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Optimal Retention of the Trajectories of a Discrete-Time Stochastic System in a Tube: One Problem Statement

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Abstract—This paper considers an optimal control problem for a time-invariant linear stochastic system with discrete time, scalar unbounded control, additive noise, and a probabilistic criterion for retaining its trajectories in a given neighborhood of zero. We use dynamic programming and two-sided Bellman function estimates to derive analytical expressions for the optimal control at two time steps and a suboptimal control on any control horizon. The effectiveness of these controls is illustrated on a numerical example.

Keywords: discrete-time systems, stochastic optimal control, probabilistic criterion, dynamic programming, Bellman function, time-invariant systems, unbounded control

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1. INTRODUCTION

Optimal control problems for stochastic systems with probabilistic criteria have been investigated since the 1960s for a wide range of applied fields: aerospace [1–5], robotics [6–10], economics [11, 12], biomedicine [13], and others [14]. The probabilistic criterion is understood as the probability of realizing some restrictions on the state vector, which often characterize the control system accuracy [1]. The interest in such problem statements is motivated by practical requirements for control systems, formalized through probabilistic constraints [1, 3, 4], and the problems of determining the sets of stochastic reachability [2, 6–8], stochastic viability [15], and absorptions [16].

Problems with the maximum probability of system retention in a given tube of trajectories as an optimality criterion were studied in [2, 6–10, 13–15, 19–21]. The papers [6–10, 13–15] were devoted to the case of discrete time; optimality conditions were obtained in the dynamic programming (DP) form, and several DP-based algorithms were proposed to solve the stochastic reachability problem in which the reachability sets were constructed by approximating the isobells of the Bellman function. The authors [19] considered a linear discrete-time system and established logarithmic concavity conditions for the Bellman function; in addition, they proposed an approximation algorithm for the isobells of the Bellman function using "open loop" program control. In [9, 10], for a wide class of systems, a numerical method was presented to find the optimal control in the class of polynomials. This method is based on reducing the original problem to the so-called moment problem; for example, see [21]. In the paper [21], two-sided (bilateral) estimates were derived for the Bellman function and an algorithm was developed to find a suboptimal control based on the lower bound. Among its advantages, we mention no need to solve the Bellman equation and an explicit relation for determining the suboptimal control accuracy [21], which makes this algorithm practicable.

This paper considers an optimal control problem for a time-invariant linear stochastic system with discrete time, scalar unbounded control, additive noise, and a probabilistic criterion for retaining its trajectories in a given neighborhood of zero. Using the results of [21], we derive explicit expressions for the isobells of levels 1 and 0 of the Bellman function and its two-sided estimates. Using the lower bounds, we also obtain an analytical expression for the suboptimal control. As demonstrated below, this control is optimal for the system trajectories lying on the isobells of levels 1 and 0 of the Bellman function. An illustrative example—control of a second-order system—is provided. For the second-order system, we show that the isobells of levels 1 and 0 are partially stationary.

2. PROBLEM STATEMENT

Consider an optimal control problem for a linear stochastic discrete-time system with an additive random noise described by

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + C\xi_k, \\ x_0 = X, \end{cases} \qquad k = \overline{0, N}, \tag{1}$$

with the following notations: $x_k \in \mathbb{R}^n$ is the state vector; $u_k \in \mathbb{R}$ is a scalar control action; $\xi_k \in \mathbb{R}$ is a random disturbance; N is the control horizon; $A \in \mathbb{R}^{n\times n}$ is the system matrix; finally, $B = (0, \ldots, b)^T$, $b \in \mathbb{R}$ and $C = (0, \ldots, c)^T$, $c \in \mathbb{R}$.

The optimality criterion is a probabilistic functional of the form

$$P_\varphi (u(\cdot)) = \mathbf{P}\left(\max_{k=\overline{0,N}} \|\Lambda x_{k+1}\|_\infty \leqslant \varphi\right), \tag{2}$$

where a matrix $\Lambda \in \mathbb{R}^{n\times n}$ and a scalar $\varphi \in \mathbb{R}$ represent the parameters of the retention set $\mathcal{F} = \{x \in \mathbb{R}^n : \|\Lambda x\|_\infty \leqslant \varphi\}$, and $\|x\|_\infty = \max_{i\in\overline{1,n}} |x^i|$ denotes the l_1 vector norm, where $x = (x^1, \ldots, x^n)^T$.

We introduce the following assumptions for system (1) and the functional (2):

1. There is complete information about the state vector x_k. Due to this fact, the control can be designed in the class of functions $u_k = \gamma_k(x_k)$, where $\gamma_k : \mathbb{R}^n \to \mathbb{R}$ is some measurable function.
2. The initial state $x_0 = X$ is a random vector from $\mathbb{R}^n$ with a known distribution $\mathbf{P}_X$.
3. The control is the set of functions $u(\cdot) = (\gamma_0(\cdot), \ldots, \gamma_N(\cdot))^T \in \mathcal{U}$, and the class of admissible controls is the set $\mathcal{U} = \mathcal{U}_0 \times \ldots \times \mathcal{U}_N$, where $\mathcal{U}_k$ is the set of measurable functions $\gamma_k(\cdot)$.
4. The random variables ξ_k, $k = \overline{0, N}$, have a distribution with a finite probability density function $f_{\xi_k}(t)$, $\mathsf{supp}\,[f_{\xi_k}(t)] = [m_\xi - \epsilon; m_\xi + \epsilon]$, where $m_\xi = \mathbf{M}[\xi_k]$ and $\epsilon > 0$; moreover, $f_{\xi_k}(t)$ is symmetric with respect to m_ξ, whereas the components of the vector $(X, \xi_0, \ldots, \xi_N)^T$ are independent.
5. The matrix $\Lambda = \mathsf{diag}\,[\lambda_1, \ldots, \lambda_n] \in \mathbb{R}^{n\times n}$ is diagonal and positive definite, and $\varphi > 0$.

The optimal control problem has the form

$$P_\varphi (u(\cdot)) \to \max_{u(\cdot)\in\mathcal{U}} \tag{3}$$

with the following physical meaning: find a positional control maximizing the probability of retaining the system trajectories (1) in the rectangular parallelepiped $\mathcal{F}$ on the time interval $\{1, \ldots, N+1\}$.

In a more general case, optimal control with criteria of the form (2) was considered in [21, 23]; in particular, DP-based optimality conditions were established, and two-sided estimates for the Bellman function were obtained. The resulting estimates were used to design a suboptimal control. We present some theoretical considerations from [21, 23] for further application to problem (3).

3. DYNAMIC PROGRAMMING AND TWO-SIDED ESTIMATES OF THE BELLMAN FUNCTION

Let us introduce the following notations:

$$f_k(x_k, u_k, \xi_k) = Ax_k + Bu_k + C\xi_k, \quad \Phi_k(x) = \|\Lambda x\|_\infty.$$

Consider the Bellman function

$$\mathsf{B}_k(x) = \sup_{\gamma_k(\cdot)\in\mathcal{U}_k,\ldots,\gamma_N(\cdot)\in\mathcal{U}_N} \mathbf{P}\left(\max_{i=\overline{k,N}} \Phi_{i+1}(x_{i+1}(x_k, \gamma_k(\cdot), \ldots, \gamma_i(\cdot), \xi_k, \ldots, \xi_i)) \leqslant \varphi \Big| x_k = x\right).$$

According to [23], the dynamic programming equations for problem (3) have the form

$$\gamma_k^*(x) = \arg\max_{u\in\mathbb{R}} \mathbf{M}_{\xi_k}\left[\mathbf{I}_{\mathcal{F}}(x)\,\mathsf{B}_{k+1}(f_k(x, u, \xi_k))\right], \tag{4}$$

$$\mathsf{B}_k(x) = \sup_{u\in\mathbb{R}} \mathbf{M}_{\xi_k}\left[\mathbf{I}_{\mathcal{F}}(x)\,\mathsf{B}_{k+1}(f_k(x, u, \xi_k))\right], \quad k = \overline{0, N}, \tag{5}$$

$$\mathsf{B}_{N+1}(x) = \mathbf{I}_{\mathcal{F}}(x), \tag{6}$$

where $\mathbf{M}_{\xi_k}[\cdot]$ is the mathematical expectation under the distribution of the random variable ξ and $\mathbf{I}_{\mathcal{F}}$ denotes the indicator function of the set $\mathcal{F}$.

