*ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2023, Vol. 84, No. 10, pp. 1243–1258.*


**TOPICAL ISSUE**

# Spectral Decompositions of Gramians and Energy Metrics of Continuous Unstable Control Systems

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Received June 19, 2023

Revised July 15, 2023

Accepted August 2, 2023

**Abstract**—Deterministic continuous finite-dimensional stationary linear dynamic control systems with many inputs and many outputs are considered. Authors assume that the dynamics matrix can be both stable and unstable, but its eigenvalues are different, do not belong to the imaginary axis, and their pairwise sum is not equal to 0. The problems of constructing spectral solutions of the equations of state and matrices of gramian controllability of these systems, as well as the associated energy functionals of the degree of stability and reachability with the aim of optimal placement of sensors and actuators of multi-connected control systems and complex networks are considered. To solve the listed problems, the article uses various models of the system in state space: a general representation, as well as a representation in various canonical forms. To calculate the spectral decompositions of controllability gramians, pseudo-Hankel matrices (Xiao matrices) are used. New methods have been proposed and algorithms have been developed for calculating controllability gramians and energy metrics of linear systems. The research results can be used for the optimal placement of sensors and actuators of multi-connected control systems or for control with minimal energy in complex networks of various natures.

*Keywords*: spectral decompositions of gramians, energy functionals, inverse matrix of gramians, stability that takes into account the interaction between modes, Lyapunov equation, unstable control systems

**DOI:** 10.25728/arcRAS.2023.99.15.001

## 1. INTRODUCTION

Monitoring the state of control objects and controlling the damping of dangerous vibrations are important areas of research in various fields of industry (energy, mechanical engineering, aviation and astronautics, robotics). New modeling technologies require the development of tools for approximating mathematical models of complex systems of various natures. When solving these problems, an important role is played by the methods of calculating the Lyapunov and Sylvester matrix equations and the study of the structural properties of solutions to these equations [1–4]. The fundamental properties of linear dynamic systems associated with solutions to these equations are controllability, observability and stability. Important results in this area were obtained for methods for calculating the gramians of systems, the models of which are presented in the canonical forms of controllability and observability. The application of gramians for constructing simplified models of high-dimensional dynamic systems and for calculating the norms of transfer functions of linear and bilinear dynamic systems is well known [1, 2, 5–8]. Controllability gramians play an important role in calculating output deviations caused by Gaussian random disturbances. In recent years, interest has arisen in the development of methods for calculating various energy indicators to analyze the stability and degree of controllability and observability of these systems.

Such indicators for linear stable systems and unstable linear systems were proposed in [1, 8–11]. Simplified models for large networks based on output controllability gramians, that allow to calculate the energy indicators, were proposed in [12]. The balanced truncation method, based on the gramians of stable and anti-stable systems, was proposed in [13]. The important problem of optimal placement of sensors and actuators based on various energy functionals, including invariant ellipsoids, and estimation of the degree of controllability was studied in [14–18]. It is important to note that all these works used the spectrum of the system dynamics matrix.

B.N. Petrov and his students developed methods, based on Lyapunov direct method, for synthesizing adaptation algorithms that guarantee the stability of the movement of a self-adjusting system relative to the movement of its reference model [19, 20]. He developed the principle of coordinate-parametric control, which implements double invariance in non-search self-tuning systems (NSTS). In the theory of NSTS, the concept of a generalized customizable object was used, which was based on identifying the structures of a specially formed main circuit and a circuit of a customizable controller. Linearized mathematical models of circuits included coordinate, parametric and coordinate-parametric models, including parametric feedbacks in controller tuning circuits. These models are called bilinear dynamic models and are used in optimization, identification theory, and adaptive control. To calculate the gramians of these systems, generalized Lyapunov equations were developed and spectral methods for solving them were proposed [2, 10, 11]. A significant contribution was made by the school of B.N. Petrov in the formation of control theory, based on the use of the structural properties of the reference model, and in other areas of control theory, in particular in the theory of invariant systems.

## 2. FORMULATION OF THE PROBLEM

Consider a continuous time-invariant linear dynamic MIMO LTI system with a simple spectrum with many inputs and many outputs

$$\Sigma_1 \colon \begin{cases} \dot{x} = Ax\,(t) + Bu\,(t)\,, \quad x(0) = 0, \\ y\,(t) = Cx\,(t)\,, \end{cases} \tag{2.1}$$

where $x(t) \in R^n, u(t) \in R^m, y(t) \in R^m$.

If all eigenvalues $s_r$ of matrix A are different, then the linear system can be reduced to diagonal form using a non-degenerate coordinate transformation

$$\begin{aligned} &x_d = Tx, \quad \dot{x}_d = A_d x_d + B_d u, \quad y_d = C_d x_d, \\ &A_d = T^{-1}AT, \quad B_d = T^{-1}B, \quad C_d = CT, \quad Q_d = T^{-1}BB^{\mathrm{T}}(T^{-1})^{\mathrm{T}}, \end{aligned}$$

or

$$\mathrm{A} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & s_n \end{bmatrix} \begin{bmatrix} \nu_1^* \\ \nu_2^* \\ \vdots \\ \nu_n^* \end{bmatrix} = T\Lambda T^{-1},$$

where the matrix $T^{-1}$ is composed of right eigenvectors $u_i$, and the matrix $T$ is composed of left eigenvectors $\nu_i^*$ corresponding to the eigenvalue $s_i$.

