underline{x}_*^- - \underline{\Delta F}_*^-\overline{x}^+ + \overline{\Delta F}_*^-\overline{x}_*^-).$$

The lemma implies $v_0(t) \geqslant 0$. Clearly, the matrices figuring in the expression above, particularly $\overline{\Delta F}_*^+$, can be always chosen so that $v_0(t) \geqslant v_{0*}$ for some $v_{0*} > 0$.

Since the matrix F_* is diagonal, we consider equation (3.5) for the first component of the error $\underline{e}_*(t)$:

$$\dot{\underline{e}}_{*1}(t) = \lambda_1\underline{e}_{*1}(t) + v_{01}(t).$$

It has the solution

$$\underline{e}_{*1}(t) = exp(\lambda_1 t)\underline{e}_{*1}(0) + exp(\lambda_1 t)\int_0^t v_{01}(\tau)exp(-\lambda_1\tau)d\tau.$$

Because $\lambda_1 < 0$ and $v_0(t) \geqslant v_{0*} > 0$, the first term will almost vanish and the error $\underline{e}_{*1}(t)$ will be positive after some time T_1, despite the possibly negative value $\underline{e}_{*1}(0)$. Similar results can be obtained for the other components of the vector $\underline{e}_*(t)$ and all components of the vector $\overline{e}_*(t)$.

Remark 4. The concepts of a confidence interval and a confidence probability from mathematical statistics can be adapted to interval observers: if $\Delta F_* \in (\underline{\Delta F}_*, \overline{\Delta F}_*)$ with some degree of belonging μ_*, then $z(t) \in (\underline{z}(t), \overline{z}(t))$ with the degree μ_* as well.

4. CONSIDERATION OF EXOGENOUS DISTURBANCES AND MEASUREMENT NOISES

The case $v(t) \neq 0$, $\rho(t) \neq 0$ was studied in detail in [13]. Note that the observer is designed based on model (2.1) with the term $J_*y(t)$ replaced by $J_*Hx(t)$. (It is necessary to consider the measurement noises according to the relation $y(t) = Hx(t) + v(t)$.) In this case, the first variant of the interval observer is described by

$$\begin{aligned}
\dot{\underline{x}}_*(t) &= (F_* + \underline{\Delta}F_* - \underline{P}H_*)\underline{x}_*(t) + \underline{P}y_*(t) + (J_* + J')y(t) + G_*u(t) \\
&\quad - (|J_*| + |J'|)E_k v_* - |L_*|E_k\rho_*, \\
\dot{\overline{x}}_*(t) &= (F_* + \overline{\Delta}F_* - \overline{P}H_*)\overline{x}_*(t) + \overline{P}y_*(t) + (J_* + J')y(t) + G_*u(t) \\
&\quad + (|J_*| + |J'|)E_k v_* + |L_*|E_k\rho_*, \\
\underline{z}(t) &= H_z\underline{x}_*(t) + Qy_0(t), \\
\overline{z}(t) &= H_z\overline{x}_*(t) + Qy_0(t), \\
\underline{x}_*(0) &= \underline{x}_{*0}, \quad \overline{x}_*(0) = \overline{x}_{*0}.
\end{aligned} \tag{4.1}$$

Here, the matrix $|A|$ consists of the absolute values of the corresponding elements of the matrix A, and E_k is a matrix of dimensions $k \times 1$ composed of unities. The variable $y_0(t)$ represents the components of the output vector $y(t)$ that are not affected by the exogenous disturbances, i.e., $y_0(t) = N_1y(t)$ for some matrix N_1. We introduce a maximum-rank matrix L_0 such that $L_0L = 0$. Since the vector $x'(t) = L_0x(t)$ is not disturbed, we obtain $y_0(t) = N_2x'(t)$ for some matrix N_2. Then the matrices N_1 and N_2 satisfy the equation $N_1H = N_2L_0$, which has a solution if

$$\operatorname{rank}\begin{pmatrix} H \\ L_0 \end{pmatrix} < \operatorname{rank}(H) + \operatorname{rank}(L_0).$$

Under this condition, the matrices N_1 and N_2 are determined from the equation

$$(N_1 \quad -N_2)\begin{pmatrix} H \\ L_0 \end{pmatrix} = 0.$$

If this condition fails, it is necessary to replace the variable $y_0(t)$ in (4.1) by $y(t)$, which will extend the interval $(\underline{z}(t), \overline{z}(t))$.