As is known [23], the solution $u^*(\cdot) = (\gamma_0^*(\cdot), \ldots, \gamma_N^*(\cdot))$ of problems (4)–(6) (if exists) is the optimal control in problem (3). Note that solving equations (4)–(6) cause much difficulty even for relatively simple problems. For the general problem of optimal retaining the trajectories of a discrete-time stochastic system in a given tube, the isobells of levels 1 and 0 of the Bellman function were adopted to derive two-sided estimates for the right-hand side of the DP equation, the Bellman function, and the optimal value of the probabilistic criterion.

Based on these theoretical results, the authors [21] developed an approximate search algorithm for the optimal control, yielding the exact solution under certain conditions. The algorithm was described in terms of the isobells of levels 1 and 0 of the Bellman function,

$$\mathcal{I}_k = \{x \in \mathbb{R}^n : \mathsf{B}_k(x) = 1\}, \quad \mathcal{O}_k = \{x \in \mathbb{R}^n : \mathsf{B}_k(x) = 0\},$$

and the set $\mathcal{B}_k = \mathbb{R}^n \setminus \{\mathcal{I}_k \cup \mathcal{O}_k\}$. For the sake of convenience, we introduce the notation $\overline{\mathcal{F}} = \mathbb{R}^n \setminus \mathcal{F}$. Obviously, by definition,

$$\mathcal{I}_k \cup \mathcal{B}_k \cup \mathcal{O}_k = \mathbb{R}^n, \quad \begin{cases} \mathsf{B}_k(x) = 1, & x \in \mathcal{I}_k, \\ \mathsf{B}_k(x) \in (0,1), & x \in \mathcal{B}_k, \\ \mathsf{B}_k(x) = 0, & x \in \mathcal{O}_k. \end{cases}$$

The recurrence relations derived in [21] are independent of the Bellman function. They lead to an explicit expression for the surfaces $\mathcal{I}_k$ and $\mathcal{O}_k$ and the set $\mathcal{B}_k$. As a result, control optimality conditions were established for $x_k \in \mathcal{I}_k \cup \mathcal{O}_k$, and two-sided estimates were found for the Bellman function. The estimates were used to propose an algorithm for calculating a suboptimal control in problem (3).

Consider the control $\underline{u}(\cdot) = \left(\underline{\gamma}_0(\cdot), \ldots, \underline{\gamma}_N(\cdot)\right)$, where $\underline{u}_k = \underline{\gamma}_k(x_k)$. At each step k, it maximizes the lower bound on the right-hand side of the dynamic programming equation:

$$\underline{\gamma}_k(x) = \arg\max_{u\in\mathbb{R}} \mathbf{P}_{\xi_k}(f_k(x, u, \xi_k) \in \mathcal{I}_{k+1}),$$

$$\mathcal{I}_k = \mathcal{F}_k \cap \{x \in \mathbb{R}^n : \exists u \in \mathbb{R} : \mathbf{P}_{\xi_k}(f_k(x, u, \xi_k) \in \mathcal{I}_{k+1}) = 1\}, \quad k = \overline{0, N}, \tag{7}$$

$$\mathcal{I}_{N+1} = \mathcal{F}.$$

According to [21], this control is optimal for $x_k \in \mathcal{I}_k \cup \mathcal{O}_k$, $k = \overline{0,N}$, and for any $x_k \in \mathbb{R}^n$ at step $k = N$. Also, the accuracy of the suboptimal control $\underline{u}(\cdot)$ was explicitly estimated therein. We employ the theoretical results of this section to solve problem (3).

4. PROBLEM SOLUTION

4.1. The Analytical Solution at Two Time Steps

Using dynamic programming, we solve problems (4)–(6) for $k = N$ and let $x = x_N$ and $u = u_N$. It follows from (5) and (6) that

$$\begin{aligned} \mathsf{B}_N(x) &= \max_{u \in \mathbb{R}} \mathbf{M}\left[\mathbf{I}_{\mathcal{F}}(x)\,\mathbf{I}_{\mathcal{F}}(f_N(x,u,\xi_N))\right] \\ &= \max_{u \in \mathbb{R}} \left\{\mathbf{I}_{\mathcal{F}}(x)\,\mathbf{P}\left(\|\Lambda(Ax + Bu + C\xi_N)\|_\infty \leqslant \varphi\right)\right\}. \end{aligned}$$

Recall that the matrix Λ is diagonal, $\Lambda = \mathsf{diag}[\lambda_1, \ldots, \lambda_n]$, and for all $i = \overline{1, n-1}$ we have $e_i^{\mathsf{T}}\Lambda b = e_i^{\mathsf{T}}\Lambda c = 0$, where e_i is the unit vector of the coordinate axis. Therefore,

$$\begin{aligned} \mathsf{B}_N(x) &= \max_{u \in \mathbb{R}} \left\{\mathbf{I}_{\mathcal{F}}(x)\,\mathbf{I}_{(-\infty,\varphi]}\left(\max_{i=\overline{1,n-1}} \left|e_i^{\mathsf{T}}\Lambda Ax\right|\right)\mathbf{P}\left(\left|e_n^{\mathsf{T}}\Lambda(Ax + Bu + C\xi_N)\right| \leqslant \varphi\right)\right\} \\ &= \begin{cases} \max\limits_{u \in \mathbb{R}} \mathbf{P}\left(\left|e_n^{\mathsf{T}}\Lambda(Ax + Bu + C\xi_N)\right| \leqslant \varphi\right), & x \in \mathcal{F} \cap \mathcal{F}', \\ 0, & x \notin \mathcal{F} \cap \mathcal{F}', \end{cases} \end{aligned}$$

where

$$\mathcal{F}' = \left\{x \in \mathbb{R}^n : \max_{i=\overline{1,n-1}} \left|e_i^{\mathsf{T}}\Lambda Ax\right| \leqslant \varphi\right\}.$$

Consider the stochastic programming problem in the first branch of the latter expression. The objective function can be transformed as follows:

$$h = \mathsf{sign}\left(e_n^T \Lambda C\right)\left(\varphi - e_n^{\mathsf{T}}\Lambda Ax - e_n^{\mathsf{T}}\Lambda Bu - e_n^{\mathsf{T}}\Lambda C m_\xi\right),$$

$$\mathbf{P}\left(\left|e_n^{\mathsf{T}}\Lambda(Ax + Bu + C\xi_N)\right| \leqslant \varphi\right) = \int\limits_{-h}^{h} f_{\mathring{\xi}_N}(t)\,dt \to \max_{u \in \mathbb{R}},$$

where $f_{\mathring{\xi}_N}(t)$ denotes the probability density function of the centered random variable $\mathring{\xi}_N = \xi_N - m_\xi$. Since $f_{\mathring{\xi}_N}(t)$ is an even function, this stochastic programming problem has [24, p. 244] a deterministic equivalent with the analytical solution $u^* = -\left(e_n^{\mathsf{T}}\Lambda B\right)^{-1}\left(e_n^{\mathsf{T}}\Lambda Ax + e_n^{\mathsf{T}}\Lambda C m_\xi\right)$. Substituting it into the objective function yields the Bellman function at step $k = N$:

$$\mathsf{B}_N(x) = \mathbf{I}_{\mathcal{F}\cap\mathcal{F}'}(x)\,\mathbf{P}\left(\left|e_n^{\mathsf{T}}\Lambda C\mathring{\xi}_N\right| \leqslant \varphi\right). \tag{8}$$

Thus, the solution of problem (3) at step $k = N$ is given by

$$\gamma_N^*(x) = -\left(e_n^{\mathsf{T}}\Lambda B\right)^{-1}\left(e_n^{\mathsf{T}}\Lambda Ax + e_n^{\mathsf{T}}\Lambda C m_\xi\right).$$

Note that the function (8) equals 1 only in the case $\left|e_n^{\mathsf{T}}\Lambda C\varepsilon\right| \leqslant \varphi$. Hence, the isobell of level 1 of the Bellman function at step $k = N$ is non-empty only under this inequality:

$$\mathcal{I}_N = \begin{cases} \mathcal{F} \cap \mathcal{F}' & \text{if } \left|e_n^{\mathsf{T}}\Lambda C\varepsilon\right| \leqslant \varphi \\ \varnothing & \text{otherwise.} \end{cases}$$