**Definition** [21]. The square matrix $Y = [y_{j\eta}]$ is called the Xiao matrix (Zero plaid structure) and has the form:

$$Y = \begin{bmatrix} y_1 & 0 & -y_2 & 0 & y_3 \\ 0 & y_2 & 0 & -y_3 & 0 \\ -y_2 & 0 & y_3 & 0 & \dots \\ 0 & -y_3 & 0 & \dots & 0 \\ y_3 & 0 & \dots & 0 & y_n \end{bmatrix},$$

its elements are specified using the elements of the Routh table [21]:

$$y_{j\eta} = \begin{cases} 0, \quad \text{if} \ \ j+\eta = 2k+1, \quad k{=}1,\dots,n; \\ y_n = \dfrac{1}{2R_{n,1}}, \\ \\ y_{n-l} = \dfrac{-\sum_{i=1}^{m-1}(-1)^i R_{n-l,i+1} y_{n-l+i}}{R_{n-l,1}}, \\ \qquad \text{if} \ \ j+\eta{=}\ 2k, \ \ k=1,\dots,n, \ \ l = \overline{1, n-1}, \end{cases}$$

where $R_{i,j}$ is the Routh table element for the system, located at the intersection of row $i$ and column $j$. In [11], a spectral decomposition of the controllability gramian of a continuous linear system with many inputs and many outputs was obtained based on the method for calculating the gramian proposed in [21, 22].

**Theorem 1** [11, 21]. *We consider a continuous linear MIMO LTI system of the form* (2.1). *Let us assume that the system is stable and all the roots of its characteristic equation are different. Then the matrices of its controllability gramian are Xiao matrices, the diagonal elements of which are defined as*

$$p_{11} = \sum_{k=1}^{n} \frac{1}{2s_k \prod_{\rho=1,\rho\neq k}^{n}\left(s_k^2 - s_\rho^2\right)},$$

$$p_{22} = \sum_{k=1}^{n} \frac{(-1)^1 (s_k)^2}{2s_k \prod_{\rho=1,\rho\neq k}^{n}\left(s_k^2 - s_\rho^2\right)},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$p_{nn} = \sum_{k=1}^{n} \frac{(-1)^{n-1} (s_k)^{2(n-1)}}{2s_k \prod_{\rho=1,\rho\neq k}^{n}\left(s_k^2 - s_\rho^2\right)}.$$

*The elements of the side diagonals of the gramian matrices are defined as:*

$$p_{j\eta} = (-1)^{\frac{j-\eta}{2}} p_{ll}, \quad j+\eta = 2l, \quad l = \overline{1, n}.$$

*The remaining elements of the gramian matrix are equal to zero.*

**Corollary 1.** Consider a stable continuous stationary linear dynamic MIMO LTI system with a simple spectrum with many inputs and many outputs of the form (2.1). Then its controllability gramian is a matrix of the form [11]

$$P_c = \sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1} P_{cj,\eta},\ P_{cj,\eta} = \omega(n, s_k, j, \eta) A_j B B^{\mathrm{T}} A_\eta^{\mathrm{T}}, \tag{2.2}$$

$$\omega\left(n, s_k, j, \eta\ \right) = \begin{cases} 0, \text{ if } j+\eta = 2k+1, \ \ k = 1, \ldots, n, \\ \displaystyle\sum_{k=1}^{n} \frac{s_k^j (-s_k)^\eta}{2 s_k \prod_{\rho=1,\rho\neq k}^{n}\left(s_k^2 - s_\rho^2\right)}, \text{ if } j+\eta = 2k, \ \ k = 1, \ldots, n. \end{cases}$$

We will call spectral decompositions (2.2) the gramian decompositions in the form of Xiao matrices. In the expansion (2.2) a scalar multiplier function $\omega\left(n, s_k, j, \eta\right)$ appears, which determines the structure of the Hadamard matrices [21].

Let us transform the system (2.1) into the upper block-diagonal Schur form with a unitary transformation matrix U [23,24].

$$\begin{gathered} x = U x_{Sch}, \ \ \dot{x}_{Sch} = A_{Sch} x_{Sch} + B_{Sch} u, \ \ y_{Sch} = C_{Sch} x_{Sch}, \\ A_{Sch} = U^{\mathrm{T}} A U, \ \ B_{Sch} = U^{\mathrm{T}} B, \ \ C_{Sch} = C U, \end{gathered} \tag{2.3}$$

$$A_{Sch} = \begin{bmatrix} A_{Sch11} & A_{Sch12} \\ 0 & A_{Sch22} \end{bmatrix}, \ \ B_{Sch} = \begin{bmatrix} B_{Sch1} \\ B_{Sch2} \end{bmatrix}, \ \ C_{Sch} = \begin{bmatrix} C_{Sch1} & C_{Sch2} \end{bmatrix}.$$

In order to obtain a block-diagonal representation, it is necessary to transform the equations (2.3) so that the place of the $A_{Sch12}$ block is replaced by a zero matrix. To do this, we perform a second transformation

$$\begin{gathered} x_{Sch} = W_{bl} x_{bl}, \ \ \dot{x_{bl}} = A_{bl} x_{bl} + B_{bl} u, \ \ y_{bl} = C_{bl} x_{bl}, \\ A_{bl} = W_{bl}^{-1} A_{Sch} W_{bl}, \ \ B_{bl} = W_{bl}^{-1} B_{Sch}, \ \ C_{bl} = C_{Sch} W_{bl}, \end{gathered} \tag{2.4}$$

$$A_{bl} = \begin{bmatrix} A_{Sch11} & 0 \\ 0 & A_{Sch22} \end{bmatrix}, \ \ B_{bl} = \begin{bmatrix} B_{bl1} \\ B_{bl2} \end{bmatrix}, \ \ C_{bl} = \begin{bmatrix} C_{bl1} & C_{bl2} \end{bmatrix},$$

$$W_{bl} = \begin{bmatrix} I_r & S \\ 0 & I_{n-r} \end{bmatrix}, W_{bl}^{-1} = \begin{bmatrix} I_r & -S \\ 0 & I_{n-r} \end{bmatrix}.$$

In order for the block $A_{Sch12}$ to be replaced by a zero matrix, the matrix S must satisfy the Sylvester equation

$$-A_{Sch11} S + S A_{Sch22} + A_{Sch12} = 0. \tag{2.5}$$

A necessary condition for the existence of a solution to this equation is the following spectral condition:

$$\lambda_s + \lambda_u \neq 0, \ \ \forall s : s = \overline{1, r}, \forall u : u = \overline{r+1, n}.$$