The equations for the estimation errors take the form

$$\begin{aligned}
\dot{\underline{e}}_*(t) &= (F_* - \underline{P}H_*)\underline{e}_*(t) + \Delta F_*x_*(t) - \underline{\Delta F}_*\underline{x}_*(t) \\
&\quad - (J_* + J')v(t) + (|J_*| + |J'|)E_k v_* - L_*\rho(t) + |L_*|E_k\rho_*, \\
\dot{\overline{e}}_*(t) &= (F_* - \overline{P}H_*)\overline{e}_*(t) + \overline{\Delta F}_*\overline{x}_*(t) - \Delta F_*x_*(t) \\
&\quad + (J_* + J')v(t) + (|J_*| + |J'|)E_k v_* + L_*\rho(t) + |L_*|E_k\rho_*.
\end{aligned}$$

The desired result directly follows from the proof of Theorem 1 and the obvious additional inequalities

$$\pm(J_* + J')v(t) + (|J_*| + |J'|)E_k v_* \geqslant 0, \quad \pm L_* \rho(t) + |L_*|E_k \rho_* \geqslant 0,$$

valid for all $t \geqslant 0$.

Remark 5. Due to (4.1), the width of the interval $(\underline{z}(t), \overline{z}(t))$ depends on the exogenous disturbances and measurement noises. To reduce this width, the matrices Φ and J_* should be determined from the equation

$$(S_i \quad - J_{*i}) \begin{pmatrix} L_0(F - \lambda_i I_n) \\ H \end{pmatrix} = 0, \quad i = 1, \dots, k, \tag{4.2}$$

which is derived from (2.8) for $\Phi = SL_0$ with some matrix S. (It ensures $L_* = 0$, i.e., the insensitivity of the model to the exogenous disturbances.) As before, here S_i denotes the ith row of the matrix S. If equation (4.2) has no solution, robust methods should be used; see below.

The sensitivity of model (2.1) to the exogenous disturbances $\rho(t)$ is often estimated by the norm $\|\Phi L\|_F$ of the matrix ΦL. To minimize this norm for $i = 1, \dots, k$, we choose an eigenvalue $\lambda_i < 0$ for which equation (2.8) has a solution and then find all solutions in the form $\Phi_i^{(1)}, \dots, \Phi_i^{(n_i)}$. These solutions can be written as

$$\Phi_{*i} = \begin{pmatrix} \Phi_i^{(1)} \\ \dots \\ \Phi_i^{(n_i)} \end{pmatrix}, \quad J_i = \begin{pmatrix} J_{*i}^{(1)} \\ \dots \\ J_{*i}^{(n_i)} \end{pmatrix}.$$

The singular-value decomposition of the matrix product $\Phi_{*i}L$ is given by

$$\Phi_{*i}L = U_L \Sigma_L V_L,$$

where U_L and V_L are orthogonal matrices,

$$\Sigma_L = (\mathrm{diag}(\sigma_1, \dots, \sigma_{c_i}) \ 0) \ \text{ or } \ \Sigma_L = \begin{pmatrix} \mathrm{diag}(\sigma_1, \dots, \sigma_{c_i}) \\ 0 \end{pmatrix}$$