Considering $\mathbf{P}\left(\left|e_n^{\mathsf{T}}\Lambda C\mathring{\xi}_N\right| \leqslant \varphi\right) \neq 0$, we express the isobell of level 0 by analogy:

$$\mathcal{O}_N = \overline{\mathcal{F}} \cup \overline{\mathcal{F}}'.$$

Now we find the optimal control and the Bellman function for $k = N-1$. Using (4) and (8), we write the chain of equalities

$$\begin{aligned}
\mathsf{B}_{N-1}(x) &= \max_{u\in\mathbb{R}} \mathbf{M}\Big[\mathbf{I}_{\mathcal{F}}(x)\,\mathsf{B}_N\left(f_{N-1}(x,u,\xi_{N-1})\right)\Big] \\
&= \max_{u\in\mathbb{R}} \mathbf{M}\left[\mathbf{I}_{\mathcal{F}}(x)\,\mathbf{I}_{\mathcal{F}\cap\mathcal{F}'}\left(f_{N-1}(x,u,\xi_{N-1})\right)\mathbf{P}\left(\left|e_n^{\mathsf{T}}\Lambda C\mathring{\xi}_N\right| \leqslant \varphi\right)\right] \\
&= \max_{u\in\mathbb{R}} \left\{\mathbf{I}_{\mathcal{F}\cap\mathcal{F}'}(x)\,\mathbf{P}\left(\left|e_n^{\mathsf{T}}\Lambda C\mathring{\xi}_N\right| \leqslant \varphi\right)\mathbf{P}\Bigg(\max\left\{\left|e_n^{\mathsf{T}}\Lambda\left(Ax+Bu+C\xi_{N-1}\right)\right|,\right.\right. \\
&\qquad \left.\left.\max_{i=\overline{1,n-1}}\left|e_i^{\mathsf{T}}\Lambda A\left(Ax+Bu+C\xi_{N-1}\right)\right|\right\} \leqslant \varphi\Bigg)\right\}.
\end{aligned}$$

Let us introduce the matrix

$$\Lambda_N = \left(e_1^{\mathsf{T}}\Lambda A,\ e_2^{\mathsf{T}}\Lambda A, \ldots,\ e_{n-1}^{\mathsf{T}}\Lambda A,\ e_n^{\mathsf{T}}\Lambda\right)^{\mathsf{T}}, \quad \Lambda_N \in \mathbb{R}^{n\times n}.$$

Then the Bellman function for $k = N-1$ can be represented as

$$\mathsf{B}_{N-1}(x) = \mathbf{I}_{\mathcal{F}\cap\mathcal{F}'}(x)\,\mathbf{P}\left(\left|e_n^{\mathsf{T}}\Lambda C\mathring{\xi}_N\right| \leqslant \varphi\right)\max_{u\in\mathbb{R}}\mathbf{P}\left(\left\|\Lambda_N\left(Ax+Bu+C\xi_{N-1}\right)\right\|_\infty \leqslant \varphi\right).$$

While proving Proposition 1 below, we show the following: the stochastic programming problem on the right-hand side of this expression had the solution

$$\gamma_{N-1}^*(x) = \arg\max_{u\in\mathbb{R}}\mathbf{P}\left(\left\|\Lambda_N\left(Ax+Bu+C\xi_{N-1}\right)\right\|_\infty \leqslant \varphi\right) = c_{N-1}(x) - \frac{c}{b}m_\xi,$$

and the Bellman function is given by

$$\mathsf{B}_{N-1}(x) = \mathbf{I}_{\mathcal{F}\cap\mathcal{F}'}(x)\,\mathbf{P}\left(\left|e_n^{\mathsf{T}}\Lambda C\mathring{\xi}_N\right| \leqslant \varphi\right)\mathbf{P}\left(\left|\frac{c}{b}\xi_{N-1}\right| \leqslant r_{N-1}(x)\right), \tag{9}$$

where the functions $c_{N-1}: \mathbb{R}^n \to \mathbb{R}$ and $r_{N-1}: \mathbb{R}^n \to \mathbb{R}$ have the form

$$c_{N-1}(x) = \frac{1}{2}\left(\overline{\varphi}_{N-1}(x) + \underline{\varphi}_{N-1}(x)\right), \quad r_{N-1}(x) = \frac{1}{2}\left(\overline{\varphi}_{N-1}(x) - \underline{\varphi}_{N-1}(x)\right),$$

and the functions $\overline{\varphi}_{N-1}: \mathbb{R}^n \to \mathbb{R}$ and $\underline{\varphi}_{N-1}: \mathbb{R}^n \to \mathbb{R}$ have the form

$$\begin{aligned}
\overline{\varphi}_{N-1}(x) &= \min_{i=\overline{1,n}}\frac{\mathsf{sign}\left(e_i^{\mathsf{T}}\Lambda_N B\right)\varphi - e_i^{\mathsf{T}}\Lambda_N Ax}{e_i^{\mathsf{T}}\Lambda_N B}, \\
\underline{\varphi}_{N-1}(x) &= \max_{i=\overline{1,n}}\frac{-\mathsf{sign}\left(e_i^{\mathsf{T}}\Lambda_N B\right)\varphi - e_i^{\mathsf{T}}\Lambda_N Ax}{e_i^{\mathsf{T}}\Lambda_N B}.
\end{aligned}$$

Let us find the isobell of level 1 of the Bellman function for $k = N-1$:

$$\mathcal{I}_{N-1} = \begin{cases}\mathcal{F}\cap\mathcal{F}'\cap\Delta\mathcal{I}_{N-1} & \text{if } \left|e_n^{\mathsf{T}}\Lambda C\varepsilon\right| \leqslant \varphi \\ \varnothing & \text{otherwise,}\end{cases}$$

where

$$\Delta\mathcal{I}_{N-1} = \left\{x \in \mathbb{R}^n : \ \left|\left(e_n^{\mathsf{T}}\Lambda_N B\right)^{-1} e_n^{\mathsf{T}}\Lambda_N C\varepsilon\right| \leqslant r_{N-1}(x)\right\}$$

$$= \left\{x \in \mathbb{R}^n : 2\left|\left(e_n^{\mathsf{T}}\Lambda_N B\right)^{-1} e_n^{\mathsf{T}}\Lambda_N C\varepsilon\right| + \max_{i=\overline{1,n}} \frac{-\mathsf{sign}\,(e_i^{\mathsf{T}}\Lambda_N B)\,\varphi - e_i^{\mathsf{T}}\Lambda_N Ax}{e_i^{\mathsf{T}}\Lambda_N B} \right.$$

$$\left. \leqslant \min_{i=\overline{1,n}} \frac{\mathsf{sign}\,(e_i^{\mathsf{T}}\Lambda_N B)\,\varphi - e_i^{\mathsf{T}}\Lambda_N Ax}{e_i^{\mathsf{T}}\Lambda_N B}\right\}$$

$$= \left\{x \in \mathbb{R}^n : 2\left|\frac{c}{b}\varepsilon\right| + \frac{-\mathsf{sign}(e_j^{\mathsf{T}}\Lambda_N B)\varphi - e_j^{\mathsf{T}}\Lambda_N Ax}{e_j^{\mathsf{T}}\Lambda_N B} \right.$$

$$\left. \leqslant \frac{\mathsf{sign}\,(e_i^{\mathsf{T}}\Lambda_N B)\,\varphi - e_i^{\mathsf{T}}\Lambda_N Ax}{e_i^{\mathsf{T}}\Lambda_N B}, \quad \forall i, j \in \{1, \ldots, n\}\right\}$$

$$= \left\{x \in \mathbb{R}^n : \ \|\Lambda_{N-1}x\|_\infty \leqslant \varphi\right\},$$

and the matrix $\Lambda_{N-1} \in \mathbb{R}^{n_{N-1}\times n}$, $n_{N-1} = n(n-1)/2$, has the following structure:

$$\Lambda_{N-1} = \left(e_1^{\mathsf{T}}\Lambda_{N-1},\ e_2^{\mathsf{T}}\Lambda_{N-1}, \ldots, e_{n_{N-1}}^{\mathsf{T}}\Lambda_{N-1}\right)^{\mathsf{T}},$$

the pth row $e_p^{\mathsf{T}}\Lambda_{N-1}$, $p = \overline{1, n_{N-1}}$, is given by

$$e_p^{\mathsf{T}}\Lambda_{N-1} = \frac{\varphi}{\widetilde{\varepsilon} + \left(\mathsf{sign}(b_{N-1}^i) + \mathsf{sign}(b_{N-1}^j)\right)\varphi}\left(\frac{\left(a_{N-1}^i\right)^{\mathsf{T}}}{b_{N-1}^i} - \frac{\left(a_{N-1}^j\right)^{\mathsf{T}}}{b_{N-1}^j}\right),$$

where

$$\begin{cases} \widetilde{\varepsilon} = 2\left|\dfrac{c}{b}\varepsilon\right|, \\ b_{N-1}^i = e_i^{\mathsf{T}}\Lambda_N B, \\ \left(a_{N-1}^i\right)^{\mathsf{T}} = e_i^{\mathsf{T}}\Lambda_N A, \\ a_{N-1}^i \in \mathbb{R}^n, \quad b_{N-1}^i \in \mathbb{R}, \quad \widetilde{\varepsilon} \in [0, +\infty), \end{cases}$$

and the numbers i, j, and p are related by

$$\begin{cases} p = (n-1)i + j - 1 \\ i, j \in \{1, \ldots, n\}, \\ i < j. \end{cases}$$