In order to transform a system (2.4) with a block diagonal matrix into a system with a diagonal matrix, it is necessary to perform a third transformation

$$x_{bl} = W_d x_d,$$

where $W_d$ is the transformation matrix of a system in block-diagonal form, which diagonal blocks have an upper-triangular shape

$$\begin{gathered} \dot{x_d} = A_d x_d + B_d u, \ \ y_d = C_d x_d, \\ A_d = W_d^{-1} A_{bl} W_d, \ \ B_d = W_d^{-1} B_{bl}, \ \ C_d = C_{bl} W_d, \end{gathered}$$

$$A_d = \begin{bmatrix} \Lambda_- & 0 \\ 0 & \Lambda_+ \end{bmatrix}, \ \ B_d = \begin{bmatrix} B_{d1} \\ B_{d2} \end{bmatrix}, \ \ C_d = \begin{bmatrix} C_{d1} & C_{d2} \end{bmatrix}, \tag{2.6}$$

where $\Lambda_-$ and $\Lambda_+$ are diagonal matrices consisting of negative and positive eigenvalues, respectively.

After the first transformation we have the relation

$$P = UP_{Sch}U^{\mathrm{T}}. \tag{2.7}$$

After the second transformation we get

$$P_{Sch} = TP_{bl}T^{\mathrm{T}},$$

or

$$P = T_2P_{bl}T_2^{\mathrm{T}}, \quad T_2 = UT. \tag{2.8}$$

After the third transformation using (2.7),(2.8) we get

$$P = UT_3P_dT_3^{\mathrm{T}}, \quad T_3 = UTW_d.$$

The structured Lyapunov equation after the second transformation has the form

$$A_{Sch11}P_1 + P_1A_{Sch11}^{\mathrm{T}} = -B_1B_1^{\mathrm{T}}, \tag{2.9}$$

$$A_{Sch22}P_2 + P_2A_{Sch22}^{\mathrm{T}} = B_2B_2^{\mathrm{T}}, \tag{2.10}$$

$$P_{cm} = T_2^{-1}\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}T_2. \tag{2.11}$$

The matrix $P_{cm}$ is called the mixed controllability gramian [13–17, 25]. The purpose of the article is to develop a method and algorithm for calculating spectral decompositions of the controllability gramians of unstable linear systems, based on the method described above, for calculating the specified gramians using the transformation of the original system into a block-diagonal form [13].

Many applications of spectral decompositions of gramians are associated with energy indicators of the structural properties of controllability, observability and stability. We consider the problem of selecting and optimizing the placement of sensors and actuators in complex automatic systems and complex networks [18, 26–28]. To solve this problem one could use the input and output energy of the system, traces of the controllability and observability gramian matrices and traces of their inverse matrices, minimum and maximum eigenvalues of the gramian. Another problem is to estimate the controllability measure of a dynamic system using controllability gramians [25]. This measure is defined as the minimum input energy required to move the system from an arbitrary initial state to an arbitrary final state.

Another goal of the article is to develop a method and algorithms for calculating spectral decompositions of energy metrics related to the above problems. It is required to find spectral decompositions of the following energy metrics from the simple (or paired) spectrum of the system dynamics matrix and the controllability and observability gramian matrices:

- metric of the input minimum energy of the system [2, 18]

$$J_1 = E_{\min}(P_c),$$

- output energy metric of the system [2, 3]

$$J_2 = E_{out},$$

- trace metric of the gramian matrix [26, 27]

$$J_3 = \mathrm{tr}\,(P_c),$$

- trace metric of inverse matrices of controllability gramians [2, 12, 18]

$$J_4 = \mathrm{tr}\,(P_c)^{-1},$$

- reachability metric

$$J_5 = \mathrm{tr}\,(P_{cm}),$$

where $P_{cm}$ is the mixed controllability gramian [18, 25].

### *2.1. Main Results*

Let us consider a finite-dimensional linear stationary continuous system with many inputs and many outputs of the form (2.1). We suppose that the spectrum of the dynamics matrix contains $r$ stable eigenvalues $\lambda_{i-} \in \mathbb{C}^-$ and $n-r$ unstable eigenvalues $\lambda_{i+} \in \mathbb{C}^+$. We will assume that the spectrum does not contain eigenvalues belonging to the imaginary axis, and the general condition is satisfied

$$\lambda_{i-} + \lambda_{j+} \neq 0, \forall i : i = \overline{1, r}, \forall j : j = \overline{r+1, n}.$$

The last condition means that the spectrum does not contain eigenvalues that are mirror images of each other relative to zero. The simplest way to calculate the spectral decompositions of gramians in the case of a simple spectrum of the dynamics matrix is to reduce it to diagonal form [1, 11]. If unstable eigenvalues appear in the spectrum, this requires several structural transformations of the (2.1) equations. Let us introduce the notation

$$B_{d11}B_{d11}^{\mathrm{T}} = [\beta_{d-\nu\eta}]_{[r\times r]},$$

$$B_{d22}B_{d22}^{\mathrm{T}} = [\beta_{d+\nu\eta}]_{[(n-r)\times(n-r)]}.$$

**Theorem 2** [8]. *Let us consider a finite-dimensional linear stationary continuous system with many inputs and many outputs of the form* (2.1)*, reduced to the diagonal form* (2.6)*. Let us assume that the system has a simple spectrum, the system is unstable, and the eigenvalues of its dynamics matrix $A$ are not on the imaginary axis, but can be in the left and/or right half-planes $\lambda_{i-} \in \mathbb{C}^-$, $i = r$; $\lambda_{i+} \in \mathbb{C}^+$, $i = n-r$.*
*In addition, assume that the condition is satisfied*

$$\lambda_i \neq -\lambda_j, \ \forall i, j : i = \overline{1, n}, \ j = \overline{1, n}.$$