depending on the number of rows and columns in the matrix $\Phi_{*i}L$, $c_i = \min(n_i, kp)$, and $0 \leqslant \sigma_1 \leqslant \dots \leqslant \sigma_{c_i}$ are the singular values of the matrix $\Phi_{*i}L$ arranged in ascending order [16]. The first transposed column of the matrix U_L is taken as the weight vector $w = (w_1 \ \dots \ w_{n_i})$, and the rows $\Phi_i = w\Phi_{*i}$ and $J_{*i} = wJ_i$ are calculated consequently; if $n_i = 1$, then $\Phi_i := \Phi_i^{(1)}$ and $J_{*i} := J_i^{(1)}$. The resulting rows $\Phi_1, \dots, \Phi_k$ and $J_{*1}, \dots, J_{*k}$ form the matrices

$$\Phi = \begin{pmatrix} \Phi_1 \\ \dots \\ \Phi_k \end{pmatrix}, \quad J_* = \begin{pmatrix} J_{*1} \\ \dots \\ J_{*k} \end{pmatrix},$$

and the feasibility of estimating the variables $z(t)$ and $y_*(t)$ by criteria (2.5) and (2.7) is checked. If the outcome is positive, the problem is successfully solved; otherwise, for some $\lambda_i < 0$, it is necessary to find another vector $w = (w_1 \ \dots \ w_{n_i})$ for a singular value $\sigma > \sigma_1$ and calculate a new matrix Φ_i.

Theorem 3. *This procedure for constructing the matrix Φ yields an optimal solution in terms of minimizing the norm $\|\Phi L\|_F$.*

Proof. The conclusion directly follows from the properties of the singular-value decomposition.

The determination of the matrices H_z, Q, ΔF_*, J', H_*, R_*, $G_* = \Phi G$, and $L_* = \Phi L$ completes the design of model (2.1) with the minimum sensitivity to the exogenous disturbances.

By analogy with [13, 14], the above results for estimating the variable $z(t)$ can be applied to estimate the full state vector $x(t)$. For this purpose, it is necessary to find the solutions of equation (4.2) with $\lambda_i < 0$ and the maximum possible value k, compile the matrix $\Phi^{(1)} =: M^{(1)}$, and construct an interval estimate for the vector $x^{(1)}(t) = M^{(1)}x(t)$ following one of the variants if criteria (2.5) and (2.7) hold. Note that this estimate will be independent of the exogenous disturbance $\rho(t)$. Next, it is necessary to determine a vector $x^{(2)}(t)$ that complements $x^{(1)}(t)$ to $x(t)$ and a matrix $M^{(2)}$ such that $x^{(2)}(t) = M^{(2)}x(t)$. The interval observer estimating the variable $x^{(2)}(t)$ is obtained based on equation (2.8). The composition of the two observers yields an interval $(\underline{x}(t), \overline{x}(t))$ that is narrower compared to the classical one [10].

5. A PRACTICAL EXAMPLE

Consider a three-tank system of the form

$$\begin{aligned}
\dot{x}_1(t) &= b_4 u_1(t)/\vartheta_1 - (b_1 - \delta_1(t))(x_1(t) - x_2(t)),\\
\dot{x}_2(t) &= b_5 u_2(t)/\vartheta_2 + (b_1 - \delta_1(t))(x_1(t) - x_2(t)) - (b_2 - \delta_2(t))(x_2(t) - x_3(t)),\\
\dot{x}_3(t) &= (b_2 - \delta_2(t))(x_2(t) - x_3(t)) - b_3(x_3(t) - b_6) + \rho(t),\\
y_1(t) &= x_2(t) + v_1(t), \quad y_2(t) = x_3(t) + v_2(t),
\end{aligned} \tag{5.1}$$