The inequality $r_{N-1}(x) > 0$ holds for all $x \in \mathbb{R}^n$. Due to (9), the isobell of level 0 of the Bellman function at step $k = N-1$ has the form

$$\mathcal{O}_{N-1} = \overline{\mathcal{F}} \cup \overline{\mathcal{F}}'.$$

It is difficult to construct the Bellman function and the optimal control at steps $k = \overline{0, N-2}$. Utilizing the results of Section 3, we find a suboptimal control $\left(\underline{\gamma}_0(\cdot), \ldots, \underline{\gamma}_{N-2}(\cdot)\right)$ and a lower bound on the Bellman function.

4.2. Suboptimal Control at Steps $k = \overline{0, N-2}$

Using Theorem 1, we obtain the isobell of level 1 of the Bellman function, its lower bound, and a suboptimal control at steps $k= \overline{0, N-2}$.

Proposition 1. *Let*

$$\max_{k=\overline{0,N-2}} \max_{i\in\overline{1,n_{k+1}+n}} \left|b_k^i\right| \leqslant 2\varphi\left(\widetilde{\varepsilon}\right)^{-1}, \tag{10}$$

where the parameters n_{k+1}, b_k^i, and $\widetilde{\varepsilon}$ are given below. Then:

1. *For $k = \overline{0, N-2}$, the isobells of level 1 of the Bellman function have the form*

$$\mathcal{I}_k = \mathcal{F} \cap \mathcal{F}' \cap \Delta\mathcal{I}_k, \tag{11}$$

where

$$\Delta\mathcal{I}_k = \left\{x \in \mathbb{R}^n : \ \|\Lambda_k x\|_\infty \leqslant \varphi\right\},$$

the matrix $\Lambda_k \in \mathbb{R}^{n_k \times n}$ with

$$\begin{cases} n_k = \dfrac{1}{2}\left(n_{k+1}+n\right)\left(n_{k+1}+n-1\right), \quad k = \overline{0, N-2}, \\ n_{N-1} = \dfrac{1}{2}n(n-1), \end{cases}$$

has the form $\Lambda_k = \left(e_1^{\mathsf{T}}\Lambda_k,\ e_2^{\mathsf{T}}\Lambda_k, \ldots, e_{n_k}^{\mathsf{T}}\Lambda_k\right)^{\mathsf{T}}$, where each pth row, $p = \overline{1, n_k}$, is given by

$$e_p^{\mathsf{T}}\Lambda_k = \frac{\varphi}{\widetilde{\varepsilon} + \left(\mathsf{sign}(b_k^i) + \mathsf{sign}(b_k^j)\right)\varphi}\left(\frac{\left(a_k^i\right)^{\mathsf{T}}}{b_k^i} - \frac{\left(a_k^j\right)^{\mathsf{T}}}{b_k^j}\right), \tag{12}$$

$$\begin{cases} \widetilde{\varepsilon} = 2\left|\dfrac{c}{b}\varepsilon\right|, \\ b_k^i = e_i^{\mathsf{T}}\widetilde{\Lambda}_{k+1}B, \\ \left(a_k^i\right)^{\mathsf{T}} = e_i^{\mathsf{T}}\widetilde{\Lambda}_{k+1}A, \\ \widetilde{\Lambda}_{k+1} = \left(\Lambda_{k+1},\ e_1^{\mathsf{T}}\Lambda A, \ldots, e_{n-1}^{\mathsf{T}}\Lambda A,\ e_n^{\mathsf{T}}\Lambda\right)^{\mathsf{T}}, \\ a_k^i \in \mathbb{R}^n, \quad b_k^i \in \mathbb{R}, \quad \widetilde{\varepsilon} \in [0, +\infty), \quad \widetilde{\Lambda}_k \in \mathbb{R}^{(n_k+n)\times n}, \end{cases} \tag{13}$$

and the numbers i, j, and p are related by

$$\begin{cases} p = (i-1)\left(n_{k+1}+n\right) + j - 1, \\ i, j \in \left\{1, \ldots, n_{k+1}+n\right\}, \\ i < j\,. \end{cases} \tag{14}$$

2. *For $k = \overline{0, N-2}$, the stochastic programming problem (7) has the solutions*

$$\underline{\gamma}_k\left(x\right) = c_k\left(x\right) - \frac{c}{b}m_\xi, \tag{15}$$

the lower bound on the Bellman function is given by

$$\underline{\mathsf{B}_k}\left(x\right) = \mathbf{I}_{\mathcal{F}\cap\mathcal{F}'}\left(x\right)\mathbf{P}\left(\left|\frac{c}{b}\mathring{\xi}_k\right| \leqslant r_k\left(x\right)\right),$$

where the functions $c_k : \mathbb{R}^n \to \mathbb{R}$, $r_k : \mathbb{R}^n \to \mathbb{R}$ *have the form*

$$c_k(x) = \frac{1}{2}\left(\overline{\varphi}_k(x) + \underline{\varphi}_k(x)\right), \quad r_k(x) = \frac{1}{2}\left(\overline{\varphi}_k(x) - \underline{\varphi}_k(x)\right)$$

and the functions $\overline{\varphi}_k : \mathbb{R}^n \to \mathbb{R}$ *and* $\underline{\varphi}_k : \mathbb{R}^n \to \mathbb{R}$ *are given by*

$$\overline{\varphi}_k(x) = \min_{i=\overline{1,n_{k+1}+n}} \frac{\mathsf{sign}\,(b_k^i)\,\varphi - (a_k^i)^{\mathrm{T}} x}{b_k^i},$$

$$\underline{\varphi}_k(x) = \max_{i=\overline{1,n_{k+1}+n}} \frac{-\mathsf{sign}\,(b_k^i)\,\varphi - (a_k^i)^{\mathrm{T}} x}{b_k^i}.$$

3. *For* $k = \overline{0, N-2}$, *the isobells of level* 0 *of the Bellman function have the form*

$$\mathcal{O}_k = \overline{\mathcal{F}} \cup \overline{\mathcal{F}}'. \tag{16}$$

4. *For* $k = \overline{0, N-2}$, *the upper bound on the Bellman function is*

$$\overline{\mathsf{B}}_k(x) = \mathbf{I}_{\mathcal{F}\cap\mathcal{F}'}(x)\,\mathbf{P}\left(\left|\frac{c}{b}\mathring{\xi}_k\right| \leqslant r_{N-1}(x)\right). \tag{17}$$

The proof of Proposition 1 is postponed to the Appendix.

Remark. If $b_k^i = 0$ *or* $b_k^j = 0$ *(see item 1 of Proposition 1), then the row vector* (12) *is eliminated from* Λ_k.

According to item 1 of Proposition 1, condition (25) is necessary for the non-empty isobells of level 1 of the Bellman function at steps $k = \overline{0, N-2}$. It follows from item 1 that the number n_k of rows in the matrix Λ_k quadratically grows with each backward time step $k = \overline{0, N-2}$. The suboptimal control (31) is a piecewise linear function of the state with maximum n_k linear segments. Note that for all $k = \overline{0, N-2}$, the upper bound on the Bellman function coincides within the random variable ξ_k with the Bellman function at step $k = N-1$ (23), and the isobells of level 0 are stationary.