*Let us define the mixed controllability gramian in the form*

$$P_{cm} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (Ij\omega - A)^{-1}BB^{\mathrm{T}}(-Ij\omega - A^{\mathrm{T}})^{-1}d\omega. \tag{2.12}$$

*The following statements are valid and equivalent.*

- *The following separable spectral decompositions of the matrices of solutions to the equation* (2.9), (2.10)*, corresponding to the stable and anti-stable subsystems, are valid.*

$$p_{c-}^{(\mu\nu)} = e_\mu^{\mathrm{T}} P_{c-} e_\nu, \quad \forall \mu, \nu = \overline{1, r},$$

$$p_{c-}^{(\mu\nu)} = \frac{-\beta_{\mu\nu-}}{\lambda_{\mu-} + \lambda_{\nu-}},$$

$$p_{c+}^{(\mu\nu)} = e_\mu^{\mathrm{T}} P_{c+} e_\nu, \quad \forall \mu, \nu = \overline{r+1, n},$$

$$p_{c+}^{(\mu\nu)} = \frac{\beta_{\mu\nu+}}{\lambda_{\mu+} + \lambda_{\nu+}};$$

- *The following separable spectral expansions of the mixed gramian of controllability in the pair and simple spectra of the matrix $A$ are valid:*

$$P_{cm} = T_3^{-1}\left[P_- \oplus P_+\right] T_3. \tag{2.13}$$

*According to the pair spectrum:*

$$P_- = \sum_{\nu=1}^{r}\sum_{\mu=1}^{r} p_{c-}^{(\nu\mu)} \mathbb{1}_{\nu\mu}, \tag{2.14}$$
$$P_+ = \sum_{\nu=r+1}^{n}\sum_{\mu=r+1}^{n} p_{c+}^{(\nu\mu)} \mathbb{1}_{\nu\mu}.$$

*According to a simple spectrum:*

$$P_- = \sum_{\nu=1}^{r} \boldsymbol{p}_{c-}^{(\nu)}, \quad \boldsymbol{p}_{c-}^{(\nu)} = \sum_{\mu=1}^{r} \boldsymbol{p}_{c-}^{(\nu\mu)} \mathbb{1}_{\nu\mu}, \tag{2.15}$$
$$P_+ = \sum_{\nu=r+1}^{n} \boldsymbol{p}_{c+}^{(\nu)}, \quad \boldsymbol{p}_{c+}^{(\nu)} = \sum_{\mu=r+1}^{n} \boldsymbol{p}_{c+}^{(\nu\mu)} \mathbb{1}_{\nu\mu}.$$

**Proof of Theorem.** The Lyapunov equations for the diagonalized system in this case have the form

$$\Lambda P_{cm} + P_{cm}\Lambda^* = -Q_d = \left[-B_- B_-^{\mathrm{T}} \oplus B_+ B_+^{\mathrm{T}}\right].$$

For a diagonalized system, this equation splits into two equations for stable and antistable subsystems

$$\Lambda_- P_{c-} + P_{c-}\Lambda_-^* = Q_{d-} = -B_- B_-^{\mathrm{T}},$$
$$\Lambda_+ P_{c+} + P_{c+}\Lambda_+^* = Q_{d+} = B_+ B_+^{\mathrm{T}}.$$

Integral formulas for solutions of Lyapunov equations [8]:

$$P_{cm} = [P_{c-} \oplus P_{c+}],$$
$$P_{c-} = \int_0^\infty e^{\Lambda_- \tau}\, B_- B_-^{\mathrm{T}} e^{\Lambda_-^* \tau} d\tau, \quad P_{c+} = \int_{-\infty}^{0} e^{\Lambda_+ \tau}\, B_+ B_+^{\mathrm{T}} e^{\Lambda_+^* \tau} d\tau. \tag{2.16}$$

Let's transform the second integral in the formula (2.16) using the replacing of variables $\tau = -t$ :

$$\int_{-\infty}^{0} e^{\Lambda_+ \tau}\, Q_{d+} e^{\Lambda_+ \tau} d\tau = -\int_0^\infty e^{-\Lambda_+ t}\, Q_{d+} e^{-\Lambda_+^* t} dt.$$

With such a change of variables, the unstable eigenvalues of the antistable subsystem become stable eigenvalues of the stable subsystem and the calculation of the second integrals is reduced to the scheme for calculating the first integrals (2.16). This implies

$$(-\Lambda_+)P_{c+} + P_{c+}(-\Lambda_+^*) = -B_+ B_+^{\mathrm{T}}.$$

The matrix $[\Lambda_- \oplus (-\Lambda_+)]$ is Hurwitz. Spectral expansions of the gramians of a stable subsystem were previously obtained in [9]. First, we obtain spectral decompositions of the gramians in (2.16), and then we obtain the spectral decomposition of the gramians of the original system according to the formula for transforming the gramians of controllability for a nondegenerate transformation of states with matrix T

$$P_{cm} = T[P_- \oplus P_+]\, T^{\mathrm{T}}. \tag{2.17}$$

The first step of spectral decompositions is based on transforming the equations of state of a stable subsystem into a diagonal canonical form. In this case, the Lyapunov equations take on a simple form and the elements $p_{c-}^{(\mu\nu)}$ of the solution matrix $P_{c-}$ can be calculated using the formulas [9]

$$p_{c-}^{(\mu\nu)} = e_\mu^{\mathrm{T}} P_{c-} e_\nu,\ \forall \mu, \nu = \overline{1, r}, \tag{2.18}$$

where $e_\mu^{\mathrm{T}}, e_\nu$ are unit vectors,

$$\begin{gathered} e_\mu^{\mathrm{T}} Q_{d-} e_\nu = \beta_{\mu\nu-},\ \forall \mu, \nu = \overline{1, r}, \\ p_{c-}^{(\mu\nu)} = \frac{-\beta_{\mu\nu-}}{\lambda_{\mu-} + \lambda_{\nu-}}. \end{gathered} \tag{2.19}$$

Since, taking into account the change of variables, the calculation of spectral decompositions of the solution matrix $P_{c+}$ is reduced to considering the approach proposed for calculating the solution matrix $P_{c-}$, we present the final formulas for calculating the spectral decompositions for this case.