where the coefficients $b_1, \ldots, b_5$ are determined by the geometrical dimensions of the system; x_1, x_2, and x_3 are the fluid levels in the corresponding tanks (Fig. 1). Fluid flows into the first and second tanks and flows out of the third tank through a pipe located at a height b_6. The parametric uncertainties $\delta_1(t)$ and $\delta_2(t)$ are related to possible clogging of the pipes connecting the tanks and, consequently, a decrease in their capacity. The exogenous disturbance $\rho(t)$ reflects possible leaks in the third tank. For simplicity, we take $b_1 = \ldots = b_5 = 1$ and $b_6 = 0$; $0 \leqslant \delta_1(t) \leqslant 0.2$, $0 \leqslant \delta_2(t) \leqslant 0.1$; $|\rho(t)| \leqslant \rho_*$, $|v_1(t)| \leqslant v_{*1}$, $|v_2(t)| \leqslant v_{*2}$.

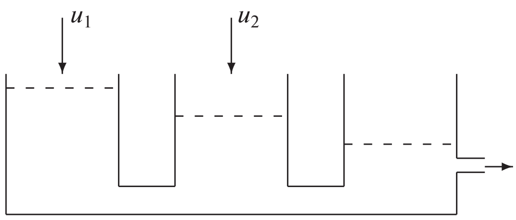

Fig. 1. Three-tank system.

This system is modeled by the matrices

$$F = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Direct comparison of the general model (1.1) and system (5.1) shows that the uncertainties are described by the matrix functions

$$\Delta_1 F(t) = \begin{pmatrix} \delta_1(t) & -\delta_1(t) & 0 \\ -\delta_1(t) & \delta_1(t) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Delta_2 F(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta_2(t) & -\delta_2(t) \\ -\delta_2(t) & \delta_2(t) & 0 \end{pmatrix}.$$

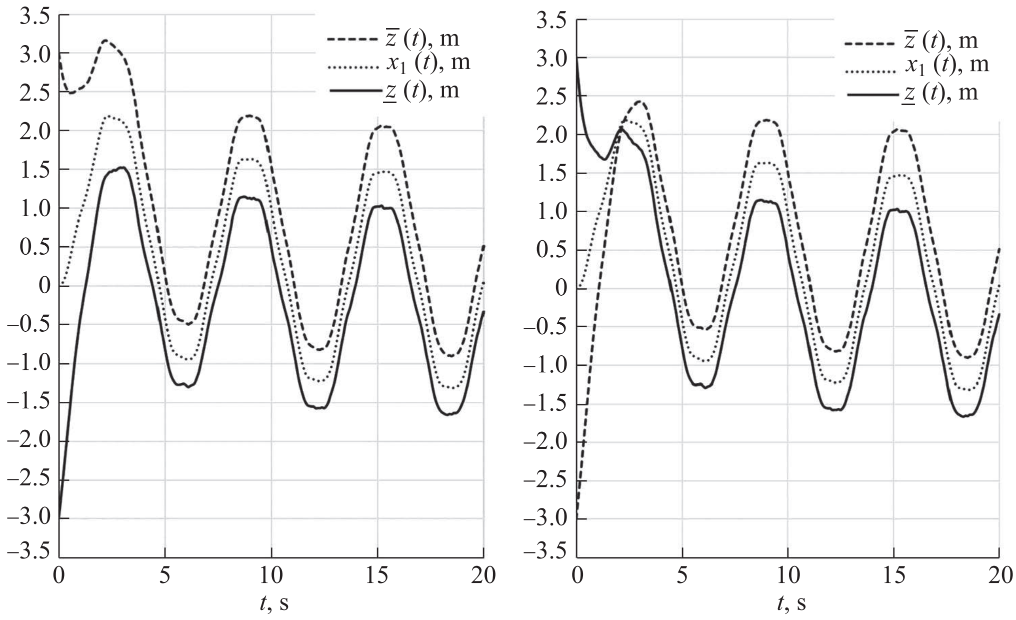

Fig. 2. The graphs of the functions $x_1(t)$, $\underline{z}(t)$, and $\overline{z}(t)$ under different initial conditions.