5. EXAMPLE 1. CONTROL OF A SECOND-ORDER SYSTEM

5.1. System Description

Consider the discrete-time system (1) with $n = 2$, which describes the dynamics of a material point:

$$\begin{cases} r_{k+1} = r_k + v_k h, \\ v_{k+1} = v_k + u_k h + \xi_k, \\ r_0 = X, v_0 = V, \end{cases} \tag{18}$$

where r_k and v_k are the coordinate and velocity of the point at step k and $\xi_k \sim U[m_\xi - \epsilon, m_\xi + \epsilon]$ are random disturbances at step k, $k = \overline{0, N}$. They satisfy the assumptions introduced in Section 2.

The role of the control action u_k is played by acceleration.

According to the notations of Section 2, we have:

$$x_k = \begin{pmatrix} r_k \\ v_k \end{pmatrix}, \; A = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ h \end{pmatrix}, \; C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let the system parameters be $N = 6$, $h = 1$, $\varphi = 1.2$, $\epsilon = 0.7$, $m_\xi = 0$, and $\Lambda = \mathsf{diag}\,[1, 1]$.

5.2. The Isobells of Level 1 of the Bellman Function

Prior to passing to the numerical experiment, we emphasize that the stationarity of the isobells of level 1 of the Bellman function in the two-dimensional system can be established for steps $k = \overline{0, N-2}$. First, consider step $k = N$. Using the results of subsection 4.1, we find $\Lambda_N = \left(e_1^{\mathsf{T}}\Lambda A,\ e_2^{\mathsf{T}}\Lambda\right)^{\mathsf{T}} \in \mathbb{R}^{2\times 2}$.

Next, consider step $k = N-1$. Based on the results of subsection 4.1, we obtain

$$n_{N-1} = n(n-1)/2 = 1 \text{ and } \Lambda_{N-1} = \left(e_1^{\mathsf{T}}\Lambda_{N-1}\right)^{\mathsf{T}} \in \mathbb{R}^{1\times n},$$

where

$$e_1^{\mathsf{T}}\Lambda_{N-1} = \frac{\varphi}{\widetilde{\varepsilon} + \left(\mathsf{sign}(b_{N-1}^1) + \mathsf{sign}(b_{N-1}^2)\right)\varphi}\left(\frac{\left(a_{N-1}^1\right)^{\mathsf{T}}}{b_{N-1}^1} - \frac{\left(a_{N-1}^2\right)^{\mathsf{T}}}{b_{N-1}^2}\right),$$

$$\begin{cases} \widetilde{\varepsilon} = 2\left|\left(e_2^{\mathsf{T}}B\right)^{-1} e_2^{\mathsf{T}}C\varepsilon\right|, \\ b_{N-1}^i = e_i^{\mathsf{T}}\Lambda_N B, \\ \left(a_{N-1}^i\right)^{\mathsf{T}} = e_i^{\mathsf{T}}\Lambda_N A, \\ a_{N-1}^i \in \mathbb{R}^n, \quad b_{N-1}^i \in \mathbb{R}, \quad \widetilde{\varepsilon} \in [0, +\infty). \end{cases}$$

Due to the structure of the matrix Λ_N and the relations $e_2^{\mathsf{T}}\Lambda e_1^{\mathsf{T}} = 0$ and $e_1^{\mathsf{T}}\Lambda e_2^{\mathsf{T}} = 0$, the expressions for the parameters a_{N-1}^i and b_{N-1}^i can be simplified as follows:

$$\left(a_{N-1}^1\right)^{\mathsf{T}} = e_1^{\mathsf{T}}\begin{pmatrix} e_1^{\mathsf{T}}\Lambda A \\ e_2^{\mathsf{T}}\Lambda \end{pmatrix} A = e_1^{\mathsf{T}}\Lambda A A, \quad b_{N-1}^1 = e_1^{\mathsf{T}}\begin{pmatrix} e_1^{\mathsf{T}}\Lambda A \\ e_2^{\mathsf{T}}\Lambda \end{pmatrix} B = e_1^{\mathsf{T}}\Lambda A B,$$

$$\left(a_{N-1}^2\right)^{\mathsf{T}} = e_2^{\mathsf{T}}\begin{pmatrix} e_1^{\mathsf{T}}\Lambda A \\ e_2^{\mathsf{T}}\Lambda \end{pmatrix} A = e_2^{\mathsf{T}}\Lambda A, \quad b_{N-1}^2 = e_2^{\mathsf{T}}\begin{pmatrix} e_1^{\mathsf{T}}\Lambda A \\ e_2^{\mathsf{T}}\Lambda \end{pmatrix} B = e_2^{\mathsf{T}}\Lambda B,$$

$$\frac{\left(a_{N-1}^1\right)^{\mathsf{T}}}{b_{N-1}^1} = \frac{e_1^{\mathsf{T}}\Lambda A A}{e_1^{\mathsf{T}}\Lambda A B} = \frac{e_1^{\mathsf{T}} A A}{e_1^{\mathsf{T}} A B} = \frac{e_1^{\mathsf{T}} A A}{e_1^{\mathsf{T}} A e_2^{\mathsf{T}} B}, \quad \frac{\left(a_{N-1}^2\right)^{\mathsf{T}}}{b_{N-1}^2} = \frac{e_2^{\mathsf{T}}\Lambda A}{e_2^{\mathsf{T}}\Lambda B} = \frac{e_2^{\mathsf{T}} A}{e_2^{\mathsf{T}} B},$$

$$\frac{\left(a_{N-1}^1\right)^{\mathsf{T}}}{b_{N-1}^1} - \frac{\left(a_{N-1}^2\right)^{\mathsf{T}}}{b_{N-1}^2} = \frac{e_1^{\mathsf{T}} A A}{e_1^{\mathsf{T}} A e_2^{\mathsf{T}} B} - \frac{e_2^{\mathsf{T}} A}{e_2^{\mathsf{T}} B} = \left(\frac{e_1^{\mathsf{T}} A}{e_1^{\mathsf{T}} A e_2^{\mathsf{T}}} - e_2^{\mathsf{T}}\right)\frac{A}{e_2^{\mathsf{T}} B} = \left(\frac{e_1^{\mathsf{T}} A e_1^{\mathsf{T}}}{e_1^{\mathsf{T}} A e_2^{\mathsf{T}}} \cdot \frac{1}{e_2^{\mathsf{T}} B}\right) e_1^{\mathsf{T}} A.$$

Finally, we arrive at the formula

$$\Lambda_{N-1} = \left(\lambda_{N-1} e_1^{\mathsf{T}} A\right)^{\mathsf{T}},$$

$$\lambda_{N-1} = \frac{\varphi}{\widetilde{\varepsilon} + \left(\mathsf{sign}(b_{N-1}^1) + \mathsf{sign}(b_{N-1}^2)\right)\varphi}\left(\frac{e_1^{\mathsf{T}} A e_1^{\mathsf{T}}}{e_1^{\mathsf{T}} A e_2^{\mathsf{T}}}\frac{1}{e_2^{\mathsf{T}} B}\right),$$

where $\lambda_{N-1} \in \mathbb{R}$ and $\lambda_{N-1} \neq 0$. Thus, by Proposition 1, the set $\Delta\mathcal{I}_{N-1}$ is

$$\Delta\mathcal{I}_{N-1} = \left\{x \in \mathbb{R}^2 :\ \|\Lambda_{N-1} x\|_\infty \leqslant \varphi\right\} = \left\{x \in \mathbb{R}^2 :\ \left|e_1^{\mathsf{T}}\Lambda_{N-1} x\right| \leqslant \varphi\right\}.$$

Now consider step $k = N - 2$. Utilizing Proposition 1, we find the values of the control parameters:

$$n_{N-2} = \frac{1}{2}\left(n_{N-1} + 2\right)\left(n_{N-1} + 1\right) = 3,$$

$$e_p^{\mathsf{T}}\Lambda_{N-2} = \frac{\varphi}{\widetilde{\varepsilon} + \left(\mathsf{sign}(b_{N-1}^i) + \mathsf{sign}(b_{N-1}^j)\right)\varphi}\left(\frac{\left(a_{N-2}^i\right)^{\mathsf{T}}}{b_{N-2}^i} - \frac{\left(a_{N-2}^j\right)^{\mathsf{T}}}{b_{N-2}^j}\right),$$

$$\begin{cases} \widetilde{\varepsilon} = 2\left|\left(e_2^{\mathsf{T}}B\right)^{-1} e_2^{\mathsf{T}}C\varepsilon\right|, \\ b_{N-2}^i = e_i^{\mathsf{T}}\widetilde{\Lambda}_{N-1}B, \\ \left(a_{N-2}^i\right)^{\mathsf{T}} = e_i^{\mathsf{T}}\widetilde{\Lambda}_{N-1}A, \\ \widetilde{\Lambda}_{N-1} = \left(\Lambda_{N-1},\ e_1^{\mathsf{T}}\Lambda A,\ e_2^{\mathsf{T}}\Lambda\right)^{\mathsf{T}}, \\ a_{N-2}^i \in \mathbb{R}^2, \quad b_{N-2}^i \in \mathbb{R}, \quad \widetilde{\varepsilon} \in [0, +\infty), \quad \widetilde{\Lambda}_{N-1} \in \mathbb{R}^{3\times 2}, \end{cases}$$

where the numbers i, j, and p are related by

$$\begin{cases} p = i \cdot 3 + j - 4, \\ i, j \in \{1, 2, 3\}, \\ i < j\,. \end{cases}$$