This approach is based on transforming the equations of state of an antistable subsystem into a diagonal canonical form. In this case, the elements $p_{c+}^{(\mu\nu)}$ of the solution matrix $P_{c+}$ are calculated using the formulas

$$p_{c+}^{(\mu\nu)} = e_\mu^{\mathrm{T}} P_{c+} e_\nu, \quad \forall \mu, \nu = \overline{r+1, n},$$

where $e_\mu^{\mathrm{T}}, e_\nu$ are unit vectors,

$$\begin{gathered} e_\mu^{\mathrm{T}} Q_{d+} e_\nu = \beta_{\mu\nu+}, \\ p_{c+}^{(\mu\nu)} = \frac{\beta_{\mu\nu+}}{\lambda_{\mu+} + \lambda_{\nu+}},\ \forall \mu, \nu = \overline{r+1, n}. \end{gathered} \tag{2.20}$$

The proof of the validity of spectral expansions for the antistable subsystem completely repeats the proof for the stable subsystem. The proof of the validity of the spectral decompositions (2.13)–(2.15) follows from the validity of the formula (2.19) and the transformation of the antistable subsystem to the form of a stable subsystem, the eigenvalues of which are a mirror image of the eigenvalues of the first subsystem with respect to the imaginary axis. Theorem 2 is proven.

**Corollary 2.** If the conditions of the theorem are satisfied, the mixed gramian is positive definite, since the matrix $[\Lambda_- \oplus (-\Lambda_+)]$ is Hurwitz. In this case, the trace of the mixed controllability gramian is equal to

$$J = \sum_{i=1}^{r} \frac{\beta_{d-ii}}{-2Re\ \lambda_i} + \sum_{i=r+1}^{n} \frac{\beta_{d+ii}}{2Re\ \lambda_i}. \tag{2.21}$$

The coefficients $\beta_{d-ii}, \beta_{d+ii}$ are always positive due to the formation of the matrices of the right sides of the Lyapunov equations. It follows that the diagonal terms of the mixed gramian matrix are positive. Then the estimates are valid

$$\begin{gathered} \max_i \beta_{d-ii}, \beta_{d+ii} = \beta_{ii} \max, \\ J \leqslant \frac{\beta_{ii}\max}{2\min\limits_i |Re\ \lambda_i|} n = \frac{\beta_{ii}\max}{\left(\frac{2\min_i |Re\ \lambda_i|}{n}\right)}. \end{gathered}$$

Thus, the trace of a mixed gramian is directly proportional to the maximum value of the diagonal element of the matrix $\left[B_{-}B_{-}^{\mathrm{T}} \oplus B_{+}B_{+}^{\mathrm{T}}\right]$ and is inversely proportional to the doubled average value of the modulus of the eigenvalue of the spectrum of the matrix $[\Lambda_{-} \oplus (-\Lambda_{+})]$, which confirms the research results of [28].

*Illustrative example.* Consider the problem of controlling a dynamic object with four inputs and four outputs. The model of the control object can be described by equations of state of the form

$$\Sigma_1 \colon \begin{cases} \dfrac{dx}{dt} = Ax\left(t\right) + Bu\left(t\right), \quad x\left(0\right) = 0, \\ y\left(t\right) = Cx\left(t\right). \end{cases}$$

$$A = \begin{bmatrix} -0{,}33 & -2{,}67 & -4 & 1{,}33 \\ 21{,}17 & -23{,}33 & -30{,}2 & 1{,}5 \\ -14{,}67 & 14 & 17{,}83 & -1{,}17 \\ 2 & -1{,}33 & -1{,}83 & -2{,}17 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 5 \\ -3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let us transform the system into the upper block-diagonal Schur form. In this case, the unitary transformation matrix will be expressed as follows:

$$U = \begin{bmatrix} 0{,}125 & 0{,}943 & -0{,}169 & -0{,}258 \\ 0{,}814 & -0{,}26 & -0{,}056 & -0{,}516 \\ -0{,}564 & -0{,}178 & -0{,}225 & -0{,}775 \\ 0{,}063 & -0{,}109 & -0{,}958 & 0{,}258 \end{bmatrix}.$$

The system will take the form

$$A_{Sch} = \begin{bmatrix} 1 & 37{,}64 & 3{,}255 & 35{,}17 \\ 0 & -4 & -0{,}97 & -0{,}212 \\ 0 & 0 & -2 & 0{,}436 \\ 0 & 0 & 0 & -3 \end{bmatrix}, \quad B_{Sch} = \begin{bmatrix} -1{,}25 \\ -0{,}137 \\ 1{,}465 \\ -5{,}939 \end{bmatrix}.$$

The next transformation occurs in such a way that the matrix $A_{Sch12}$ becomes zero. We select the transformation matrix $W_{bl}$ so that the matrix $A_{bl}$ is divided into two blocks, stable and antistable subsystems.

$$W_{bl} = \begin{bmatrix} 1 & -7{,}53 & 1{,}35 & -8{,}25 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_{bl} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & -0{,}97 & -0{,}21 \\ 0 & 0 & -2 & 0{,}436 \\ 0 & 0 & 0 & -3 \end{bmatrix},$$

$$B_{bl} = \begin{bmatrix} -53{,}2 \\ -0{,}14 \\ 1{,}47 \\ -5{,}94 \end{bmatrix}.$$

Let's check the execution of Sylvester's equation (2.5). We transpose all components of the equation

$$\begin{bmatrix} -7{,}529 \\ 1{,}35 \\ -8{,}25 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 0 \\ -0{,}97 & -2 & 0 \\ -0{,}212 & 0{,}436 & -3 \end{bmatrix} \times \begin{bmatrix} 7{,}529 \\ -1{,}35 \\ 8{,}25 \end{bmatrix} + \begin{bmatrix} 37{,}64 \\ 3{,}255 \\ 35{,}167 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