We construct an interval observer estimating the variable $z(t) = x_1(t)$. In this case, $M = (1\ \ 0\ \ 0)$ and $L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Equation (4.2) takes the form

$$(S_i \quad - J_{*i}) \begin{pmatrix} -1-\lambda_i & 1 & 0 \\ 1 & -2-\lambda_i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0.$$

Letting $\lambda = -1$ gives $S_i = J_{*i} = (1\ \ 0)$, $\Phi = (1\ \ 0\ \ 0)$, and $G_* = (1\ \ 0)$. The uncertainty $\Delta F_*(t)$ and the matrix J' are determined from equation (2.9); as a result, we obtain $\Delta_1 F_*(t) = \delta_1(t)$, $J'_1 = (-\delta_1(t)\ \ 0)$, $\Delta_2 F_*(t) = 0$, and $J'_2 = 0$. According to physical considerations, $x_*(t) \geqslant 0$ and $\delta_1(t) \geqslant 0$ for all $t \geqslant 0$; hence, the first variant can be used to design the observer. In view of $0 \leqslant \delta_1(t) \leqslant 0.2$, we assume that $P = 0$.

Since the variable $x_3(t)$ is affected by the disturbance $\rho(t)$ and, in addition, $y_2(t) = x_3(t) + v_2(t)$, we take $y_0(t) = y_1(t)$. Obviously, condition (2.5) holds and $H_z = 1$, $Q = 0$. The model becomes

$$\begin{aligned} \dot{x}_*(t) &= (\delta_1(t) - 1)x_*(t) + (1 - \delta_1(t))H_1 x(t) + u_1(t), \\ z(t) &= x_*(t), \end{aligned}$$

where $x_* = \Phi x(t) = x_1(t)$. The interval observer (4.1) take the form

$$\begin{aligned} \dot{\underline{x}}_*(t) &= (\underline{\delta_1} - 1)\underline{x}_*(t) + (1 - \underline{\delta_1})y_1(t) + u_1(t) - v_*, \\ \dot{\overline{x}}_*(t) &= (\overline{\delta_1} - 1)\overline{x}_*(t) + (1 - \overline{\delta_1})y_1(t) + u_1(t) + v_*, \\ \underline{z}(t) &= \underline{x}_*(t), \quad \overline{z}(t) = \overline{x}_*(t). \end{aligned}$$

Due to $\underline{\delta}_1 - 1 < 0$ and $\overline{\delta}_1 - 1 < 0$, the observer is stable.

Numerical simulation was performed for $u_1(t) = 2\sin(t)$, $u_2(t) = 2\sin(5t)$, the noises $v_1(t)$, $v_2(t)$ and disturbance $\rho(t)$ described by random independent variables with variances 0.1, and the uncertainty $\delta_1(t) = 0.1(1 + \sin(10t))$ with $\underline{\delta_1} = 0$ and $\overline{\delta_1} = 0.2$. The simulation results are presented in

Fig. 2: the graphs of the functions $x_1(t)$, $\underline{z}(t)$, and $\overline{z}(t)$ under the initial conditions $x(0) = (0\ \ 0\ \ 0)^T$, $\underline{x}_*(0) = -3$, and $\overline{x}_*(0) = 3$ (Fig. 2a) and $x(0) = (0\ \ 0\ \ 0)^T$, $\underline{x}_*(0) = 3$, and $\overline{x}_*(0) = -3$ (Fig. 2b). As has been demonstrated in Section 3, the "atypical" initial conditions are rather quickly "forgotten" and have no effect on the estimation process for $t \geqslant 2$.

6. CONCLUSIONS

In this paper, we have constructed interval observers for linear dynamic systems described by continuous-time models with exogenous disturbances, measurement noises, and parametric uncertainties. Jordan canonical form-based relations have been presented for a minimum-dimension interval observer estimating the set of admissible values of a given linear function of the system state vector. Two observer design variants have been considered and compared. The theoretical results have been illustrated by a practical example.

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