Obviously, we have the equalities

$$a_{N-2}^2 = a_{N-1}^1,\ a_{N-2}^3 = a_{N-1}^2,\ b_{N-2}^2 = b_{N-1}^1,\ b_{N-2}^3 = b_{N-1}^2, \tag{19}$$

$$\frac{\left(a_{N-2}^1\right)^{\mathsf{T}}}{b_{N-2}^1} = \frac{\lambda_{N-1}e_1^{\mathsf{T}}AA}{\lambda_{N-1}e_1^{\mathsf{T}}AB} = \frac{e_1^{\mathsf{T}}AA}{e_1^{\mathsf{T}}AB} = \frac{\left(a_{N-2}^2\right)^{\mathsf{T}}}{b_{N-2}^2} = \frac{\left(a_{N-1}^1\right)^{\mathsf{T}}}{b_{N-1}^1}. \tag{20}$$

Due to (19), row $p = 1$ ($i = 1$, $j = 2$) of the matrix Λ_{N-2} turns out a zero vector because

$$\frac{\left(a_{N-2}^1\right)^{\mathsf{T}}}{b_{N-2}^1} - \frac{\left(a_{N-2}^2\right)^{\mathsf{T}}}{b_{N-2}^2} = (0,\ 0)\,.$$

By analogy, rows $p = 2$ ($i = 1$, $j = 3$) and $p = 3$ ($i = 2$, $j = 3$) of the matrix Λ_{N-2} coincide since

$$\frac{\left(a_{N-2}^1\right)^{\mathsf{T}}}{b_{N-2}^1} - \frac{\left(a_{N-2}^3\right)^{\mathsf{T}}}{b_{N-2}^3} = \frac{\left(a_{N-2}^2\right)^{\mathsf{T}}}{b_{N-2}^2} - \frac{\left(a_{N-2}^3\right)^{\mathsf{T}}}{b_{N-2}^3};$$

moreover, they equal $e_1^{\mathsf{T}}\Lambda_{N-1}$ by (19) and (21). Therefore, the matrix Λ_{N-2} takes the form

$$\Lambda_{N-2} = \left((0,\ 0)\,,\ e_1^{\mathsf{T}}\Lambda_{N-1},\ e_1^{\mathsf{T}}\Lambda_{N-1}\right)^{\mathsf{T}},$$

and the set $\Delta\mathcal{I}_{N-2}$ is

$$\Delta\mathcal{I}_{N-2} = \left\{x \in \mathbb{R}^2 :\ \|\Lambda_{N-2}x\|_\infty \leqslant \varphi\right\} = \left\{x \in \mathbb{R}^2 :\ \left|e_1^{\mathsf{T}}\Lambda_{N-1}x\right| \leqslant \varphi\right\} = \Delta\mathcal{I}_{N-1}.$$

By induction, we establish

$$\Delta\mathcal{I}_k = \Delta\mathcal{I}_{N-1},$$

for all $k = \overline{0, N-2}$.

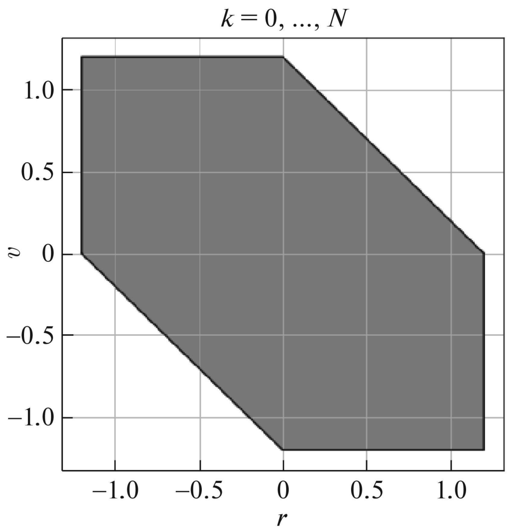

Fig. 1. The isobell of level 1 of the Bellman function for $k = 0, \ldots, N$.

Hence, for the steps under consideration, the isobells of level 1 of the Bellman function coincide. According to the above result and item 3 of Proposition 1, the isobells of levels 1 and 0 of the Bellman function are stationary for steps $k = \overline{0r, N-2}$. As an illustrative example, Fig. 1 presents the isobell of level 1 of the Bellman function for system (18).

We proceed to the numerical experiment for system (18). The goal is to analyze the solution based on the results of subsections 4.1 and 4.2.

5.3. A Numerical Experiment

The initial state of the system is not fixed but generated with a uniform distribution. For the position r_0 and velocity v_0 of the system, the distribution is $U[-1.1,\ 1.1]$. Note that the supports of the random variables r_0 and v_0 are chosen to generate points inside and outside the retention set. Let us simulate $M = 100$ trajectories $r_k(i), v_k(i)$, and $u_k(i), k = \overline{0, N}, i = \overline{1, M}$, of the stochastic system.

Figure 2 shows each system trajectory separately. According to the graphs, the system is retained for cases where the initial state is inside and outside the retention set; starting from step 3, the system trajectories pass close to the origin. Obviously, some trajectories $r_k(i)$ and $v_k(i)$ lie outside the retention set until step 2, but all trajectories are inside it afterwards.

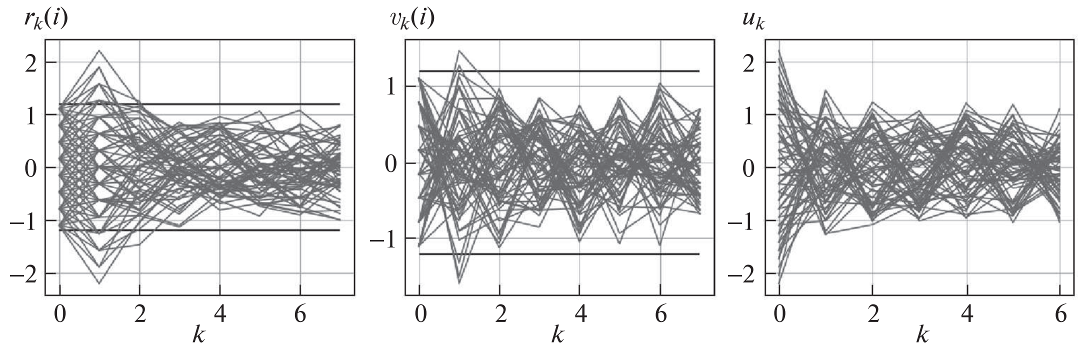

Fig. 2. The system trajectories $r_k(i), v_k(i)$, and $u_k(i)$.

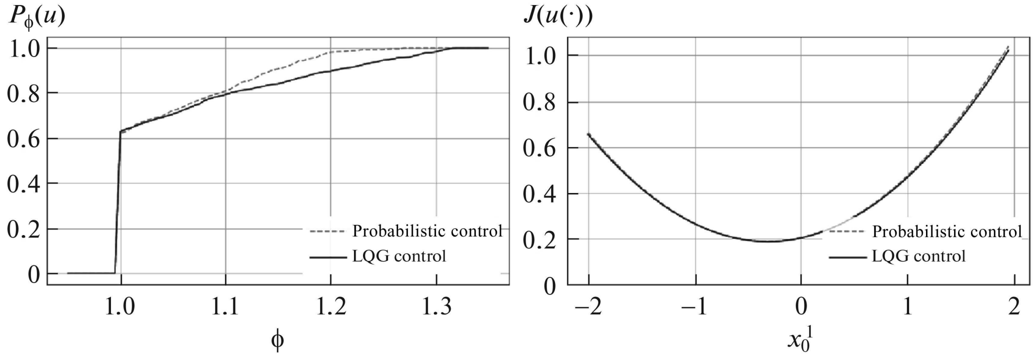

Fig. 3. The value of the criteria $P_\varphi(u(\cdot))$ and $J(u(\cdot))$ under the probabilistic and LQG controls.