For the system in this case, the mixed gramian is given by the equation (2.11)

$$P_{cm} = T_2^{-1} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} T_2.$$

$$P_{cm} = \begin{bmatrix} 5{,}32 & -5{,}32 & -7{,}98 & 2{,}66 \\ 0{,}94 & -0{,}26 & -0{,}18 & -0{,}11 \\ -0{,}17 & -0{,}056 & -0{,}23 & -0{,}96 \\ -0{,}26 & -0{,}52 & -0{,}78 & 0{,}26 \end{bmatrix} \times \begin{bmatrix} 1417 & 0 & 0 & 0 \\ 0 & 0{,}0067 & -0{,}057 & 0{,}18 \\ 0 & -0{,}057 & 0{,}52 & -1{,}72 \\ 0 & 0{,}18 & -1{,}72 & 5{,}88 \end{bmatrix} \times$$

$$\times \begin{bmatrix} 0{,}13 & 0 & 0 & -1{,}29 \\ 0{,}81 & -6{,}39 & 1{,}04 & -7{,}23 \\ -0{,}56 & 4{,}07 & -0{,}99 & 3{,}87 \\ 0{,}063 & -0{,}58 & -0{,}87 & -0{,}26 \end{bmatrix} = \begin{bmatrix} 32 & -203 & -132{,}5 & 11{,}3 \\ -203 & 1290 & -844 & 73 \\ -132{,}5 & -844 & -349 & -50{,}5 \\ 11{,}3 & 73 & -50{,}5 & 5{,}57 \end{bmatrix}.$$

Let's check the correctness of the gramian calculation. The matrix of the third transformation and the system itself will take the form

$$W_d = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -0{,}44 & 0{,}19 \\ 0 & 0 & 0{,}9 & -0{,}39 \\ 0 & 0 & 0 & 0{,}9 \end{bmatrix}, \quad A_d = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}, \quad B_d = \begin{bmatrix} -53{,}2 \\ 0{,}57 \\ -1{,}25 \\ -6{,}6 \end{bmatrix}.$$

Then the gramian for the diagonalized system becomes equal to

$$[P_- \oplus P_+] = \begin{bmatrix} 1417 & 0 & 0 & 0 \\ 0 & 0{,}04 & -0{,}12 & -0{,}54 \\ 0 & -0{,}12 & 0{,}39 & 1{,}65 \\ 0 & -0{,}54 & 1{,}65 & 7{,}26 \end{bmatrix}.$$

The general expression of the mixed gramian after the third transformation will be written as follows:

$$P_{cm} = \begin{bmatrix} 5{,}32 & -5{,}32 & -7{,}98 & 2{,}66 \\ 0{,}86 & -0{,}29 & -0{,}29 & -0{,}57 \\ -0{,}31 & -0{,}31 & -0{,}63 & -0{,}94 \\ -0{,}29 & -0{,}57 & -0{,}86 & 0{,}29 \end{bmatrix} \times \begin{bmatrix} 1417 & 0 & 0 & 0 \\ 0 & 0{,}041 & -0{,}12 & -0{,}54 \\ 0 & -0{,}12 & 0{,}39 & 1{,}65 \\ 0 & -0{,}54 & 1{,}65 & 7{,}26 \end{bmatrix} \times$$

$$\times \begin{bmatrix} 0{,}125 & 0 & 0 & -1{,}16 \\ 0{,}81 & -6{,}39 & 3{,}73 & -8{,}13 \\ -0{,}56 & 4{,}07 & -2{,}66 & 4{,}65 \\ 0{,}063 & -0{,}58 & -0{,}53 & 0 \end{bmatrix} = \begin{bmatrix} 32 & -203 & -132{,}5 & 11{,}3 \\ -203 & 1290 & -844 & 73 \\ -132{,}5 & -844 & -349 & -50{,}5 \\ 11{,}3 & 73 & -50{,}5 & 5{,}57 \end{bmatrix}.$$

The mixed gramians coincided. Let us check whether the Sylvester criterion is satisfied for the gramian of stable and antistable systems. To do this, the matrices $P_1$ and $P_2$ must be positive definite. For compactness, we write them into one matrix.

$$[P_1 \oplus P_2] = \begin{bmatrix} 1417 & 0 & 0 & 0 \\ 0 & 0{,}0067 & -0{,}057 & 0{,}18 \\ 0 & -0{,}057 & 0{,}52 & -1{,}72 \\ 0 & 0{,}18 & -1{,}72 & 5{,}88 \end{bmatrix}, \quad \lambda_{P_1} = 1417, \quad \lambda_{P_2} = \begin{bmatrix} 0{,}0001 \\ 0{,}018 \\ 6{,}39 \end{bmatrix}.$$

All eigenvalues are greater than zero. The criterion is met. Let's calculate the trace using the formula (2.21)

$$J = \sum_{i=1}^{r} \frac{\beta_{d-ii}}{-2Re\ \lambda_i} + \sum_{i=r+1}^{n} \frac{\beta_{d+ii}}{2Re\ \lambda_i} = 0{,}0067 + 0{,}52 + 5{,}88 + 1417 \approx 1423.$$

Let us compare the value of the spectrum trace with the estimate

$$J = 1423 \leqslant \frac{2834}{\frac{2*1}{4}} = 5668.$$

The reciprocal of the average value of the modules of the eigenvalues of the dynamics matrix estimates the degree of dispersion of the real parts of the eigenvalues relative to the imaginary axis. The smaller this value is, the higher its influence on the trace of the mixed controllability gramian. The formula for the spectral decomposition of the trace allows us to perform a more refined analysis of the influence of the distribution of eigenvalues on the energy metric of the degree of reachability [25, 27].

## 3. SPECTRAL EXPANSIONS OF ENERGY METRICS OF CONTROLLABILITY AND OBSERVABILITY GRAMIANS

We consider the application of the obtained results to solve some problems of state estimation and control. We obtain spectral decompositions of energy metrics.