To analyze the control proposed in Section 4 (the probabilistic control), we compare its performance with the Linear Quadratic Gaussian (LQG) control with an identity matrix Q and the criterion $J(u(\cdot)) = M[\sum_{k=0}^{N+1} x_k^T x_k]$.

The left-hand graph in Fig. 3 shows the frequency-based estimate of the probabilistic criterion depending on the parameter φ under the probabilistic (dashed line) and LQG (bold line) controls. To estimate the criterion, we simulated $M = 1000$ trajectories $r_k(i)$, $v_k(i)$, and $u_k(i)$, $k = \overline{0, N}$, $i = \overline{1, M}$, of the stochastic system for a given value φ. Clearly, for $\varphi \in [0, 1) \cup (1.35,\ +\infty]$, the values of the criteria coincide; on the interval [1, 1.35], however, the estimated probability of retaining the system trajectories is higher under the probabilistic control for the main part of the points φ.

The right-hand graph in Fig. 3 shows the frequency-based estimate of the criterion $J(u(\cdot))$ depending on the first coordinate x_0^1 of the initial state under the probabilistic (dashed line) and LQG (bold line) controls. (Here, the second coordinate has the fixed value $x_0^2 = 0.5$.) Clearly, the values of the criteria are very close, but the curve for the LQG criterion always runs slightly below its counterpart for the probabilistic criterion.

6. CONCLUSIONS

This paper has considered an optimal control problem for a time-invariant linear stochastic system with discrete time, scalar unbounded control, additive noise, and a probabilistic criterion for retaining its trajectories in a given neighborhood of zero. The isobells of levels 1 and 0 of the Bellman function and its two-sided estimates have been analytically expressed. The lower bound has served to design a suboptimal control, which becomes optimal if the system state lies on the isobells of levels 1 and 0. The effectiveness of these controls has been illustrated on a numerical example. The partial stationarity of the isobells of level 1 has been established for a particular case of a second-order system.

APPENDIX

Proof of Statement 1. At some step $k+1$, where $k = \overline{0, N-2}$, let the isobell of level 1 of the Bellman function have the form $\mathcal{I}_{k+1} = \mathcal{F} \cap \mathcal{F}' \cap \Delta\mathcal{I}_{k+1}$,

$$\Delta\mathcal{I}_{k+1} = \{x \in \mathbb{R}^n : \ \|\Lambda_{k+1} x\|_\infty \leqslant \varphi\},$$

where $\Lambda_{k+1} \in \mathbb{R}^{n_{k+1} \times n}$, $n_{k+1} \in \mathbb{N}$, and $n_{k+1} \geqslant n$ is some integer. Utilizing item 1 of Theorem 1, we find the isobell of level 1 at step k:

$$\begin{aligned}\mathcal{I}_k &= \mathcal{F} \cap \{x \in \mathbb{R}^n\colon \exists u \in \mathbb{R}\colon \mathbf{P}\left([Ax + Bu + C\xi_k] \in \{\mathcal{F} \cap \mathcal{F}' \cap \Delta\mathcal{I}_{k+1}\}\right) = 1\}\\ &= \mathcal{F} \cap \mathcal{F}' \cap \{x \in \mathbb{R}^n\colon \exists u \in \mathbb{R}\colon \mathbf{P}\left([Ax + Bu + C\xi_k] \in \{\mathcal{F}_n' \cap \mathcal{F}' \cap \Delta\mathcal{I}_{k+1}\}\right) = 1\},\end{aligned}$$

where $\mathcal{F}'_n = \{x \in \mathbb{R}^n : \ |e_n^{\mathsf{T}} \Lambda x| \leqslant \varphi\}$. Let us introduce the notations

$$\widetilde{\mathcal{I}}_{k+1} = \{\mathcal{F}'_n \cap \mathcal{F}' \cap \Delta\mathcal{I}_{k+1}\} = \left\{x \in \mathbb{R}^n : \ \max\left\{\left|e_n^{\mathsf{T}}\Lambda x\right|, \ \max_{i=\overline{1,n-1}}\left|e_i^{\mathsf{T}}\Lambda A x\right|, \ \|\Lambda_{k+1}x\|_\infty\right\} \leqslant \varphi\right\}$$

and construct a matrix $\widetilde{\Lambda}_{k+1} \in \mathbb{R}^{(n_{k+1}+n)\times n}$ of the form

$$\widetilde{\Lambda}_{k+1} = \left(\Lambda_{k+1}, \ e_1^{\mathsf{T}}\Lambda A, \ldots, e_{n-1}^{\mathsf{T}}\Lambda A, \ e_n^{\mathsf{T}}\Lambda\right)^{\mathsf{T}}.$$

Then

$$\widetilde{\mathcal{I}}_{k+1} = \left\{x \in \mathbb{R}^n : \ \left\|\widetilde{\Lambda}_{k+1}x\right\|_\infty \leqslant \varphi\right\}$$

and the isobell of level 1 of the Bellman function at step k is given by

$$\begin{aligned}\mathcal{I}_k &= \mathcal{F} \cap \mathcal{F}' \cap \left\{x \in \mathbb{R}^n : \exists u \in \mathbb{R} : \ \mathbf{P}\left(\left\|\widetilde{\Lambda}_{k+1}\left(Ax + Bu + C\xi_k\right)\right\|_\infty \leqslant \varphi\right) = 1\right\} \\ &= \mathcal{F} \cap \mathcal{F}' \cap \left\{x \in \mathbb{R}^n : \ \max_u \mathbf{P}\left(\left\|\widetilde{\Lambda}_{k+1}\left(Ax + Bu + C\xi_k\right)\right\|_\infty \leqslant \varphi\right) = 1\right\}.\end{aligned}$$

Consider the stochastic programming problem

$$\mathbf{P}\left(\left\|\widetilde{\Lambda}_{k+1}\left(Ax + Bu + C\xi_k\right)\right\|_\infty \leqslant \varphi\right) \to \max_u, \tag{A.1}$$

which serves to find the isobell of level 1, the lower bound on the Bellman function and the control (3) at step $k = \overline{0, N-2}$. In view of the notations (12)–(14), the objective function (A.1) is transformed as follows:

$$\begin{aligned}&\mathbf{P}\left(\left\|\widetilde{\Lambda}_{k+1}\left(Ax + Bu + C\xi_k\right)\right\|_\infty \leqslant \varphi\right) \\ &= \mathbf{P}\left(\max_{i=\overline{1,n_{k+1}+n}}\left|e_i^{\mathsf{T}}\widetilde{\Lambda}_{k+1}\left(Ax + Bu + C\xi_k\right)\right| \leqslant \varphi\right) \\ &= \mathbf{P}\left(\max_{i=\overline{1,n_{k+1}+n}}\left|\left(a_k^i\right)^{\mathsf{T}}x + b_k^i u + c_k^i\xi_k\right| \leqslant \varphi\right) \\ &= \mathbf{P}\left(\frac{-\mathsf{sign}\,(b_k^i)\,\varphi - (a_k^i)^{\mathsf{T}}x}{b_k^i} \leqslant u + \frac{c_k^i}{b_k^i}\xi_k \leqslant \frac{\mathsf{sign}\,(b_k^i)\,\varphi - (a_k^i)^{\mathsf{T}}x}{b_k^i}, \ \ \forall i = \overline{1, n_{k+1}+n}\right),\end{aligned} \tag{A.2}$$

where $c_k^i = e_i^{\mathsf{T}}\widetilde{\Lambda}_{k+1}C$. Note that $c_k^i/b_k^i = e_n^{\mathsf{T}}C/e_n^{\mathsf{T}}B$ for any $i = \overline{1, n_{k+1}+n}$. Then the objective function can be written as

$$\begin{aligned}&\mathbf{P}\left(\underline{\varphi}_k(x) \leqslant u + \frac{e_n^{\mathsf{T}}C}{e_n^{\mathsf{T}}B}\xi_k \leqslant \overline{\varphi}_k(x)\right) \\ &= \mathbf{P}\Bigg(-\frac{1}{2}\left(\overline{\varphi}_k(x) - \underline{\varphi}_k(x)\right) + \frac{1}{2}\left(\overline{\varphi}_k(x) + \underline{\varphi}_k(x)\right) \leqslant u + \frac{e_n^{\mathsf{T}}C}{e_n^{\mathsf{T}}B}\xi_k \\ &\qquad \leqslant \frac{1}{2}\left(\overline{\varphi}_k(x) - \underline{\varphi}_k(x)\right) + \frac{1}{2}\left(\overline{\varphi}_k(x) + \underline{\varphi}_k(x)\right)\Bigg) \\ &= \mathbf{P}\left(\left|u + \frac{e_n^{\mathsf{T}}C}{e_n^{\mathsf{T}}B}\xi_k - c_k(x)\right| \leqslant r_k(x)\right) \\ &= \mathbf{P}\left(\left|u + \frac{e_n^{\mathsf{T}}C}{e_n^{\mathsf{T}}B}\mathring{\xi}_k + \frac{e_n^{\mathsf{T}}C}{e_n^{\mathsf{T}}B}m_\xi - c_k(x)\right| \leqslant r_k(x)\right).\end{aligned} \tag{A.3}$$