**Theorem 3** [8]. *Let us consider a finite-dimensional linear stationary continuous system with many inputs and many outputs of the form* (2.1)*, reduced to the diagonal form* (2.6)*. Let us assume that the system has a simple spectrum, the system is completely controllable and unstable, and the eigenvalues of its dynamics matrix $A$ are not on the imaginary axis, but can be in the left and/or right half-planes*

$$\lambda_{i-} \in \mathbb{C}^{-}, \quad i = r; \quad \lambda_{i+} \in \mathbb{C}^{+}, \quad i = n - r.$$

*In addition, we assume that the condition is satisfied*

$$\lambda_i \neq -\lambda_j, \ \forall i, j : i = \overline{1, n}, \ j = \overline{1, n}.$$

*The following spectral expansions of energy functionals are valid and equivalent* [18]*:*

$$J_1 = E_{\min}(\infty) = \begin{bmatrix} x_{f-} & x_{f+} \end{bmatrix}^{\mathrm{T}} (P_{cm})^{-1} \begin{bmatrix} x_{f-} & x_{f+} \end{bmatrix} =$$
$$= \begin{bmatrix} x_{f-} & x_{f+} \end{bmatrix}^{\mathrm{T}} \left[\sum_{i=1}^{n} V_c^{*} |\sigma_i|^{-1} \mathbb{1}_{ii} U_c\right] \begin{bmatrix} x_{f-} & x_{f+} \end{bmatrix}.$$

$$J_3 \ (\textit{for SISO LTI stable systems}) = \operatorname{tr} \sum_{k=1}^{n} P_{c,k} = \sum_{k=1}^{n} \operatorname{tr} P_{c,k} =$$
$$\left( \frac{1}{\sum_{k=1}^{n} \dot{N}(s_k) N(-s_k)} - \frac{\sum_{k=1}^{n} s_k^2}{\sum_{k=1}^{n} \dot{N}(s_k) N(-s_k)} + \ldots \right.$$
$$\left. \ldots + \frac{(-1)^{n-1} \sum_{k=1}^{n} s_k^{2n}}{\sum_{k=1}^{n} \dot{N}(s_k) N(-s_k)} \right),$$

$$J_4 = \operatorname{tr} \sum_{i=1}^{n} (P_c)_i^{-1} = \sum_{i=1}^{n} \operatorname{tr} \ (P_c)_i^{-1} = \left[\sum_{i=1}^{n} \operatorname{tr} \left[V_c^{*} |\sigma_i|^{-1} \mathbb{1}_{ii} U_c\right]\right],$$

*where $N(s)$ is characteristic polynomial of the system*(2.1)*.*

**Proof of Theorem.** Let us return to stable continuous MIMO LTI systems with a simple spectrum and note that the controllability and observability gramians are symmetric complex-valued matrices. In this case, there are their singular decompositions of the form [1]

$$P_c = V_c \Lambda V_c^{*},$$

where the matrix $V_c$ is formed by the right singular vectors of the matrix $P_c$, and the matrix $\Lambda$ is a diagonal matrix of the form

$$\Lambda = \operatorname{diag}\left\{|\sigma_1|\,|\sigma_2|\dots|\sigma_{\mathrm{n}}|\right\}.$$

We define matrices S and U in the form

$$S = \operatorname{diag}\left\{\operatorname{sgn}\sigma_1 \quad \operatorname{sgn}\sigma_2 \ \dots \ \operatorname{sgn}\sigma_{\mathrm{n}} \ \right\}, \ U_c = V_c S,$$
$$\operatorname{sgn}\sigma = \begin{cases} +1, & \text{if} \ \ \sigma \geqslant 0 \\ -1, & \text{if} \ \ \sigma < 0. \end{cases}$$

Then

$$P_c = U_c \Lambda V_c^*, \tag{3.1}$$

where the matrix $U_c$ is formed by the left singular vectors of the matrix $P_c$. Since $\Lambda, U_c, V_c$ are nonsingular matrices, then

$$(P_c)^{-1} = (U_c)^{-1}\Lambda^{-1}(V_c^*)^{-1} = V_c^*\Lambda^{-1}U_c. \tag{3.2}$$

Since the matrix $\Lambda$ is diagonal, its inverse matrix can be represented as

$$\Lambda^{-1} = \left[|\sigma_1|^{-1}\mathbb{1}_{11} + |\sigma_2|^{-1}\mathbb{1}_{22} + \dots + |\sigma_n|^{-1}\mathbb{1}_{nn}\right]. \tag{3.3}$$

Substituting (3.3) into (3.1), (3.2), we obtain the following spectral expansions of the inverse controllability gramians in a simple spectrum:

$$(P_c)^{-1} = (P_c)_1^{-1} + (P_c)_2^{-1} + \dots + (P_c)_n^{-1},$$
$$(P_c)_1^{-1} = V_c^*|\sigma_1|^{-1}\mathbb{1}_{11}U_c, \ (P_c)_2^{-1} = V_c^*|\sigma_2|^{-1}\mathbb{1}_{22}U_c, \ \dots, \ (P_c)_n^{-1} = V_c^*|\sigma_n|^{-1}\mathbb{1}_{nn}U_c.$$

This implies the following spectral expansions of energy functionals [11]:

$$J_1 = E_{\min}(\infty) = \begin{bmatrix} x_{f-} & x_{f+} \end{bmatrix}^{\mathrm{T}} (P_c)^{-1} \begin{bmatrix} x_{f-} & x_{f+} \end{bmatrix} =$$
$$= \begin{bmatrix} x_{f-} & x_{f+} \end{bmatrix}^{\mathrm{T}} \left[\sum_{i=1}^{n} V_c^*|\sigma_i|^{-1}1_{ii}U_c\right] \begin{bmatrix} x_{f-} & x_{f+} \end{bmatrix},$$
$$J_2 = \operatorname{tr}\sum_{i=1}^{n}(P_c)_i^{-1} = \sum_{i=1}^{n}\operatorname{tr}\,(P_c)_i^{-1} = \left[\sum_{i=1}^{n}\operatorname{tr}\left[V_c^*|\sigma_i|^{-1}\mathbb{1}_{ii}U_c\right]\right],$$
$$J_3 \text{ (for SISO LTI systems)} = \operatorname{tr}\sum_{k=1}^{n}P_{c,k} = \sum_{k=1}^{n}\operatorname{tr}P_{c,k} =$$
$$\left(\frac{1}{\sum_{k=1}^{n}\dot{N}(s_k)\,N(-s_k)} - \frac{\sum_{k=1}^{n}s_k^2}{\sum_{k=1}^{n}\dot{N}(s_k)\,N(-s_k)} + \dots\right.$$
$$\left.\dots + (-1)^{n-1}\frac{\sum_{k=1}^{n}s_k^{2n}}{\sum_{k=1}^{n}\dot{N}(s_k)\,N(-s_k)}\right),$$
$$J_5 = \operatorname{tr}\,(P_{cm}).$$

Theorem 3 is proven.

**Theorem 4** [2]. *Let us consider a finite-dimensional linear stationary continuous system with many inputs and many outputs of the general form* (2.1)*. Let us assume that the system has a simple spectrum, is completely controllable and stable. Then the following spectral expansions of the energy functionals of the input and output energies $\hat{J}_1$ and $\hat{J}_2$ are valid and equivalent over the simple spectrum of the controllability gramian:*

$$\hat{J}_1 = \sum_{i=1}^{n} x_0^T \left[ V_c^* |\sigma_i|^{-1} 1_{ii} U_c \right] x_0, \tag{3.4}$$

*or a simple spectrum of the dynamics matrix $A$:*

$$\hat{J}_2 = \sum_{i=1}^{n} x_0^T \left[ \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \frac{\lambda_i^j (-\lambda_i)^\eta}{\dot{N}(\lambda_i) N(-\lambda_i)} A_j^{\mathrm{T}} C^{\mathrm{T}} C A_\eta \right] x_0. \tag{3.5}$$

**Proof of Theorem.** It was proven in [2] that the energy functionals of the input and output energies $\hat{J}_1$ and $\hat{J}_2$ are equal

$$\hat{J}_1 = \inf_{u,x} \int_{-\infty}^{0} \|u(t)\|^2 dt, \quad \hat{J}_2 = \int_{0}^{\infty} \|y(t), 0, \ x_0\|^2 dt.$$

Under the conditions of the theorem, they can be represented in the form of quadratic forms

$$\hat{J}_1 = E_c(x_0) = x_0^{\mathrm{T}} P_c^{\#} x_0, \tag{3.6}$$

$$\hat{J}_2 = E_o(x_0) = x_0^{\mathrm{T}} P_o x_0, \tag{3.7}$$

where $P_c^{\#}$ is the Moore-Penrose pseudo-inversion of the gramian controllability matrix, and $P_o$ is the gramian observability matrix. Under the conditions of the theorem, the gramian controllability matrix is a non-singular matrix, therefore the equality

$$P_c^{\#} = P_c^{-1}.$$

Substituting the spectral decomposition of the inverse gramian matrix into the formula (3.6), we obtain the desired spectral decomposition of the input energy functional. In [11], a spectral decomposition of the observability gramian of system was obtained in the form of Xiao Hankel matrices [11, 22, 23]

$$P_o = \sum_{i=1}^{n} \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \frac{\lambda_i^j (-\lambda_i)^\eta}{\dot{N}(\lambda_i) N(-\lambda_i)} A_j^{\mathrm{T}} C^{\mathrm{T}} C A_\eta.$$

Substituting the spectral decomposition of the gramian matrix $P_o$ (3.7), we obtain the desired spectral decomposition of the output energy functional. Theorem 4 is proven.

The functionals $\hat{J}_1$ and $\hat{J}_2$ were used in [10] to analyze the degree of stability of a linear system based on the analysis of anomalies of the square $H_2$ is the norm of the transfer function of the system, caused by the influence of the following weakly stable modes:

- modes close to the origin of coordinates,
- modes close to the imaginary axis,
- several aperiodic and oscillatory modes close to each other.

As the main tool for anomaly analysis, it was proposed to use asymptotic models of spectral expansions of the functionals $J_1$ and $J_2$ over the simple and/or pair spectrum of the system dynamics matrix. A similar approach can be extended to the analysis of anomalies in the spectral decompositions of the metrics of the traces of the gramians $J_3$ and $J_4$, as well as to the analysis of the degree of reachability of a linear system based on the anomalies of the spectral decompositions of the metrics of the mixed gramians $J_5$. Note that the spectral decompositions of the metrics depend on the eigenvalues of the dynamics matrix, which are tied to a specific node in the system graph, which makes it possible to associate the problem of optimal placement of sensors and actuators with certain nodes in the system graph.

## 4. CONCLUSION

The article generalizes the known results of gramian decomposition for unstable continuous linear systems to calculate their spectral decompositions of the simplest case of decompositions over the pair spectrum of the dynamics matrix. Most energy metrics associated with the use of gramians are based on calculating the spectrum of dynamics matrices and measures of the minimum energy required for the system to transition from the initial to the final point. The paper shows that spectral decompositions of controllability gramians and their inverse gramians make it possible to calculate the energy components corresponding to the characteristic eigenvalues of the gramian matrices, which determine the main contribution to the value of the reachability metric and the energy metric of stability. These spectral decompositions are presented in the form of formulas that allow one to analyze the influence of various nodes of the system graph on the formation of energy metrics of reachability and stability. The results obtained can find application in problems of localization and optimal placement of sensors and actuators on the graph of a complex multi-connected control system or in problems of placement of control nodes in the graph of a complex social, transport, energy or biological network [25].

## FUNDING

This work was carried out with financial support from the Russian Science Foundation, project no. 19-19-00673.

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*This paper was recommended for publication by V.M. Glumov, a member of the Editorial Board*