Since the probability density function $f_{\mathring{\xi}_k}(t)$ of the centered random variable $\mathring{\xi}_k$ is an even function, problem (A.1) has the solution

$$u^* = \underline{\gamma}_k(x) = c_k(x) - \frac{c}{b} m_\xi, \tag{A.4}$$

and the optimal value of the objective function is given by

$$\mathbf{P}\left(\left\|\widetilde{\Lambda}_{k+1}(Ax + Bu^* + C\xi_k)\right\|_\infty \leqslant \varphi\right) = \mathbf{P}\left(\left|\frac{c}{b}\mathring{\xi}_k\right| \leqslant r_k(x)\right). \tag{A.5}$$

We write the transformed isobell of level 1 of the Bellman function at step k :

$$\begin{aligned}
\mathcal{I}_k &= \mathcal{F} \cap \mathcal{F}' \cap \left\{x \in \mathbb{R}^n : \mathbf{P}\left(\left|\frac{c}{b}\mathring{\xi}_k\right| \leqslant r_k(x)\right) = 1\right\} \\
&= \mathcal{F} \cap \mathcal{F}' \cap \left\{x \in \mathbb{R}^n : \left|\frac{c}{b}\varepsilon\right| \leqslant r_k(x)\right\} \\
&= \mathcal{F} \cap \mathcal{F}' \cap \left\{x \in \mathbb{R}^n : \underline{\varphi}_k(x) + 2\left|\frac{c}{b}\varepsilon\right| \leqslant \overline{\varphi}_k(x)\right\} \\
&= \mathcal{F} \cap \mathcal{F}' \cap \left\{x \in \mathbb{R}^n : \max_{i=\overline{1,n_{k+1}}} \frac{-\mathsf{sign}\,(b_k^i)\,\varphi - (a_k^i)^{\mathrm{T}} x}{b_k^i} + 2\left|\frac{c}{b}\varepsilon\right| \leqslant \min_{i=\overline{1,n_{k+1}+n}} \frac{\mathsf{sign}\,(b_k^i)\,\varphi - (a_k^i)^{\mathrm{T}} x}{b_k^i}\right\} \\
&= \mathcal{F} \cap \mathcal{F}' \cap \left\{x \in \mathbb{R}^n : \frac{-\mathsf{sign}\,(b_k^i)\,\varphi - (a_k^i)^{\mathrm{T}} x}{b_k^i} + \widetilde{\varepsilon} \leqslant \frac{\mathsf{sign}\left(b_k^j\right)\varphi - \left(a_k^j\right)^{\mathrm{T}} x}{b_k^j},\ \forall i,j \in \{1,\ldots,n_{k+1}+n\}\right\} \\
&= \mathcal{F} \cap \mathcal{F}' \cap \left\{x \in \mathbb{R}^n : \frac{-\mathsf{sign}\,(b_k^i)\,\varphi - (a_k^i)^{\mathrm{T}} x}{b_k^i} + \widetilde{\varepsilon} \leqslant \frac{\mathsf{sign}\left(b_k^j\right)\varphi - \left(a_k^j\right)^{\mathrm{T}} x}{b_k^j},\ \forall i,j \in \{1,\ldots,n_{k+1}+n\}\right\}.
\end{aligned}$$

Obviously, $\mathcal{I}_k \neq \varnothing$ if

$$\max_{i \in \overline{1,n_{k+1}+n}} \left|b_k^i\right| \leqslant 2\varphi\,(\widetilde{\varepsilon})^{-1}. \tag{A.6}$$

Due to (A.4), we obtain

$$\begin{aligned}
\mathcal{I}_k = \mathcal{F} \cap \mathcal{F}' \cap \Bigg\{x \in \mathbb{R}^n : &\frac{-\mathsf{sign}\,(b_k^i)\,\varphi - (a_k^i)^{\mathrm{T}} x}{b_k^i} + \widetilde{\varepsilon} \\
&\leqslant \frac{\mathsf{sign}\left(b_k^j\right)\varphi - \left(a_k^j\right)^{\mathrm{T}} x}{b_k^j},\ \forall i,j \in \{1,\ldots,n_{k+1}+n\}, i < j\Bigg\} \\
= \mathcal{F} \cap \mathcal{F}' \cap \Bigg\{x \in \mathbb{R}^n : &\left|\frac{\varphi}{\widetilde{\varepsilon} + \left(\mathsf{sign}(b_k^i) + \mathsf{sign}(b_k^j)\right)\varphi}\left(\frac{(a_k^i)^{\mathrm{T}}}{b_k^i} - \frac{\left(a_k^j\right)^{\mathrm{T}}}{b_k^j}\right)x\right| \leqslant \varphi, \\
&\forall i,j \in \{1,\ldots,n_{k+1}+n\}, i < j\Bigg\} \\
= \mathcal{F} \cap \mathcal{F}' \cap \{x \in \mathbb{R}^n :\ &\|\Lambda_k x\|_\infty \leqslant \varphi\} = \mathcal{F} \cap \mathcal{F}' \cap \Delta\mathcal{I}_k.
\end{aligned}$$

To complete the proof of item 1, we verify that at step $k = N - 1$, the isobell of level 1 of the Bellman function has the form (11). This result follows from subsection 4.1.

Item 1 of Proposition 1 is established.

Item 2 of Proposition 1 follows from (A.4) and (A.5).

Assume that at some step $k+1$, $k = \overline{0, N-2}$, the isobell of level 0 of the Bellman function has the form $\mathcal{O}_{k+1} = \overline{\mathcal{F}} \cup \overline{\mathcal{F}}'$. Using item 2 of Theorem 1, we find the isobell of level 0 at step k :

$$\begin{aligned}\mathcal{O}_k &= \overline{\mathcal{F}} \cup \Big\{x \in \mathbb{R}^n : \forall u \in \mathbb{R} : \mathbf{P}_{\xi_k}\Big([Ax + Bu + C\xi_k] \in \overline{\mathcal{F}} \cup \overline{\mathcal{F}}'\Big) = 1\Big\} \\ &= \overline{\mathcal{F}} \cup \overline{\mathcal{F}}' \cup \Big\{x \in \mathbb{R}^n : \forall u \in U_k : \mathbf{P}_{\xi_k}\Big([Ax + Bu + C\xi_k] \in \overline{\mathcal{F}} \cup \overline{\mathcal{F}}'_n\Big) = 1\Big\}, \quad \text{(A.7)}\end{aligned}$$

where $\overline{\mathcal{F}}'_n = \mathbb{R}^n \setminus \mathcal{F}'_n$ (see the proof of item 1 of Proposition 1). Obviously,

$$\Big\{x \in \mathbb{R}^n : \forall u \in \mathbb{R} : \mathbf{P}_{\xi_k}\Big([Ax + Bu + C\xi_k] \in \overline{\mathcal{F}} \cup \overline{\mathcal{F}}'_n\Big) = 1\Big\} = \varnothing,$$

which leads to $\mathcal{O}_k = \overline{\mathcal{F}} \cup \overline{\mathcal{F}}'$ by (A.7). Since this equality holds for $k = N - 2$, it will hold for all $k = \overline{0, N-2}$ as well.

Item 3 of Proposition 1 is established.

Item 4 of Proposition 1 follows from (9), (10), and item 3 of Proposition 1.

The proof of Proposition 1 is complete.

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